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COLLECTIVE-COORDINATE VS FIELD APPROACH TO THE TRANSIENT DYNAMICS OF DRIVEN AND DAMPED Φ^4 KINK

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We compare results of two alternative approaches to the transient dynamics of perturbed Φ^4 kinks: method of collective coordinates and a field approach using a generalized solitary wave-kink Ansatz including excited states for perturbed solution of the driven and damped Φ^4 equation. The field method is shown to give more rigorous results for transient regime of excited states. However, general characteristics of the transient regime - fast relaxation of the excited modes and exponential relaxation of the kink velocity are similar.

1. Introduction

In low dimensional (1+1 D quantum or 2D classical) many-body systems the fundamental concepts of 3D physics as mean field and the related perturbation theories fail. Namely, in 3D many particle systems nonlinearities in respective dynamic equations stemming from particle interactions can be foremost included into the mean fields and the remaining correlations can be accounted for within the perturbation theories (e.g. Hartree-Fock equations in many electron systems). Low dimensional many-body systems do not yield necessary conditions for to use the mean field concept and therefore dynamic equations remain basically nonlinear. Therefore, systems in low dimensions are very sensitive to various interactions: electron-phonon, electron-electron, interactions with impurities, external fields, etc. These interactions can drive various phase transitions, excited states, variety of dynamic effects, transient effects, etc.

Nonlinear dynamic equations in low dimensional physics often possess basically non-perturbative soliton or solitary wave solutions. Solitons as travelling stationary wave packets are widely accepted as quasiparticles of respective nonlinear field models representing lowest excited states above the ground state (see e.g. [1]). They are often of importance for determining the physical properties of the related systems. There exists number of comprehensive reviews on various aspects of soliton physics [1-8] .

The class of low dimensional field fermion models with SU(2) symmetry allows for a unified boson or phase description by sine-Gordon model which can be alternatively introduced by a special transformation of original fermion operators at certain additional conditions (often fulfilled at low temperatures). In solid state physics these models are continuous versions of the respective lattice models. Let us mention a few systems allowing for the bosonization: 1+1 quantum and 2D classical electron gas, electron-phonon system with "off-diagonal" interaction (Peierls), quantum 1D Heisenberg model of spins 1/2, classical 2D X-Y model, long Josephson junction, two-level models. There exists also a number of systems described by "Φ^4" model which can be considered as a short range approximation of the sine-Gordon model: electron-phonon system with "diagonal" interaction, domain walls in ferromagnets and ferroelectrics, an equilibrium Landau-Ginzburg model under the critical temperature.

Weak interactions present in real physical systems (impurities, phonons, external fields) perturb the soliton dynamics and can also destroy their stability. A comprehensive review of various aspects of the perturbed soliton dynamics was presented by Kivshar and Malomed [2].

In this paper we shall focus on transient effects of the soliton dynamics caused mostly by interplay of various competing interactions after switching on the interactions. If the competing interactions conserve a total energy of the system then an equilibrium state is reached asymptotically for large times. If the system does not conserve the total energy, then the soliton either gains or loses energy and it collapses after some time [8-11].

Let us assume that at time $t = t_0$ there exists a Lorentz invariant solution of the equation

$$\phi_{tt} - \phi_{xx} = -U_\phi[\phi(x,t)] \tag{1}$$

of the form $\phi[(x - vt_0 - x_0)\gamma]$, $\gamma = (1 - v^2)^{-1/2}$, where v is an arbitrary constant velocity. At $t = t_0$ we switch on a reservoir with a friction coefficient Γ and a constant external field f which are supposed to conserve the total energy, i. e.

$$\int_{-\infty}^{\infty} dx f \phi_t(x,t) = \int_{-\infty}^{\infty} dx \Gamma \phi_t^2(x,t), \tag{2}$$

and on the r.h.s. of equation (1) there appear respective force terms $-\Gamma\phi_t + f$. It was shown first by Collins et al. for ϕ^4 kink [12] that the condition (2) at small fields allows for a stable driven Lorentz invariant domain wall solution $\phi[(x - vt - x_0)\gamma]$ with a constant velocity $v \propto f/\Gamma$. Moreover, this result was shown to be more general: the function $v = v(f/\Gamma)$ is of a universal form for various functionals $U[\phi(x,t)]$ in equation (1) and is implied by the energy balance between the damping and driving forces [18].

We have shown [13] that the above result with a constant velocity is achieved only asymptotically for large times: there is certain time necessary for a system to achieve an equilibrium due to the acting competing forces. During this time the velocity $v(t)$ is time-dependent and phonons relax fastly than exponentially : only in the asymptotical region the relaxation becomes exponential leaving the kink stable and the velocity tends to a constant value . This transient effect is analogical to dressing of polarons or excitons interacting with phonons during their transport in solids.

In order to generalize the dynamics of the kinks in the case with external fields we have used the WKB Ansatz for the kink with time dependent parameters $v(t)$ and $\Omega(t)$ as follows

$$\Phi(x,t) = \Phi_K(\xi\gamma) + \phi(\xi,t) \tag{3}$$

where $\Phi_K(\xi\gamma)$ is a solitary wave solution, $\phi(\xi,t)$ is a small perturbation which satisfies equation (1) linearized about the single kink solution $\Phi_K(\xi\gamma)$. Further, $\xi(t) = x - x_0(t)$, $\gamma = (1 - \dot{x}_0^2(t))^{-1/2}$, $\phi(\xi,t) = \sum_n \phi_n(\xi)\exp(iQ_n(t))$, $v(t) = \dot{x}_0(t)$, $\Omega_n = \dot{Q}_n(t)$. The functions $v(t)$ or $x_0(t)$ and $\Omega_n(t)$ or $Q_n(t)$ are to be determined. The problem outlined above was solved in the paper [13] for the Φ^4 problem. It has been found solutions for $v(t)$ and $\Omega(t)$ showing an exponential relaxation of the velocity and a faster-than-exponential relaxation of $\phi(t)$ which becomes exponential in far asymptotic region. Similar generalization is known in the theory of systems with continuous symmetry, where the respective zero mode is excluded by introducing a dynamical variable (collective variable) as an alternative degree of freedom. In this paper we shall compare the results of both approaches, i.e. of the generalized WKB Ansatz (3) and the collective coordinate Ansatz for a Φ^4 kink perturbed by the external fields f and Γ at the condition (2).

2. Collective coordinate approach.

In systems with continuous symmetry there was developed an alternative approach to soliton or solitary wave dynamics based on collective coordinates as dynamical variables which represent the degree of freedom (zero mode) manifesting implicitly invariance of the respective Lagrangian against the transformation of the continuous symmetry. Rice [14] has found a unified way of description of Φ^4 and sine-Gordon kinks in collective coordinates, representing them as particle-like deformable objects with center-of-mass coordinate and the width as coupled collective degrees of freedom. One bound linear mode for both cases was found differing only by constant parameters. In the Φ^4 case this internal mode is an exact eigenstate of the linearized Φ^4 equation (linearized about the single-kink solution) whose eigenfrequency lies in the gap below the phonon gap edge. In the SG case there is no exact bound eigenstate (other than the zero-frequency Goldstone mode). This puzzle was solved by Boesch and Willis [15] who have elaborated a collective mode formalism which accounts for also linear modes. The frequency calculated by Rice corresponding to the quasi-internal mode for the SG system was found in the phonon continuum.

We shall compare results of the Rice's collective coordinate method [16] and the direct field method of the solution of the Φ^4 case damped by a reservoir and in a constant external field [13] using the Ansatz (3) and the condition (2). We show that also in Φ^4 case there are differences for excited states when compared the results obtained by both approaches. However, the essential characteristics of the transient regime are the same in both cases.

In what follows we shall summarize shortly the results of [16]. The density of

Lagrangian of our system is

$$L[\Phi(x,t),\Phi_t,\Phi_x,q_k,\dot q_k]=\frac{1}{2}\Phi_t^2-\frac{c_0^2}{2}\Phi_x^2-\omega_0^2V(\Phi)-F\Phi+\frac{1}{2}\sum_k M_k[\dot q_k^2-\omega_k^2(q_k-\frac{\gamma_k\Phi}{\omega_k^2})^2], \quad (4)$$

where the field $\Phi(x,t)$ is linearly coupled to the heat bath represented as a system of harmonic oscillators, c_0 and ω_0 are constants of the field, M_k,ω_k are constants of the heat bath, $V(\Phi)=\frac{1}{8}(1-\Phi^2)^2$. In the presence of small perturbations we start from the Ansatz

$$\Phi(x,t)=\sigma\tanh[2(x-x_0(t))/l(t)]+\Phi_s, \quad (5)$$

where $x_0(t)$ and $l(t)$ are generalized collective coordinates , center-of-mass and the width of solitons, respectively, $\sigma=\pm1$ and Φ_s is a constant stationary solution. The respective dynamic equations are derived from the total Lagrangian

$$L=L(x_0,\dot x_0,l,\dot l,q_k,\dot q_k)=\int dx L[\Phi(x,t),\Phi_t,\Phi_x,q_k,\dot q_k], \quad (6)$$

where $\Phi(x,t)$ is given by (5). If we assume markoffian approximation for phonons of the heat bath with the broad quadratic distribution of the phonon density $\rho(\omega)=A\omega^2,\quad \omega\le\omega_D$, then we can exclude memory term from the solution for $q_k(t)$ [16]. The heat bath is then represented by a friction force $\Gamma=\frac{1}{2}\pi A\gamma^2M$, where γ and M are the same for all oscillators. Then we get for collective coordinates dynamic equations

$$\frac{d}{dt}(\frac{\dot l}{l})+\frac{1}{2}(\frac{\dot l}{l})^2+\Gamma\frac{\dot l}{l}+\frac{c_0^2}{2\alpha}(\frac{\cos\Phi_s}{l_0^2}-\frac{1}{l^2})+2\beta^2(1-\exp(-\Gamma t))^2=0, \quad (7a)$$

where $\beta=f/(2\sqrt{\alpha}\Gamma m_sl_0)$, $\quad\alpha=\frac{\pi^2-6}{48}$, $\quad m_s=\frac{2}{3}c_0^{-1}\omega_0$, $\quad l_0=4$ and

$$\dot x_0(t)=\frac{1}{(m_sl_0)}l(t)[\frac{f}{\Gamma}(1-e^{-\Gamma t})+\zeta(t)]. \quad (7b)$$

Here, $\zeta(t)$ is a random force related to the reservoir which was calculated explicitly for the system of harmonic oscillators of the reservoir [16]. In accordance with our expectation it is evident from eqs. (7) that the problem is solved by finding the solution of eq. (7a). Eq. (7a) can be rewritten by using Ansatz

$$l(t)=g^2(t) \quad (8)$$

as

$$\ddot g+\Gamma\dot g+\frac{\Omega^2}{4}g-\frac{c_0^2}{4\alpha}g^{-3}-\beta^2e^{-\Gamma t}(2-e^{-\Gamma t})g=0, \quad (9)$$

where

$$\Omega^2=\frac{c_0^2}{\alpha l_0^2}[\cos\Phi_s+(\frac{f}{\Gamma m_sc_0})^2].$$

Equation (9) represents the driven and damped harmonic oscillator with the constant asymptotic value $g_s^2=l_s(p_x^{(\infty)})$. It can be solved perturbatively using the Ansatz

$g(t)=g_s+g_1(t)$ for small $g_1(t)$ where we linearize $g^{-3}\approx g_s^{-3}(1-3g_1/g_s)$. The approximation is valid for $|g_1|<<g_s,t>>\Gamma^{-1}$. When introducing a new variable $\xi=\exp(-\Gamma t)$, equation (9) can be rewritten as

$$\xi^2g''+\Gamma^{-2}[\Omega^2-\beta^2\xi(2-\xi)]g=\Gamma^{-2}\Omega^2g_s. \quad (10)$$

Solution to the homogeneous version of (10) can be expressed as

$$g(t)=g_s+\xi^{(\frac{1}{2}+\nu)}\exp(\pm i\frac{\beta}{\Gamma}\xi)w(\xi). \quad (11)$$

Here, $w(\xi)$ is a linear combination of functions

$$w_1=\phi(\frac{1}{2}+\nu\pm i\frac{\beta}{\Gamma},1+2\nu;x) \quad (12a)$$

$$w_2=x^{-2\nu}\phi(\frac{1}{2}-\nu\pm i\frac{\beta}{\Gamma},1-2\nu;x), \quad (12b)$$

where $\phi(\alpha,\gamma;x)={}_1F_1(\alpha,\gamma;x)$ is the degenerate hypergeometric function, $x=\pm\frac{2\beta}{i\Gamma}\xi$, $\nu=\pm\frac{1}{2}\sqrt{1-4\Omega^2/\Gamma^2}$. Finally, solution for inhomogeneous eq.(10) reads

$$g(t)=g_s+\xi^{1/2+\nu}[g_I+C_1w_1(\xi)+C_2w_2(\xi)] \quad (13)$$

where w_1 and w_2 are homogeneous solutions given by (12a) and (12b), respectively, and g_I is a particular solution to the inhomogeneous equation (10) which can be expressed by linear combination of products of the hypergeometric and Bessel functions [16].

The solution (13) determines very fast time dependence except of the asymptotic region $\xi<<1$, where the relaxation becomes exponential

$$g(t)\approx g_s+C_1e^{-\Gamma t/2}\sin(t\sqrt{\Omega^2-\Gamma^2/4})$$

$$+\frac{g_sf^2}{4\nu}e^{-\Gamma t/2}\Big[\frac{e^{-\Gamma t/2}}{2\nu+3}-\frac{2}{2\nu+1}-\frac{f}{2}e^{-\Gamma t}(\frac{e^{-\Gamma t/2}}{3-2\nu}-\frac{2}{1-2\nu})\Big]. \quad (14)$$

In the asymptotic region the mean square displacement for the kink center reads

$$<x^2>^0=v_s^2t^2+2D(t-\frac{e^{-\Gamma t}-1}{\Gamma}), \quad (15)$$

where $v_s=\frac{f}{\Gamma m_sl_0}$ and $D=4\Gamma k_BT(\frac{l(p_x^{(\infty)})}{m_sl_0\Gamma})^2$.

3. Field approach to the dynamics of the Φ^4 kink and of its fluctuations.

We shall briefly summarize relevant results of the paper [13] related to the transient behaviour in question: When inserting the generalized WKB Ansatz (3) with $\Phi_K(x,t)$ given by

$$\Phi_K(\xi\gamma)=\Phi_0+\Phi_1\tanh(q\gamma\xi), \quad (16)$$

where ξ, γ and $\phi(\xi, t)$ are defined below equation (3) and Φ_0, Φ_1, q, $x_0(t)$ and $Q_n(t)$ are to be determined. Then, from equation (1) with the r.h.s. $= \Phi - \Phi^3 + \lambda\Phi_t - f$ one gets the constants Φ_0, Φ_1, λ, q . The damping force λ is relevant for the time behaviour of the velocity: When introducing the Ansatz (16) into equation (3) with the r.h.s. defined above, we get

$$\lambda = \frac{\ddot{x}_0 + \Gamma\dot{x}_0}{(1-\dot{x}_0^2)^{1/2}}. \tag{17}$$

From the condition of the constant friction λ (17) we get equation for $v(t)$,

$$\dot{y} + \Gamma(y - v_c/\sqrt{2})(1+y^2) = 0, \tag{18}$$

where $y = \tan v(t)$ and $v_c = \pm 3f/\Gamma\sqrt{2}$. By integration of (18) it is easy to find

$$\log|\sin v \pm v_c \cos v| \pm v_c v = -\Gamma(1+v_c^2)(t-t_0). \tag{19a}$$

This can be further simplified for $|v| << 1$ as

$$\dot{x}_0(t) = v(t) = [v_c \pm \exp(-\Gamma(1+v_c^2)(t-t_0)][1 \pm v_c \exp(-\Gamma(1+v_c^2)(t-t_0))]^{-1}. \tag{19a}$$

For the width of the kink one gets

$$l(t) = [\gamma\Phi_1/\sqrt{2}]^{-1} = \sqrt{2}(1-v(t)^2)^{1/2}[1 - \frac{(\dot{v}(t)+\Gamma v(t))^2}{(1-v(t)^2)}]^{-1/2}, \tag{20}$$

where $v(t)$ is given by (19). $x_0(t)$ can be obtained from (19) as $x_0(t) = \int_{t_0}^{t} v(t')dt'$.

Time behaviour of the linear modes defined by the WKB Ansatz (3) with

$$\phi(\xi) = \sum_n \phi_n \exp(iQ_n(t)) \tag{21}$$

is described by the linearized equation for $\Psi_n(\xi) = \exp(\frac{\alpha}{2}\xi)\phi_n$ which reads

$$\Psi_n'' + (\beta_n(t) - \frac{\alpha_n^2(t)}{4} - W)\Psi_n = 0, \tag{22}$$

where $\alpha_n(t) = f\gamma + 2iv(t)\gamma^2\dot{Q}_n$, $\beta_n(t) = [\dot{Q}_n^2 - i\ddot{Q}_n - i\Gamma\dot{Q}_n - 2(1-\frac{3}{4}f^2)]\gamma^2$, $W = \gamma^2[3f(1-\frac{3}{4}f^2)\tanh(\Phi_1\gamma\xi/\sqrt{2}) - 3(1-\frac{3}{4}f^2)\cosh^{-2}(\Phi_1\gamma\xi/\sqrt{2})]$ and γ is defined by the formula (3). For the solution of (22) we use the bare condition

$$\beta_n(t) - \frac{\alpha_n^2(t)}{4} = E_n, \tag{23}$$

which requires asymptotic stationary solutions of eq.(22). Here, E_n is known as an eigenenergy of the non-symmetric double potential well with Rosen-Morse potential [17],

$$E_n = -\frac{\gamma^2}{2}[(\frac{3f}{2} - n)^2 + (4-3f^2)(1-\frac{n}{2})^2] \tag{24}$$

with two bound states : $n = 0$ and $n = 1$ for $f < f_c$, where f_c is a critical field below which the bound state $n = 1$ survives. Equation for the time behaviour of the linear modes $\Omega_n(t) = \dot{Q}_n(t)$ then reads

$$\dot{\Omega}_n + \gamma(1 + v_c\gamma_c v\gamma)\Omega_n + \gamma^2\Omega_n^2 + f(n) = 0. \tag{25}$$

Here, v_c is given by (18) , $\gamma_c = (1-v_c^2)^{\frac{-1}{2}}$ and $f(0) = 0$, $f(1) = \frac{3}{2}(1-3f^2)$. For $n = 0$ we get splitting of the double degenerated [17] zero mode: in the limit $t \to \infty$ we get $\Omega_{(0,1)} = 0$, $\Omega_{(0,2)} = -\Gamma$. Hence, of two split modes, one is asymptotically unchanged, the second is the zero inertia mode. For the transient behaviour of the zero inertia mode for $v << 1$ we get

$$\exp(iQ_0) \approx v(t)\exp(\frac{1}{2}v(t)^2). \tag{26}$$

For $n = 1$ we get equation

$$\frac{d^2g}{dv^2} + \Omega(v)g = 0, \tag{27}$$

where $g(v)$ and $\Omega(v)$ are related to $\Omega_n(t)$ and $v(t)$, respectively, by rather complicated transformations [13]. However, for small $v(t)$ one is able to get analytical results. Finally, the low field time behaviour for $n = 1$ is

$$iQ_1(v) \approx -\frac{f(1)}{\Gamma^2(b-1)}\log|v - v_c|, \tag{28}$$

where $b - 1 = -\frac{1}{2} \pm \frac{1}{2}(1 - 4f(1)/\Gamma^2)^{\frac{1}{2}}$. Therefore, the bound state $n = 1$ is split due to the perturbations as well. There are two possible kinds of relaxation regimes:

(a) the regime of damped oscillations for $f(1) > \frac{\Gamma^2}{4}$ with two branches

$$iQ_1(v(t)) \approx [\frac{1}{2} \pm \frac{if^{1/2}}{\Gamma}(1 - \frac{\Gamma^2}{4f(1)})^{\frac{1}{2}}]\log|v(t) - v_c|, \tag{29a}$$

(b) the purely damped regime for $\frac{\Gamma^2}{4} > f(1)$ with two branches,

$$iQ_1(v(t)) \approx [\frac{1}{2} \pm \frac{1}{2}(1 - 4\frac{f(1)}{\Gamma^2})^{\frac{1}{2}}]\log|v(t) - v_c|. \tag{29b}$$

Here, $v(t)$ is an exponential function of t according to equation (19).

4. Conclusion

To find common features of the results of both descriptions presented in Sections 2 and 3 we have to emphasize that Rice's collective coordinate formalism describes a classical dynamics of a kink as a coupled center-of-mass coordinate and a width of the kink. Effect of the perturbations on the collective motion of the kink manifests itself as transient oscillations of the kink width and due to the coupling (7b) also as transient oscillations of the center-of-mass motion. The frequency of the oscillations is a complicated function of time (equations (11)-(13)). From eqs. (12) and (13) and also from the numerical solution of the equation (7a) given in [16] it is evident that the small time behaviour of the kink-width is much faster than exponential. In the asymptotic limit of the large time the relaxation of the width becomes exponential (14).

In the field approach of the Sect.3 internal state of the kink is described as a set of quantum excited states of the kink described as a solution of the related Schrödinger equation, (see equations (21)- (25)). Equation (25) describes the transient time dependence of the eigenfrequencies which shows up again faster than exponential time behaviour of the respective wave functions. Only in the asymptotic regime for $t \to \infty$ they become exponentially damped so that the fluctuations disappear leaving the kink stable. The stability is implied by the energy balance condition for the competing constant force and the damping force due to the reservoir. The description of the excited states provided by the field method of the Sect.3 is evidently much more rigorous than the description by the collective coordinate method. The transient regime with the dressing of solitons is analogous to the dressing of polarons and excitons interacting with phonons during their transport in solids. The balance of the competing friction and driving forces implying the transient regime is also reminiscent of the motion of a particle moving in a viscous fluid in the gravitation field.

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References

[1] A.R. Bishop, J.A. Krumhansl, S.E. Trullinger: *Physica* **1D** (1980) 1;

[2] Y.S. Kivshar, B.A. Malomed: *Rev. Mod. Phys.* **61** (1989) 763;

[3] A.J. Heeger, S. Kivelson, J.R. Schrieffer, W.-P. Su: *Rev. Mod. Phys.* **60** (1988) 781;

[4] R. Rajaraman: *Solitons and Instantons* Amsterdam, New York, Oxford: North-Holland 1982;

[5] *Interacting electrons in reduced dimensions.* Eds. D. Baeriswyl, D. K. Campbell, Plenum, NATO ASI Series, Series B: Physics Vol.213, 1989;

[6] V.J. Emery: in *Highly conducting one-dimensionl solids* Ed. J. Devreese, R. Evrard and V. van Doren . New York: Plenum p.247;

[7] A.S. Davydov: *Solitons in Molecular systems.* D. Reidel Publ. Company 1985;

[8] M. Inoue, S.G. Chung: *J. Phys.Soc. Jpn.* **46** (1979) 1594;

[9] E. Majerníková, G. Drobný: *Phys. Rev.* **E47** (1993) 3677;

[10] E. Majerníková, Y.B. Gaididei, O.M. Braun: *Phys. Rev.* **E** (in press) ;

[11] E. Majerníková: *Phys. Rev.* **E49** (1994) 3360;

[12] M.A. Collins, A. Blumen, J.F. Currie: *Phys. Rev.* **B19** (1979) 3630;

[13] E. Majerníková: *Z. Phys.B Condensed Matter* **78** (1990) 507;

[14] M.J. Rice: *Phys. Rev.* **B28** (1983) 3587;

[15] R. Boesch, C. R. Willis: *Phys. Rev.* **B42** (1990) 2290;

[16] E. Majerníková, G. Drobný: *Z. Phys. B-Condens. Matter* **89** (1992) 123;

[17] E. Magyari: *Z. Phys. B-Condens. Matter* **55** (1984) 137;

[18] M. Büttiger, H. Thomas: *Phys. Rev. A* **37** (1988) 235;