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PHASE DISTRIBUTIONS IN QUANTUM OPTICS VIA GENERALIZED $SU(1,1)$ COHERENT STATES

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We present an operational approach for a description of phase distributions of quantum states of a single mode of the radiation field. These operational phase distributions are defined via $SU(1,1)$ generalized coherent states. In particular, we study generalized coherent states based on the bosonic representations of the $SU(1,1)$ Lie algebra characterized by the Bargmann index k equal to 1/2 which have been recently introduced by Bužek [V.Bužek, *Phys. Rev. A* **39** (1989) 3196]. We discuss very appealing phase properties of these nonclassical states of light and we analyze phase distributions defined via them. We analyze the interaction of a two-level atom with a single-mode cavity field prepared initially in the $SU(1,1)$ coherent state under consideration. We find a phase-locking effect via which the phase of the atomic coherent state can be determined.

1. Introduction

Even though the theory of nonrelativistic quantum mechanics was completed almost seventy years ago there are fundamental problems in this theory which only recently have been clarified. One of these problems is related to the existence of a Hermitian phase operator of the harmonic oscillator (or a single mode of the electromagnetic field). The classical electromagnetic field can be described by its amplitude, i.e., the square root of the intensity of the field, and its phase. In the quantum theory the amplitude of the field is proportional to the square root of the photon number operator, but the question is how to define the phase operator.

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There exist two different (nevertheless, intrinsically closely related) concepts of phase in modern quantum optics. The first concept is based on a definition of a Hermitean phase operator (observable). The second approach is based on an operational definition of phase states through which phase distributions are defined (measured).

1.1. Phase operators

From the complementarity principle [1,2] it follows that for each degree of freedom the dynamical variables are a pair of complementary observables [3]. This implies that there should be a Hermitian operator conjugate to the excitation (photon) number operator such that a precise knowledge of one of them implies that all possible outcomes of measuring the other are equally probable. Dirac [4] was the first to introduce a Hermitian phase operator of the electromagnetic field. He utilized the Poisson-bracket-commutator correspondence principle [5] and suggested that the photon number operator $\hat{n}$ and the phase operator $\hat{\Phi}$ should obey the canonical commutation relation

$$[\hat{n}, \hat{\Phi}] = -i, \tag{1.1}$$

and that the annihilation $\hat{a}$ and the creation $\hat{a}^\dagger$ operators of the single mode of the electromagnetic field (for which $[\hat{a}, \hat{a}^\dagger] = 1$) can be expressed in the polar form

$$\hat{a} = \exp(i\hat{\Phi})\sqrt{\hat{n}} \quad ; \quad \hat{a}^\dagger = \sqrt{\hat{n}}\exp(-i\hat{\Phi}). \tag{1.2}$$

It was shown by Louisell [6] and Susskind and Glogower [7] that the number-phase commutator (1.1) is not consistent with the existence of a well-defined Hermitian phase operator (see also Refs. [8,9]). Later there were several attempts to define Hermitian phase operators consistently by introducing periodic functions of the phase [6,10,11]. These attempts did not however solve the problem (for details see Refs. [7,9,12]). Susskind and Glogower [7] proposed exponential operators $\widehat{\exp}(i\Phi)$ and $\widehat{\exp}(-i\Phi)$ which are not functions of a common phase operator $\hat{\Phi}$ [13] (for more discussion see Section 4 of our paper). The Susskind-Glogower (SG) phase operators have been applied in a variety of problems in quantum optics. In particular, using this operator Carruthers and Nieto [14] have studied the phase properties of coherent states.

Susskind and Glogower [7] realized that the main problem in the proper definition of a phase operator lies in the existence of a cut-off in the spectrum of the number operator which excludes the negative integers. In fact there are two possible ways to overcome this problem of the semiboundedness of the energy spectrum of the harmonic oscillator and hence to define the phase operator consistently. One possibility is to extend the normal harmonic-oscillator Hilbert space to include negative number states (i.e., the spectrum of the harmonic oscillator is unbounded, but simultaneously it is assumed that the negative-energy states are decoupled from the positive-energy ground state [15]). Recently it has been shown that this approach suffers from some inconsistencies [16] which are due to the unbounded state space. The second possibility to treat the problem of the phase operator is to suppose the spectrum of the harmonic oscillator to be bounded, that is to consider a finite-dimensional Hilbert space of the harmonic oscillator [17].

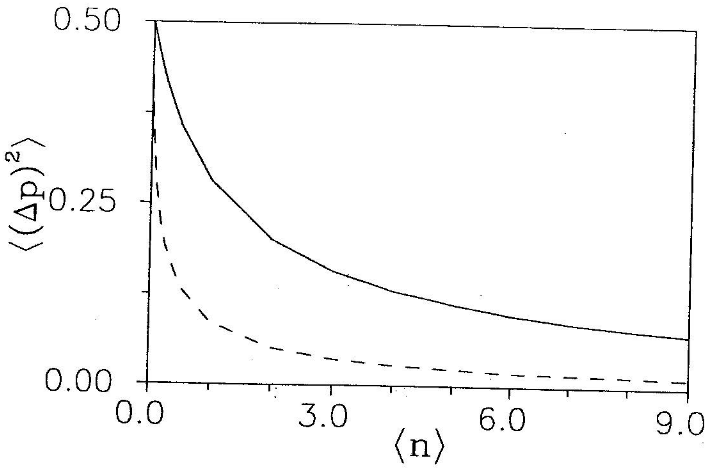

Fig.1. We plot the value of the variance of the squeezed quadrature of the squeezed vacuum state (dashed line) and the $SU(1,1)$ GCS (solid line) as a function of the mean photon number in the particular state. We see, that for a given value of the mean photon number the squeezed vacuum state is more squeezed than the $SU(1,1)$ GCS under consideration. This result can be understood as a consequence of the fact that part of the mean energy (i.e., $\bar{n}$) of the $SU(1,1)$ GCS is associated with a displacement in the phase space so not whole energy is used for squeezing (see Fig.2a). In the limit of infinite squeezing both states are infinitely squeezed. We should stress here that the $SU(1,1)$ GCS given by Eq.(2.27) is not a minimum uncertainty state.

Recently Pegg and Barnett [12,16,18] defined the Hermitian phase operator in a finite-dimensional state space. They used the fact that in this state space one can define phase states rigorously. The phase operator is then defined as the projection operator on the particular phase state multiplied by the corresponding value of the phase. The main idea of the Pegg-Barnett (PB) formalism consists in evaluation of all expectation values of physical variables in a finite-dimensional Hilbert space. This gives a real number which depends parametrically on the dimension of the Hilbert space. Because a complete description of a real harmonic oscillator involves an infinite set of number states, the infinite limit must be taken (for more details see paper by Bužek et al. [19]). This limit is taken only *after* the physical results (mean values of observables) are evaluated thereby leading to a proper limit which corresponds to the results obtainable in ordinary quantum mechanics (for further work concerning the relation between the SG and the PB formalisms see recent papers by Lukš and Peřinová [20]). It can be used for investigation of the phase properties of quantum states of the single mode of the electromagnetic field. In last few years the PB formalism has been applied to various problems in quantum optics. In particular, it has been shown that the uncertainty product of the number and the phase fluctuations of a highly excited coherent state is minimized with increasing intensity of the coherent field [12]; it has been also shown that the number states of the single mode of the electromagnetic field are the minimum uncertainty states [21] of light. Phase properties of a single mode squeezed vacuum have been analyzed and the relation between the squeezing parameter and the form of the phase probability distribution has been found [22-24]. In addition phase properties of the cotangent states were recently analyzed [25] and the PB formalism has been used for investigation of the phase correlations between two modes of the electromagnetic field [26]. In particular, the phase properties of the two-mode squeezed vacuum have been studied [27] and the interesting feature of phase locking has been revealed. The PB formalism has been adopted to describe optical phase diffusion [28], phase-difference fluctuations in a quantum-beat laser [29], phase fluctuations in a laser with an atomic memory [30] and phase properties of coherent light interacting with a two-level atom [31] or nonlinear medium [32].

1.2 Operational definition of phase

In the operational approach the phase is a quantity measured in an experiment which is considered to measure phase (at least in a classical limit). We can consider two variants of the operational approach. The first one is more abstract and is related to a quantum-mechanical description of an overlap between the measured state and the reference (phase) state. The second variant of the operational approach is more pragmatic and is related to a direct measurement schemes via which phase information about the system under consideration can be obtained.

1.2.1 Phase via overlaps with phase states

Following the von Neumann theory of measurement we can associate phase with a phase-dependent distributions obtained via a quantum-mechanical overlap (scalar product) between the reference (phase) state and the state under consideration. As a prototype of this approach we can consider the Vogel-Schleich operational phase distribution [33] defined as:

$$P^{(\mathrm{VS})}_{|\Psi\rangle}(\phi) \equiv \mathcal{N}\,|\langle\Psi|\Phi(\phi)\rangle_{\mathrm{VS}}|^2\,, \tag{1.3}$$

where $\mathcal{N}$ is a normalization constant and $|\Psi\rangle$ is the state to be "measured". The Vogel-Schleich "phase-state" $|\Phi(\phi)\rangle_{\mathrm{VS}}$ is defined as a rotated eigenstate of the position operator $\hat{q}$ with the mean value of the position equal to zero, i.e.,

$$|\Phi\rangle_{\mathrm{VS}} \equiv \hat{U}(\phi - \pi/2)|q\rangle, \tag{1.4}$$

where $\hat{U}(\phi)$ is the rotation operator defined as usually

$$\hat{U}(\phi) = \mathrm{e}^{-i\hat{n}\phi}\,, \tag{1.5}$$

and $|q\rangle$ is an eigenstate of the position operator ($\hat{q}|q\rangle = q|q\rangle$). The particular choice of the phase $\phi - \pi/2$ in Eq.(1.4) is related to a choice of the reference phase equal to zero. We remind us that the phase distribution (1.3) can in principle be measured in a noise-free process, because fluctuations related to the measurement (filtering) are eliminated due to the specific choice of the reference state which is characterized by zero phase fluctuations.

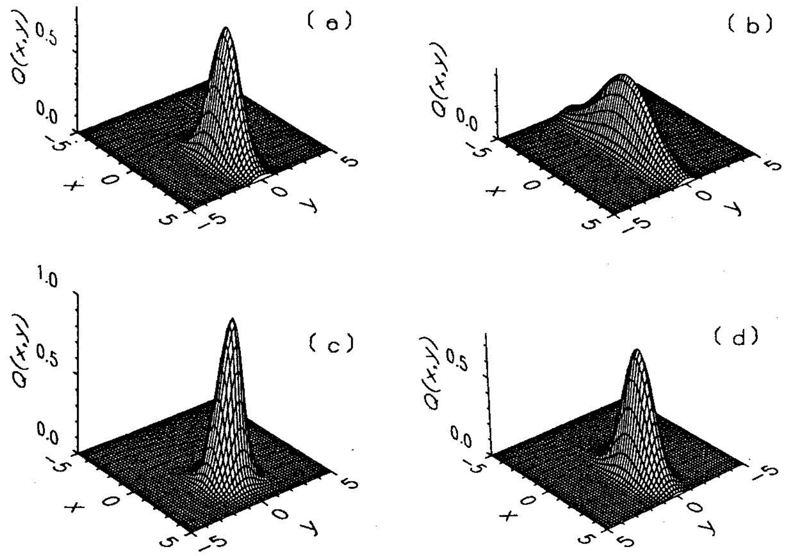

Fig.2. We plot the Q-function of the $SU(1,1)$ GCS given by Eq.(2.27) [Fig.(a)]; the Q function of the squeezed vacuum given by Eq.(2.25) [Fig.(b)] and the Q-function of the coherent state $|\alpha\rangle$ [Fig.(c)]. The mean value of the number operator in all these states is equal to 4. For comparison purposes we also plot the Q-function of the displaced squeezed state $\hat{D}(\alpha)\hat{S}(\eta)|0\rangle$ for such value of parameter, that α is equal to the mean amplitude $\langle\hat{a}\rangle$ of the $SU(1,1)$ GCS with 4 photons. The squeezing parameter η is chosen to be given by the relation $|\eta|^2/(1-|\eta|^2) = 4 - \langle\hat{a}\rangle^2$, so that the total number of photons in the displaced squeezed state is equal to 4 [Fig.(d)].

1.2.2 Phase-space measurements

There exists a more pragmatic approach to the phase measurement, when the phase is associated with a quantity measured in experiments which are considered to be quantum counterparts of experiments assigned to measure phase in the classical regime [34-38]. We can consider several types of these experiments. In particular, phase can be measured via amplification, via heterodyning and via beam splitting.

Amplification

The first to propose a realistic scheme for a phase measurement in quantum optics were Bandilla and Paul [39]. They suggested to strongly amplify (with the help of a linear laser amplifier or a parametric amplifier) the microscopic initial (signal) field and to apply afterwards well known classical interference techniques to measure the phase on the signal field amplified to a macroscopic level. Due to the unavoidable presence of amplifier noise, the measuring process under consideration is "noisy". Therefore the measured phase distribution is biased (i.e., is broadened) compared to an ideal scheme of the measurement modelled by a direct scalar product of the reference phase state and the measured state [compare with Eq.(1.3)] It has been shown by Schleich et al.

[40] that amplification scheme proposed by Bandilla and Paul is effectively equal to a measurement of the Husimi Q function of the signal field. This means that the method proposed by Bandilla and Paul corresponds to a simultaneous phase-space measurement of canonically conjugated observables (for details see [41]). It is well know (see, for instance, [41] and references therein) that the Q function can be interpreted as the propensity corresponding to a measured state providing the quantum filter (which gives origin to quantum-mechanical noise due to which measured data are biased) is in a vacuum state.

Heterodyning

Another method for a simultaneous measurement of phase and amplitude has been suggested by Shapiro and Wagner [36] who have discussed the following heterodyning scheme: By means of a beam splitter the signal at the frequency $\omega_0 + \Delta\omega$ is mixed with a strong coherent local oscillator field at the frequency ω_0 and is sent to a (unit-efficiency) photodetector. On the measured a.c. photocurrent oscillating on the difference frequency $\Delta\omega$ amplitudes A_1 and A_2 of two components oscillating out of phase (as $\sin \Delta\omega t$ and $\cos \Delta\omega t$) are separately measured. From the measured data the Q function of the signal can be reconstructed. The Q function contains information (biased by the measurement process) about the phase of the signal.

Beam splitting

A recent operational definition of phase due to Mandel et al. [38] utilizes an eight-port homodyne detection scheme [35]. In this set-up the signal is divided with the help of a lossless 5L : 50 beam splitter into two parts. On each of them a homodyne measurement is carried out, whereby the reference beams differ mutually in their phases by $\pi/2$. If the reference beams are very strong coherent fields, then the homodyne measurement on the split beam corresponds to a (noisy) measurement of two quadratures of the signal field. From here again the Q function of the signal can be reconstructed and the information about the phase of the signal can be obtained.

All three processes described above represent a model description of a realistic quantum-mechanical phase-space measurement of the Q function of the signal under consideration.

In the present paper we focus our attention on an operational definition of a phase state with the help of which we define a phase distribution (i.e., we confine ourself within a framework of the approach introduced in subsection 1.2.1). We utilize a group-theoretical approach for a definition of operational phase-states. In particular, we analyze a bosonic representation of the $SU(1,1)$ Lie algebra with Bargmann index equal to 1/2. We construct generalized coherent states (GCS) corresponding to this algebra. We show that these states can be used for a very precise phase-shift measurements. In Section II we briefly describe $SU(1,1)$ GCS. Section III is devoted to an investigation of phase properties of our operational phase states. We conclude the paper with discussion in Section IV.

2. Generalized coherent states and their application to phase -shift measurement

It is well known that the phase noise $\Delta\phi$ in a coherent state $|\alpha\rangle$ with large mean photon

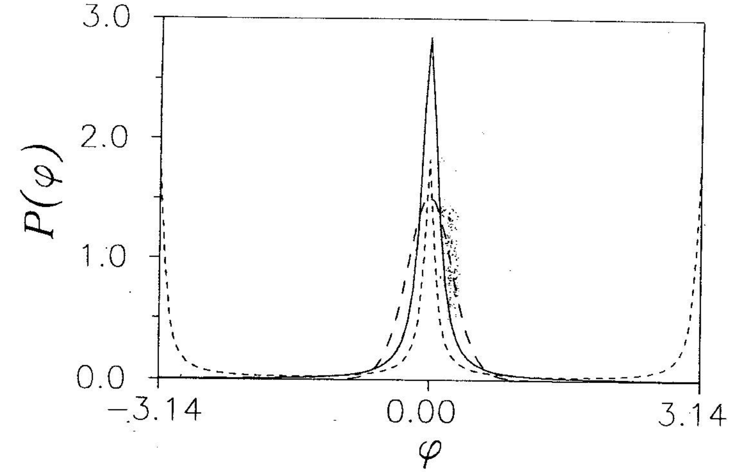

Fig.3. The Pegg-Barnett phase probability distribution of the $SU(1,1)$ GCS (solid line); the squeezed vacuum state (short-dashed line) and the coherent state (long-dashed line). In all cases the mean photon number is equal to 4. We assume $\theta = 0$.

number $\bar{n}$ is proportional to $1/\sqrt{\bar{n}}$ (see below). To be more specific, we consider the Glauber-Sudarshan coherent state [42] defined as

$$|\alpha\rangle = \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]|0\rangle; \qquad \alpha = |\alpha|e^{i\theta}, \tag{2.1}$$

where $|0\rangle$ is the vacuum state of a harmonic oscillator which models a single mode of the electromagnetic field in a cavity. The mean photon number of the coherent state (2.1) is $\langle\hat{n}\rangle = |\alpha|^2$. If we want to perform a measurement of phase shifts with the help of coherent states, then we are interested in a phase distribution $P^{(coh)}(\phi)$ which is defined as

$$P^{(coh)}(\phi) = \mathcal{N}\tilde{P}^{(coh)}(\phi), \tag{2.2a}$$

where $\mathcal{N}$ is the normalization constant the un-normalized distribution $\tilde{P}^{(coh)}(\phi)$ is given by the relation

$$\tilde{P}^{(coh)}(\phi) = \left|\langle\alpha|\hat{U}(\phi)|\alpha\rangle\right|^2; \tag{2.2b}$$

$\hat{U}(\phi)$ is the rotation operator (1.5). After some algebra we find for the phase distribution (2.2b) the expression (here we assume $\theta = 0$, i.e., amplitude of the coherent state (2.1) is real):

$$\tilde{P}^{(coh)}(\phi) = \exp\left[-2\bar{n}(1-\cos\phi)\right]; \tag{2.3a}$$

while the normalization constant reads

$$\mathcal{N} = \int_{-\pi}^{\pi} \exp\left[-2\bar{n}(1-\cos\phi)\right]\, d\phi. \tag{2.3b}$$

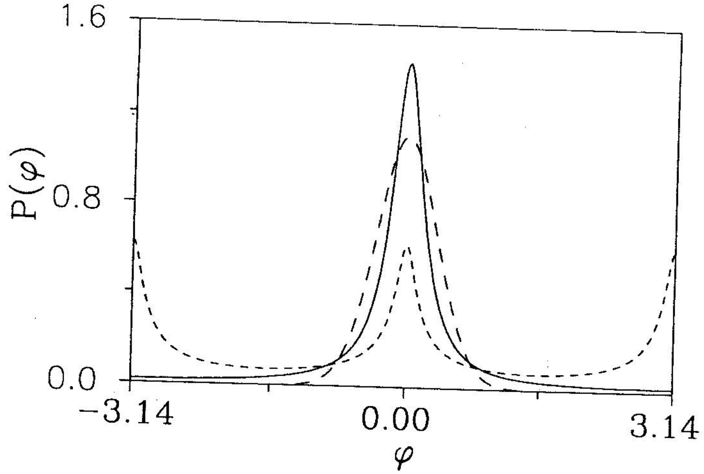

Fig.4. The phase probability distribution $P(\phi) = \mathcal{N}\tilde{P}(\phi)$ defined with the help of the mean value of the rotation operator (1.5) in the $SU(1,1)$ GCS (solid line), the coherent state (long-dashed line), and the squeezed vacuum state (short-dashed line). We assume $\theta = 0$, the mean photon number equal to 4.

From Eq.(2.3) it follows that $P^{(coh)}(\phi)$ is 2π-periodic as one should expect for a proper phase distribution. With the help of (2.3) we can evaluate mean value of phase in the coherent state (2.1)

$$\bar{\phi} = \int_{-\pi}^{\pi} \phi P^{(coh)}(\phi)\, d\phi = 0; \tag{2.4a}$$

and the corresponding variance

$$\overline{(\Delta\phi)^2} \equiv \overline{\phi^2} - (\bar{\phi})^2 = \int_{-\pi}^{\pi} \phi^2 P^{(coh)}(\phi)\, d\phi. \tag{2.4b}$$

In the large $\bar{n}$ limit the phase distribution can be approximated as

$$P^{(coh)}(\phi) \simeq \sqrt{\frac{\bar{n}}{\pi}} e^{-\bar{n}\phi^2}, \tag{2.5}$$

where for simplicity we assume the phase distribution (2.5) to be normalized on the interval $(-\infty, \infty)$. From Eq.(2.5) it directly follows that $\overline{(\Delta\phi)^2} \simeq 1/\bar{n}$ and consequently the phase noise is proportional to $1/\sqrt{\bar{n}}$.

Phase noise can be reduced bellow the coherent-state level providing squeezed vacuum state is used to perform phase-shift "measurements". The squeezed vacuum state can be defined as (see, for instance, [43] and Section 2.2)

$$|\xi\rangle_{SV} = \exp\left[\frac{r}{2}\left(\hat{a}^{\dagger 2} - \hat{a}^2\right)\right] |0\rangle. \tag{2.6}$$

With this particular definition of the squeezed vacuum we find that variances is the $\hat{p}$-quadrature are reduced below the vacuum level. To see this we remind us that the quadrature operators $\hat{q}$ and $\hat{p}$ are defined as (we use units such that $\hbar = 1$)

$$\hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}; \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}, \tag{2.7}$$

so the corresponding variances

$$\langle\xi|\,(\Delta\hat{x})^2\,|\xi\rangle = \langle\xi|\hat{x}^2|\xi\rangle - \langle\xi|\hat{x}|\xi\rangle^2 \equiv \sigma_x^2; \qquad x = q, p \tag{2.8}$$

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$$\sigma_q^2 = \frac{1}{2}e^{2r}; \qquad \sigma_p^2 = \frac{1}{2}e^{-2r}. \tag{2.9}$$

From Eq.(2.9) we clearly see that fluctuations in the momentum ($\hat{p}$) are reduced below the vacuum level (corresponding to $\sigma_p^2 = 1/2$). The phase distribution of the form (2.2) for the squeezed vacuum state (2.6) can be evaluated in an explicit form:

$$P^{(sq)}(\phi) = \mathcal{N}\tilde{P}^{(sq)}(\phi); \qquad \tilde{P}^{(sq)}(\phi) = \frac{1}{\left[1 + (\sigma_q^2 - \sigma_p^2)^2 \sin^2\phi\right]^{1/2}}. \tag{2.10}$$

Unlike the phase distribution $P^{(coh)}(\phi)$ the distribution $P^{(sq)}(\phi)$ is just π-periodic, which means that it can be utilized only for small phase shifts. On the other hand, in the large $\bar{n} = \sinh^2 r$ limit, the phase noise obtained from (2.10) is proportional to $1/\bar{n}$. This means that with the help of the squeezed vacuum state more precise (compared with the coherent states) phase-shift measurements can be performed.

Now the question is whether we can find a quantum-mechanical state of light such that the phase distribution of the form (2.2) is 2π-periodic and simultaneously the corresponding phase noise is proportional to $1/\bar{n}$, i.e., the corresponding phase distribution is much narrower that $P^{(coh)}(\phi)$.

From the point of view of the Perelomov group-theoretical approach [44] for a description of generalized coherent states, the Glauber-Sudarshan coherent states are associated with the Weyl-Heisenberg algebra. On the other hand, the squeezed vacuum state (2.6) is associated with a particular bosonic representation of the $SU(1,1)$ Lie algebra (see below). Therefore we concentrate our attention on the investigation of $SU(1,1)$ GCS. We should expect to find $SU(1,1)$ GCS for which the phase noise is scaled as $1/\bar{n}$ and which can provide us with 2π-periodic phase distributions.

2.1 $SU(1.1)$ generalized coherent states

Here we present some brief remarks on $SU(1,1)$ Lie algebra and its application to quantum optics [44-46]. The $SU(1,1)$ Lie algebra consists of three generators $\hat{K}_0$, $\hat{K}_+$ and $\hat{K}_-$ satisfying the commutation relations:

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm \quad ; \quad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0. \tag{2.11a}$$

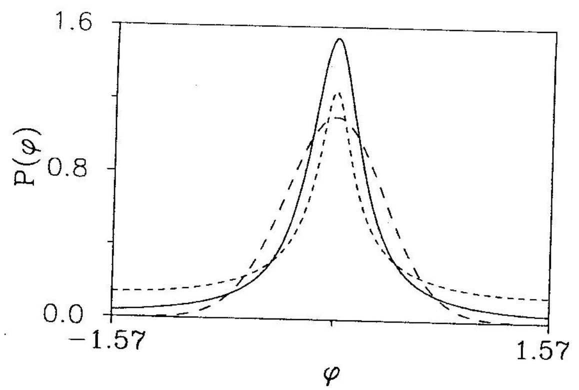

Fig.5 The same as in Fig.4 but the phase distributions are normalized on the interval $[-\pi/2, \pi/2]$.

The discrete series unitary representations of the Lie algebra under consideration are labelled by the eigenvalues of the Casimir operator $\hat{C}$:

$$\hat{C} = \hat{K}_0^2 - \frac{\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+}{2}. \tag{2.11b}$$

The eigenvalue of $\hat{C}$ is equal to $k(k-1)$, where the parameter k is called the Bargmann index. For the representations of interest the states $|m,k\rangle$ diagonalize the compact operator $\hat{K}_0$:

$$\hat{K}_0|m,k\rangle = (m+k)|m,k\rangle \quad ; \quad k>0 \text{ and } m = 0,1,2,\dots \tag{2.12}$$

The operators $\hat{K}_+$ and $\hat{K}_-$ are Hermitian conjugates of each other and act as raising and lowering operators of the quantum number m:

$$\hat{K}_-|m,k\rangle = [m(m+2k-1)]^{1/2}|m-1,k\rangle; \tag{2.13a}$$

$$\hat{K}_+|m,k\rangle = [(m+1)(m+2k)]^{1/2}|m+1,k\rangle. \tag{2.13b}$$

Now we can proceed to a construction of the $SU(1,1)$ GCS. There are two possible ways to construct these states. One of them consist of displacing the vacuum state by the unitary operator (see [44,45]). The second is based on solving the eigenvalue problem for the generalized annihilation operator $\hat{K}_-$ (see [46]).

Following the work of Perelomov [44] we will displace the vacuum state $|0,k\rangle$, defined as:

$$\hat{K}_-|0,k\rangle = 0 \tag{2.14}$$

by a unitary operator $\hat{S}(\alpha)$

$$\hat{S}(\alpha) = \exp(\alpha\hat{K}_+ - \alpha^*\hat{K}_-) \tag{2.15}$$

to obtain the $SU(1,1)$ GCS $|\xi,k\rangle$:

$$|\xi,k\rangle = \hat{S}(\alpha)|0,k\rangle. \tag{2.16}$$

Using the disentangling theorem for $SU(1,1)$ Lie algebra [45] the operator $\hat{S}(\alpha)$ can be rewritten in the following form:

$$\hat{S}(\xi) = \exp(\xi\hat{K}_+)\exp(\Gamma\hat{K}_0)\exp(-\xi^*\hat{K}_-), \tag{2.17}$$

where $\alpha = -\frac{1}{2}\theta\exp(-i\phi)$; $\xi = -\tanh(\theta/2)\exp(-i\phi)$ and $\Gamma = \ln(1-|\xi|^2)$. The range of the parameters θ ,ϕ and $|\xi|$ is

$$\theta \in (-\infty,\infty) \quad ; \quad \phi \in (0,2\pi) \quad ; \quad |\xi| \in (0,1). \tag{2.18}$$

It is easy to see that the $SU(1,1)$ GCS (2.16) can be expanded into the basis $|m,k\rangle$ as

$$|\xi,k\rangle = (1-|\xi|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{1/2} \xi^m |m,k\rangle. \tag{2.19}$$

2.2.1 Squeezed-vacuum state

In what follows we present two different classes of $SU(1,1)$ GCS. In particular, we turn our attention to two possible bosonic realizations of the $SU(1,1)$ Lie algebra. One possible realization is that with the generators $\hat{K}_0$ and $\hat{K}_\pm$ given in the following way:

$$\hat{K}_0 = \frac{\hat{a}^\dagger\hat{a}+1/2}{2} \quad ; \quad \hat{K}_+ = \frac{(\hat{a}^\dagger)^2}{2} \quad ; \quad \hat{K}_- = \frac{\hat{a}^2}{2}. \tag{2.20}$$

In this case the eigenvalue of the Casimir operator is equal to $-3/16$ and the Bargmann index is equal to $1/4$ or $3/4$. For $k = 1/4$ we obtain the even parity states and for $k = 3/4$ the odd parity ones. The vacuum state for the harmonic oscillator is the state $|0,1/4\rangle$ and the states $|m,1/4\rangle$ are equal to Fock states $|m\rangle$ of the harmonic oscillator. In this case the $SU(1,1)$ GCS $|\xi,1/4\rangle$ is equal to the squeezed vacuum state (see [43])

$$|\xi,1/4\rangle \equiv |\xi\rangle_{SV} = (1-|\xi|^2)^{1/4} \sum_{m=0}^{\infty} \frac{[(2m)!]^{1/2}}{2^m m!} \xi^m |m\rangle. \tag{2.21}$$

This definition is equivalent to Eq.(2.6) with the parameters r and ξ related as $\xi = \tanh r$ (here, for simplicity, we assume ξ and r to be real). Statistical properties of the squeezed vacuum state have been extensively analyzed in literature (for details see review articles [43]). Here we just remind the reader that the mean photon number $\bar{n}$ in the squeezed vacuum state (2.21) is given by the relation

$$\bar{n} \equiv \langle\xi|\hat{n}|\xi\rangle = \frac{\xi^2}{1-\xi^2}. \tag{2.22}$$

We also evaluate mean values of the variances of the quadrature operators $\hat{q}$ and $\hat{p}$ which can be expressed as

$$\langle\xi|(\Delta\hat{q})^2|\xi\rangle = \sigma_q^2 = \frac{1}{2}\left(\frac{1+\xi}{1-\xi}\right) = \frac{1}{2}\left(\frac{\sqrt{\bar{n}+1}+\sqrt{\bar{n}}}{\sqrt{\bar{n}+1}-\sqrt{\bar{n}}}\right) > \frac{1}{2}, \tag{2.23a}$$

and

$$\langle\xi|(\Delta\hat{p})^2|\xi\rangle = \sigma_p^2 = \frac{1}{2}\left(\frac{1-\xi}{1+\xi}\right) = \frac{1}{2}\left(\frac{\sqrt{\bar{n}+1}-\sqrt{\bar{n}}}{\sqrt{\bar{n}+1}+\sqrt{\bar{n}}}\right) < \frac{1}{2}. \tag{2.23b}$$

The last equation reflects the fact that quadrature fluctuations of the $\hat{p}$ operator are squeezed below the level associated with vacuum fluctuations (see Fig.1).

This reduction of fluctuations can also be seen if we write down the explicit expression for the Q-function [47] of the squeezed-vacuum state (2.21). The Q-function is a probability density distribution in the phase space (i.e., parametric space) corresponding to the complex variable $\beta = x + iy$:

$$Q(\beta) = \langle\beta|\hat{\rho}|\beta\rangle, \tag{2.24}$$

where $\hat{\rho}$ is a density operator describing the state of the harmonic oscillator under consideration and $|\beta\rangle$ is a coherent state with a complex amplitude β. The Q-function of the squeezed vacuum state (2.21) reads (see Fig.2b)

$$Q(x,y) = \sqrt{1-\xi^2}\exp\left[-(1-\xi)x^2 - (1+\xi)y^2\right]. \tag{2.25}$$

2.1.2 SU(1,1) GCS with $k = 1/2$

Another possible realization of the $SU(1,1)$ Lie algebra is that with the Bargmann index equal to 1/2. In this case the generators of the $SU(1,1)$ Lie algebra can be expressed through the bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{K}_0 = \hat{a}^\dagger\hat{a} + 1/2 \quad ; \quad \hat{K}_+ = \left(\hat{a}^\dagger\hat{a}\right)^{1/2}\hat{a}^\dagger \quad ; \quad \hat{K}_- = \hat{a}\left(\hat{a}^\dagger\hat{a}\right)^{1/2}. \tag{2.26}$$

This particular realization of the $SU(1,1)$ Lie algebra has been employed in quantum optics by Buck and Sukumar [48] (see also paper by Singh [49]). The GCS corresponding to this realization of the $SU(1,1)$ lie algebra has been introduced by Bužek [50]. This GCS in the Fock basis reads

$$|\xi, 1/2\rangle \equiv |\xi\rangle = (1-|\xi|^2)^{1/2}\sum_{m=0}^{\infty}\xi^m|m\rangle. \tag{2.27}$$

This states exhibits several interesting statistical properties. Firstly, we evaluate the mean photon number in the state (2.27) for which we find formally the same expression as for the squeezed vacuum, i.e., (here we again assume ξ to be real)

$$\bar{n} = \frac{\xi^2}{1-\xi^2}. \tag{2.28}$$

The photon number distribution $P_n = |\langle n|\xi\rangle|^2$ of the state (2.27) reads

$$P_n = (1-\xi^2)\xi^{2n}. \tag{2.29a}$$

If we rewrite this photon number distribution in terms of the mean photon number (2.28) we find that

$$P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}, \tag{2.29b}$$

which means that the $SU(1,1)$ GCS given by Eq.(2.27) has a thermal photon number distribution. Simultaneously we underline that this is a *pure* state. On the other hand this state has non-vanishing mean values of the amplitude operators $\hat{a}^k$:

$$\langle\xi|\hat{a}^k|\xi\rangle = (1-|\xi|^2)\xi^k\sum_m^{\infty}|\xi|^{2m}\left[\frac{(m+k)!}{m!}\right]^{1/2}, \tag{2.30}$$

which in particular results in a reduction of quadrature fluctuations in this state. That is, the state under consideration is a squeezed state for which the degree of squeezing increases with the increase of the mean photon number (see Fig.1).

Unlike the squeezed vacuum (2.21) the state (2.27) is not a minimum uncertainty state and is described by a non-Gaussian Q-function (see Fig.2)

$$Q(\beta) = e^{-|\beta|^2}(1-\xi^2)\left|\sum_m^{\infty}\frac{(\beta^*\xi)^n}{\sqrt{n!}}\right|^2. \tag{2.31}$$

As seen from Fig.2 the $SU(1,1)$ GCS (2.27) has nonzero mean amplitude (i.e., $\langle\xi|\hat{a}|\xi\rangle$) moreover, this state is squeezed, so one can approximate it by the displaced squeezed state $\hat{D}(\alpha)\hat{S}(\eta)|0\rangle$ with properly chosen parameters of displacement (α) and squeezing (η). We have recently studied a possibility to utilize such displaced squeezed states in a theoretical schemes of precise measurement of phase shifts (i.e., we have used them as an approximation of phase states [51]). As we will show later the $SU(1,1)$ GCS (2.27) are even better candidates for a phase-shift measurement. One can see this from the shape of the Q function of this state – the Q-function is squeezed in the y-direction (i.e., $\hat{p}$-quadrature) and asymmetric with respect to its maximum value in x-direction ($\hat{q}$-quadrature) so that the value of the Q function at the origin of the phase space is approximately equal to zero and the function is perfectly localized in one quadrant of the phase space (unlike the Vogel-Schleich states [33]) or squeezed vacuum states.

3. Phase properties of $SU(1,1)$ GCS

To study phase properties of the states under consideration we will firstly analyze their Pegg-Barnett phase-probability distributions $P^{(PB)}(\phi)$. The function $P(\phi)$ is formally defined as (for details see [18])

$$P^{(PB)}_{|\xi\rangle}(\phi) = \lim_{s\to\infty}\frac{s+1}{2\pi}|\langle\phi_m|\xi\rangle|^2, \tag{3.1}$$

where the state $|\phi_m\rangle$ in the Fock basis reads

$$|\phi_m\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\phi_m)|n\rangle, \tag{3.2a}$$

and the phase ϕ_m is defined as

$$\phi_m = \phi_0 + 2\pi\frac{m}{s+1}. \tag{3.2b}$$

For the $SU(1,1)$ GCS given by Eq.(2.27) we find the explicit expression for the Pegg-Barnett phase probability distribution in a closed compact form:

$$P^{(PB)}_{|\xi\rangle}(\phi) = \frac{1}{2\pi}\frac{1-|\xi|^2}{[1-2|\xi|\cos(\theta-\phi)+|\xi|^2]}; \qquad \xi = |\xi|e^{i\theta}. \tag{3.3}$$

This distribution function is properly normalized to unity and is 2π-periodic. We note that it is equal to the Poissonian Kernel. It has a maximum at $\theta = \phi$ while it takes minimum value at $\phi = \theta + \pi$. In the limit $|\xi| \to 1$, i.e., in the limit of infinite photon number which is accompanied with an infinite quadrature squeezing the Poissonian Kernel tends to a δ-function:

$$\lim_{|\xi|\to 1} P^{(PB)}_{|\xi\rangle}(\phi) = \delta(\theta-\phi). \tag{3.4}$$

As we have said the phase distribution (3.3) is 2π-periodic. On the contrary the Pegg-Barnett phase distribution function of the squeezed vacuum state is π-periodic. We plot both these distributions in Fig.3. from which it is clearly seen that $SU(1,1)$ GCS as defined by Eq.(2.27) has very well defined phase and can be used as a very good approximation of phase states which is much better than the approximation proposed by Vogel and Schleich [33]. For comparison purposes we also plot the distribution corresponding to a coherent state $|\alpha\rangle$ with $\alpha = 2$.

To illustrate phase properties of the states under consideration in more detail we evaluate the square of the modulus of the mean value of the rotation operator $\hat{U}(\phi)$ given by Eq.(1.5), i.e., we evaluate the square of the scalar product of the state $|\xi\rangle$ and the rotated state $\hat{U}(\phi)|\xi\rangle$. This expression is equal to a non-normalize phase distribution as defined by Eq.(2,2b). We find the explicit expression for $\tilde{P}(\phi)$ for the $SU(1,1)$ GCS given by Eq.(2.27) in a form:

$$\tilde{P}(\phi) = \left|\langle\xi|\hat{U}(\phi)|\xi\rangle\right|^2 = \frac{(1-|\xi|^2)^2}{1-2|\xi|^2\cos(\phi-\theta)+|\xi|^4}. \tag{3.5}$$

The expression for the phase distribution $\tilde{P}^{(sq)}(\phi)$ for the squeezed vacuum state (2.21) is given by Eq.(2.10b) but we can rewrite in a different parameterization:

$$\tilde{P}^{(sq)}(\phi) = \frac{(1-|\xi|^2)}{[1-2|\xi|^2\cos 2(\phi-\theta)+|\xi|^4]^{1/2}}. \tag{3.6}$$

Finally, the phase distribution $\tilde{P}^{(coh)}(\phi)$ for the coherent state $|\alpha\rangle$ is given by Eq.(2.3a). We plot the corresponding normalized phase distributions $P(\phi)$ in Fig.4 from which we see that $SU(1,1)$ GCS are much more sensitive with respect to phase shifts than coherent states or squeezed vacuum states. In Fig.4 we consider the phase to range from $-\pi$ to π and in correspondence with this the phase distributions are normalized on the 2π interval. Due to the π-periodicity of the phase distribution (3.6) for the squeezed vacuum state $P^{(sq)}(\phi)$ has at $\phi = 0$ the value even smaller than the coherent-state phase distribution $P^{(coh)}(\phi)$. For comparison purposes we plot in Fig.5 the same phase distributions but assume the normalization interval to range from $-\pi/2$ to $\pi/2$. We see that with this normalization condition small phase shifts can be measured more precisely with the help of squeezed vacuum states than with the help of coherent states. Nevertheless, the best performance can be obtained if $SU(1,1)$ GCS with the Bargmann index equal to 1/2 are used.

4. Discussion and conclusions

To understand phase properties of the $SU(1,1)$ GCS with the Bargmann index equal to 1/2 we turn our attention back to the Susskind-Glogower phase operators $\widehat{\exp}(i\Phi)$ and $\widehat{\exp}(-i\Phi)$ (see [7]). These operators in the Fock basis read:

$$\hat{E}_- = \widehat{\exp}(i\Phi) = \sum_{n=0}^{\infty}|n\rangle\langle n+1|, \tag{4.1a}$$

$$\hat{E}_+ = \widehat{\exp}(-i\Phi) = \sum_{n=0}^{\infty}|n+1\rangle\langle n|. \tag{4.1b}$$

It can be shown that:

$$\hat{E}_-\hat{E}_+ = 1; \qquad \hat{E}_+\hat{E}_- = 1 - |0\rangle\langle 0|, \tag{4.2}$$

which means that $\hat{E}_+$ is an isometric but non-unitary operator. We can easily check that the $SU(1,1)$ GCS with the Bargmann index equal to 1/2 is an eigenstate of the Susskind-Glogower operator $\hat{E}_-$

$$\hat{E}_-|\xi\rangle = \xi|\xi\rangle, \tag{4.3}$$

which explains exceptional phase properties of the $SU(1,1)$ generalized coherent state under consideration.

We can conclude our analysis with the remark that the $SU(1,1)$ GCS discussed in the present paper can be adopted as the operational phase states with $|\xi|$ as the "regularization" parameter. As seen from Eqs.(3.3) and (3.4) in the limit $|\xi| \to 1$ these states have precisely defined phase. On the other hand, as shown by Daeubler et al. [52] for $|\xi| < 1$ these states minimize the phase uncertainty (quantified with the help of the Sussmann measure) for a given norm and a given mean photon number, i.e., these states can be considered as *realistic* phase states and can be used for investigations of phase properties of physical system. One of examples we can consider is the problem of a measurement of the phase of the atomic coherent state. To be specific, let us assume a two-level atom interacting with a single mode cavity field. This system can be described within the framework of the well-known Jaynes-Cummings model [53]. The atom is supposed to be initially ($t = 0$) prepared in the atomic coherent state

$$|A\rangle = \frac{1}{\sqrt{2}}|-\rangle - \frac{1}{\sqrt{2}}e^{i\varphi}|+\rangle, \tag{4.4}$$

where the vectors $|+\rangle$ and $|-\rangle$ describe the upper and lower levels of the two-level atom, respectively. The phase φ has to be determined. If we assume the cavity field initially to be prepared in the $SU(1,1)$ GCS described by Eq.(2.27), then at time $t > 0$ the atom-field state vector can be expressed as:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}[C_n(t)|+;n\rangle + D_n(t)|-;n\rangle], \tag{4.5}$$

where the probability amplitudes $C_n(t)$ and $D_n(t)$ are given by relations

$$C_n(t) = \frac{(1-|\xi|^2)^{1/2}\xi^n}{\sqrt{2}}\left[i\xi \sin\sqrt{(n+1)}\tau - e^{i\varphi}\cos\sqrt{(n+1)}\tau\right]; \tag{4.6a}$$

$$D_n(t) = \frac{(1-|\xi|^2)^{1/2}\xi^{n-1}}{\sqrt{2}}\left[ie^{i\varphi}\sin\sqrt{n}\tau + \xi\cos\sqrt{n}\tau\right], \tag{4.6b}$$

where τ is the scaled time ($\tau = \lambda t$ and λ is the atom-field coupling constant in the dipole approximation). From Eq.(4.6) it follows that the atomic inversion $W(t)$ defined as

$$W(t) = \sum_{n=0}^{\infty}\left(|C_n(t)|^2 - |D_n(t)|^2\right), \tag{4.7}$$

is in the limit $|\xi| \to 1$ equal to its initial value for any $t > 0$ providing the atomic-coherent state φ is equal to the phase θ of the GCS under consideration. Otherwise $W(t)$ oscillates in time. Using this phase-locking effect one can determine the phase of the atomic coherent state (4.4).

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