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EIGENSTATES OF SQUEEZING AND OTHER QUADRATIC BOSON OPERATORS[1]

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The eigenvalue problem for arbitrary linear combinations of the quadratic boson operators of a single mode is discussed and reduced by using group theory for the $SU(1,1)$ group and its complex extension to standard problems for squeezing–like, rotation–like and cone–like operators in the real case. It is shown that the Killing form has a great importance for this reduction. Furthermore, it is discussed how the solution of the root equations allows to displace the eigenvalues of squeezing–like operators and to determine the new eigenvectors. These root solutions are explicitly found. The general considerations are applied to solve the eigenvalue problem for the Hermitean squeezing operator $K_2 = -\frac{1}{4\hbar}(QP + PQ)$

1. Introduction

The solution of the eigenvalue problem for arbitrary linear combinations of a boson annihilation operator a and a boson creation operator $a^\dagger$ leads to squeezed coherent states and is explicitly given in the nonunitary approach by [1–3]

$$(a + \zeta a^\dagger)|\beta;\zeta\rangle = \beta|\beta;\zeta\rangle,$$

$$|\beta;\zeta\rangle = \exp\left(\beta a^\dagger - \frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle = \exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|\beta;0\rangle, \quad \beta,\zeta \in C, \qquad (1)$$

where $|\beta;0\rangle$ is a nonnormalized coherent state and $|0\rangle$ the vacuum state. The same result can also be obtained in the unitary approach by applying a unitary squeezing–like operator with linear combinations of the quadratic boson operators in the exponent onto coherent states with transformed parameters in comparison to (1) [4]. The eigenvalue problem for arbitrary linear combinations of quadratic boson operators is therefore the

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eigenvalue problem of squeezing-like and rotation-like operators. The solution of this eigenvalue problem contains a variety of interesting states including the squeezed coherent states $|\beta;\zeta\rangle$ and the Schrödinger cat states formed by arbitrary superpositions of $|\beta;\zeta\rangle$ and $|-\beta;\zeta\rangle$ in the degenerate case when these linear combinations are representable as the square $(a+\zeta a^\dagger)^2$. It was shown the negative result that the unitary squeezing operators do not possess proper normalizable eigenstates [5, 6]. Positive results for the explicit form of the eigenstates of unitary squeezing operators which have a continuous spectrum and are normalizable by means of the delta function can be obtained [7 – 9]. It is also possible to regularize such eigenstates and to make them normalizable, for example, by infinitely small changes of the unitary squeezing operator to certain nonunitary squeezing operators [10] or simply by regularizing the singularities in the explicit solutions for the eigenstates.

In the present paper we mainly show that the application of the group theory to the Lie algebra of the quadratic combinations of boson operators allows to reduce the general eigenvalue problem to some standard cases, where the other cases can be obtained from the solution for the standard cases by applying some explicitly given operators. This concerns the transition from the eigenvalue problem of one squeezing operator to other squeezing operators for the same eigenvalue and the transition from one eigenvalue to other eigenvalues of the same squeezing operator using the solutions for the roots of the squeezing operators.

2. Basic notions and problem reduction

We consider a single boson mode with the following connection between the annihilation and creation operator $(a, a^\dagger)$ and the canonical operators (Q, P)

$$a = \frac{Q+iP}{\sqrt{2\hbar}}, \quad a^\dagger = \frac{Q-iP}{\sqrt{2\hbar}}, \quad [a, a^\dagger] = I, \quad [Q, P] = i\hbar I, \tag{2}$$

where I denotes the unity operator in the Fock space. As was already said the eigenvalue problem for arbitrary linear combinations of the operators $(a, a^\dagger)$ or (Q, P) can be completely solved and leads to squeezed coherent states and their limiting cases (for example, eigenstates $|q\rangle$ of Q or $|p\rangle$ of P) but in the nonunitary approach one can also make the transition to the cases of nonnormalizable eigenstates, for example, of the creation operator $a^\dagger$ (see [1, 3]).

We now pose the analogous eigenvalue problem for arbitrary linear combinations of the quadratic boson operators $a^2, a^{\dagger 2}$ and $aa^\dagger + a^\dagger a$ in the form (we use sum convention)

$$x|\omega\rangle = \omega|\omega\rangle, \quad x = K_1 x^1 + K_2 x^2 + K_3 x^3 \equiv K_j x^j, \tag{3}$$

where (K_1, K_2, K_3) are three Hermitean basis operators defined by

$$\begin{aligned} K_1 &\equiv \frac{1}{4}(a^2 + a^{\dagger 2}) = \frac{1}{4\hbar}(Q^2 - P^2), \\ K_2 &\equiv i\frac{1}{4}(a^2 - a^{\dagger 2}) = -\frac{1}{4\hbar}(QP + PQ), \quad (K_j = K_j^\dagger), \\ K_3 &\equiv \frac{1}{4}(aa^\dagger + a^\dagger a) = \frac{1}{4\hbar}(Q^2 + P^2)), \end{aligned} \tag{4}$$

and satisfying the commutation relations

$$[K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2, \quad [K_1, K_2] = -iK_3. \tag{5}$$

As abstract operators, the operators (K_1, K_2, K_3) (not necessarily Hermitean operators) with the commutation relations (5) define a basis of a Lie algebra which is the Lie algebra $su(1,1) \sim sl(2,R) \sim sp(2,R)$ in case of real coefficients (x^1, x^2, x^3) or its complex extension $sl(2,C) \sim sp(2,C)$ in case of complex coefficients (x^1, x^2, x^3). The identification of (K_1, K_2, K_3) according to (4) gives one of the possible realizations of this Lie algebra (other realization by single and multi-Boson operators see [11–17]). The corresponding Lie groups of elements X obtained by the exponential mapping $x \to X = \exp(ix)$ are $SU(1,1) \sim SL(2,R) \sim Sp(2,R)$ in the real or $SL(2,C) \sim Sp(2,C)$ in the complex case (see [16, 18–20]). The commutation relations (5) are in analogy to the commutation relations for the Lie algebra $su(2) \sim so(3,R)$ of basis operators (J_1, J_2, J_3) with one changed sign in the third commutation relation. However, the groups $SU(1,1)$ and $SU(2)$ are very different because the first is a noncompact group, whereas the second is a compact group. Instead of (K_1, K_2, K_3) another basis (K_+, K_-, K_3) of the Lie algebra $su(1,1)$ is often used, where K_+ and K_- are defined by

$$\begin{aligned} K_\pm &\equiv K_1 \pm iK_2, \quad K_- \equiv \frac{1}{2}a^2, \quad K_+ \equiv \frac{1}{2}a^{\dagger 2}, \\ x &= K_- x^- + K_+ x^+ + K_3 x^3, \quad x^\pm \equiv \frac{1}{2}(x^1 \mp ix_2), \end{aligned} \tag{6}$$

with the commutation relations

$$[K_3, K_+] = +K_+, \quad [K_3, K_-] = -K_3, \quad [K_-, K_+] = 2K_3. \tag{7}$$

Before attacking the eigenvalue problem (3) we make some general considerations leading to some reduction of the problem.

The solution of the eigenvalue problem for the operators x leads at once to the solution of the eigenvalue problem for the operators $\exp(ix)$ which are the unitary squeezing and rotation operators for real (x^1, x^2, x^3) and the nonunitary generalized squeezing operators for complex (x^1, x^2, x^3). These squeezing operators play a two-fold role for our problem. First, they are the operators for which we intend to solve the eigenvalue problem and second, they are the operators which make the general linear transformation of the basis operators (K_1, K_2, K_3) preserving the commutation relations (5) or (6). Consider the similarity transformation of an arbitrary operator x of the Lie algebra $sl(2,C)$ with an arbitrary (in general nonunitary) squeezing operator $S \equiv \exp(is)$ according to

$$\begin{aligned} x' &= \mathrm{e}^{is} x \mathrm{e}^{-is} = e^{is} K_j x^j \mathrm{e}^{-is} = K'_j x^j = K_i S^i_j x^j = K_i x'^i, \\ K'_j &\equiv K_i S^i_j, \quad x'^i \equiv S^i_j x^j. \end{aligned} \tag{8}$$

The explicit form of the matrix S^i_j when the element s is represented in the basis (K_1, K_2, K_3) is given by

$$(K'_1, K'_2, K'_3) = \mathrm{e}^{is}(K_1, K_2, K_3)\mathrm{e}^{-is} = (K_1, K_2, K_3)$$

$$\times \begin{pmatrix} 1+((s^2)^2-(s^3)^2)\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & s^3\frac{\mathrm{sh}\sigma}{\sigma}-s^1s^2\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & -s^2\frac{\mathrm{sh}\sigma}{\sigma}+s^1s^3\frac{\mathrm{ch}\sigma-1}{\sigma^2} \\ -s^3\frac{\mathrm{sh}\sigma}{\sigma}-s^1s^2\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & 1+((s^1)^2-(s^3)^2)\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & s^1\frac{\mathrm{sh}\sigma}{\sigma}+s^2s^3\frac{\mathrm{ch}\sigma-1}{\sigma^2} \\ -s^2\frac{\mathrm{sh}\sigma}{\sigma}-s^1s^3\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & s^1\frac{\mathrm{sh}\sigma}{\sigma}-s^2s^3\frac{\mathrm{ch}\sigma-1}{\sigma^2}, & 1+((s^1)^2+(s^2)^2)\frac{\mathrm{ch}\sigma-1}{\sigma^2} \end{pmatrix},$$

$$\sigma \equiv \sqrt{(s^1)^2+(s^2)^2-(s^3)^2}, \quad s \equiv K_j s^j . \qquad (9)$$

The matrix in (9) gives the so-called regular representation of an element $\exp(is)$ of the Lie group $SL(2,C)$ in the basis (K_1,K_2,K_3). Its determinant is equal to 1 but this is only a consequence of a more special property of this matrix which can be expressed by the preservation of the so-called Killing form (x,y) which is a symmetric bilinear form of two arbitrary elements x and y of the Lie algebra and which has in our case the following special form

$$(x,y) \equiv g_{ij}x^iy^j = -x^1y^1 - x^2y^2 + x^3y^3 . \qquad (10)$$

In general, the metric tensor $g_{ij} = g_{ji}$ of the Killing form can be defined by the structure coefficients c^l_{ik} in the general commutation relations of the Lie algebra $[K_i,K_k] = c^l_{ik}K_l$ by $g_{ij} \equiv \frac{1}{2}c^l_{ik}c^k_{jl}$ (mostly defined without factor $\frac{1}{2}$, see [18–20]). The preservation of the Killing form $(x,y) = (x',y')$ with the invariant metric tensor g_{ij} under transformations of the form (8) can be proved for arbitrary Lie algebras but one can also easily check this for the special transformation matrices (9) together with the special Killing form (10).

The invariance of the Killing form (10) means in particular the invariance of

$$(x',x') = (x'^3)^2 - (x'^1)^2 - (x'^2)^2 = (x^3)^2 - (x^1)^2 - (x^2)^2 \equiv (x,x) . \qquad (11)$$

This is for real (x^1,x^2,x^3) equivalent to the squared distance in a pseudo-Euclidian space with two space-like and one time-like coordinates and the transformation matrices in (9) form for real (s^1,s^2,s^3) the proper Lorentz group $SO(2,1)$ of this space. In analogy to the terminology of space-like, time-like and light-cone vectors, one can call a transformation $\exp(ix)$ with real (x^1,x^2,x^3) squeezing-like for $(x,x)<0$, rotation-like for $(x,x)>0$ and cone-like for $(x,x)=0$. The totality of complex vectors (x^1,x^2,x^3) forms a three-dimensional complex Euclidian space and the transformation matrices in (9) with complex vectors (s^1,s^2,s^3) provides a certain parametrization of the proper three-dimensional complex orthogonal group $SO(3,C)$. We call the, in general, nonunitary operators $e^{is} = \exp(iK_js^j) \equiv S(s^1,s^2,s^3)$ in this case generalized squeezing operators.

The above considerations show that if one has the solution of the eigenvalue problem for one operator x according to (3) then one gets, at least formally, the solution of the eigenvalue problem for all operators x' connected with x by the similarity transformations (8) as follows

$$x|\omega\rangle = \omega|\omega\rangle, \quad x'e^{is}|\omega\rangle = e^{is}xe^{-is}e^{is}|\omega\rangle = \omega e^{is}|\omega\rangle , \qquad (12)$$

and we know already that only operators x and x' can be connected if they possess the same value of the Killing form (11). The restriction to real vectors (s^1,s^2,s^3) and therefore Hermitean operators s and unitary operators e^{is} can be called the unitary approach and its extension to complex vectors (s^1,s^2,s^3) the nonunitary approach. The unitary approach connects Hermitean operators x in every case only with Hermitean operators x'. Since the eigenvalue problem of an operator is not essentially changed by multiplication of this operator with a complex number we have to solve this eigenvalue problem only for one representative of each such class. However the multiplication with a complex number destroys the Hermitecity of an operator and one can multiply it only with real numbers to keep the Hermitecity and this cannot change the sign of the Killing form (x,x). Hence, squeezing-like and rotation-like operators are disconnected in the unitary approach. In the unitary approach, one has to consider the eigenvalue problem for one representative of Hermitean operators from each of the three classes $(x,x)<0, (x,x)=0$ and $(x,x)>0$ and for a larger two-dimensional manifold of representatives of non-Hermitean operators. The nonunitary approach can, in principle, connect also these classes but the action of the nonunitary operators e^{is} is mostly more difficult to handle. An advantage of the nonunitary approach is that it unifies the treatment of normalizable and nonnormalizable eigenstates of operators.

The parity operator $\Pi = (-1)^{a^\dagger a}$ commutes with arbitrary operators x defined by (3), i.e. $(N \equiv a^\dagger a)$

$$[\Pi, x] = 0\,, \Pi = \Pi^{-1} \equiv (-1)^N = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n| . \qquad (13)$$

It has the eigenvalue $+1$ for all superpositions of even number states $|2m\rangle$ and -1 for all superpositions of odd number states $|2m+1\rangle, (m=0,1,2,\ldots)$. The eigenstates of all operators x to a given eigenvalue ω are two-fold degenerate in the whole Fock space and one can choose a basis of even and odd eigenstates to arbitrary eigenvalues ω

$$x|\omega_\pm\rangle = \omega|\omega_\pm\rangle, \quad \Pi|\omega_\pm\rangle = \pm|\omega_\pm\rangle . \qquad (14)$$

The Fock space is reducible with respect to the action of the operators x and can be decomposed into an even-number and an odd-number Fock space but this decomposition possess a certain arbitrariness because in both partial spaces the same irreducible representation of the Lie algebra of operators x is acting. This can be seen from the Casimir operator C

$$C \equiv g^{ij}K_iK_j = K_3^2 - K_1^2 - K_2^2 , \qquad (15)$$

which according to the definition has to commute with all operators x and is for the considered representation proportional to the unity operator I, i.e.

$$C = -\frac{3}{16}I \equiv k(k-1)I, \quad k = \frac{3}{4} , \qquad (16)$$

despite the reducibility of the Fock space.

3. Roots of the quadratic boson operators

We now solve for arbitrary operators x given by (3) the following root equation

$$[x, e_\alpha] = \alpha e_\alpha\,, \qquad (17)$$

i.e. we determine the possible roots (complex numbers α) and the corresponding root operators e_α. This equation has the form of an eigenvalue equation in linear algebra formulated for the projection operators to subspaces of the eigenvalues where the usual matrix multiplication is here substituted by the product definition in a Lie algebra that means by forming the commutator. The solutions of this problem leads to a further reduction of the eigenvalue problem for the operator x because with the knowledge of one eigenvector $|\omega\rangle$ to an eigenvalue ω one can construct new eigenvectors to other eigenvalues according to $(l = 0, 1, 2, \ldots)$

$$\begin{aligned} x e_\alpha|\omega\rangle &= (e_\alpha x + \alpha e_\alpha)|\omega\rangle = (\omega + \alpha) e_\alpha|\omega\rangle\,, \\ x (e_\alpha)^l|\omega\rangle &= (\omega + l\alpha)(e_\alpha)^l|\omega\rangle, \quad e^{ix}(e_\alpha)^l|\omega\rangle = e^{i(\omega + l\alpha)}(e_\alpha)^l|\omega\rangle\,. \end{aligned} \qquad (18)$$

The systems of roots play a great role in the classification of Lie algebras.

With $x = K_j x^j$ and with the ansatz $e_\alpha = K_j e_\alpha^j$ and using the commutation relations (5) one obtains from (17) a homogeneous linear system of three equations which can be readily solved with the result

$$\begin{aligned} \alpha &= \pm\sqrt{(x^3)^2 - (x^1)^2 - (x^2)^2} \equiv \pm\sqrt{(x, x)}\,, \\ e_\alpha &= K_1(x^1 x^3 - i\alpha x^2) + K_2(x^2 x^3 + i\alpha x^1) + K_3((x^1)^2 + (x^2)^2)\,, \end{aligned} \qquad (19)$$

where e_α is only determined up to an arbitrary proportionality factor. There is also the trivial solution $\alpha = 0$ with $e_0 = x$. It is remarkable that the root operators e_α are cone–like operators according to their Killing form

$$(e_\alpha, e_\alpha) = ((x^1)^2 + (x^2)^2)^2 - (x^1 x^3 - i\alpha x^2)^2 - (x^2 x^3 + i\alpha x^1)^2 = 0\,. \qquad (20)$$

The solution (19) shows that for real (x^1, x^2, x^3) and therefore Hermitean operators x the roots α are either real or imaginary in dependence on the sign of the Killing form (x, x). For squeezing–like operators e^{ix} is $(x, x) < 0$ and the roots α become imaginary. This has the consequence that the root operators e_α according to (19) become Hermitean operators and it is then possible to define the powers $(e_\alpha)^\lambda$ uniquely not only for nonnegative integer λ but for arbitrary real and even complex λ. This means that for squeezing–like operators e^{ix} an eigenvalue ω of the operator x can be displaced to arbitrary complex eigenvalues and the corresponding eigenvectors can be constructed using the powers $(e_\alpha)^\lambda$ of e_α. Squeezing–like operators possess a continuous spectrum of eigenvalues with eigenvectors which are normalizable on the real axis by means of the delta function. This is not so in case of rotation–like operators e^{ix} for $(x, x) > 0$ which possess on the real axis a discrete spectrum of normalizable eigenstates.

We consider some special cases. For the operator K_1 one finds from (19)

$$\begin{aligned} x &= K_1, \quad (x^1 = 1, x^2 = 0, x^3 = 0)\,, \\ \alpha &= +i, \quad e_\alpha = K_3 - K_2 = \frac{1}{4\hbar}(Q + P)^2, \quad \alpha = -i, \quad e_\alpha = K_3 + K_2 = \frac{1}{4\hbar}(Q - P)^2, \end{aligned} \qquad (21)$$

and for K_2

$$\begin{aligned} x &= K_2, \quad (x^1 = 0, x^2 = 1, x^3 = 0)\,, \\ \alpha &= +i, \quad e_\alpha = K_3 + K_1 = \frac{1}{2\hbar}Q^2, \quad \alpha = -i, \quad e_\alpha = K_3 - K_1 = \frac{1}{2\hbar}P^2\,. \end{aligned} \qquad (22)$$

We later investigate the eigenvalue problem for the operator K_2 as a representative for a squeezing–like Hermitean operator. For the operator K_3 one obtains by a limiting procedure

$$\begin{aligned} x &= K_3, \quad (x^1 = 0, x^2 = 0, x^3 = 1)\,, \\ \alpha &= +1, \quad e'_\alpha = K_1 + iK_2 \equiv K_+ = \frac{1}{2}a^{\dagger 2}, \quad \alpha = -1, \quad e'_\alpha = K_1 - iK_2 \equiv K_- = \frac{1}{2}a^2. \end{aligned} \qquad (23)$$

The operators K_+ and K_- are raising and lowering operators in the subspaces of even and odd number states of the Fock space.

4. Cone–like operators

The degenerate case of cone–like operators x defined by $(x, x) = 0$ leads with (3) and (4) to the following special structure of squared linear combinations of boson operators

$$x = \frac{1}{4}\left(a\sqrt{x^1 + ix^2} + a^\dagger\sqrt{x^1 - ix^2}\right)^2 = \frac{1}{4\hbar}\left(Q\sqrt{x^3 + x^1} - P\sqrt{x^3 - x^1}\right)^2\,. \qquad (24)$$

If we introduce the abbreviation $\zeta \equiv \sqrt{\frac{x^1 - ix^2}{x^1 + ix^2}}$ one can solve the eigenvalue problem in the following equivalent form in the nonunitary approach (unitary approach, see [21])

$$(a + a^\dagger\zeta)^2|\beta_\pm\rangle = \beta^2|\beta_\pm\rangle, \quad \Pi|\beta_\pm\rangle = \pm|\beta_\pm\rangle\,,$$

$$\begin{aligned} |\beta_\pm\rangle &= \frac{1}{2}(\exp(\beta a^\dagger) \pm \exp(-\beta a^\dagger))\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle \\ &\equiv \frac{1}{2}(|\beta;\zeta\rangle \pm |-\beta;\zeta\rangle)\,, \end{aligned} \qquad (25)$$

where $|\beta;\zeta\rangle$ are the (nonnormalized) squeezed coherent states in the notation introduced in [1, 3]. The states $|\beta_\pm\rangle$ are given here in the nonnormalized form but they are

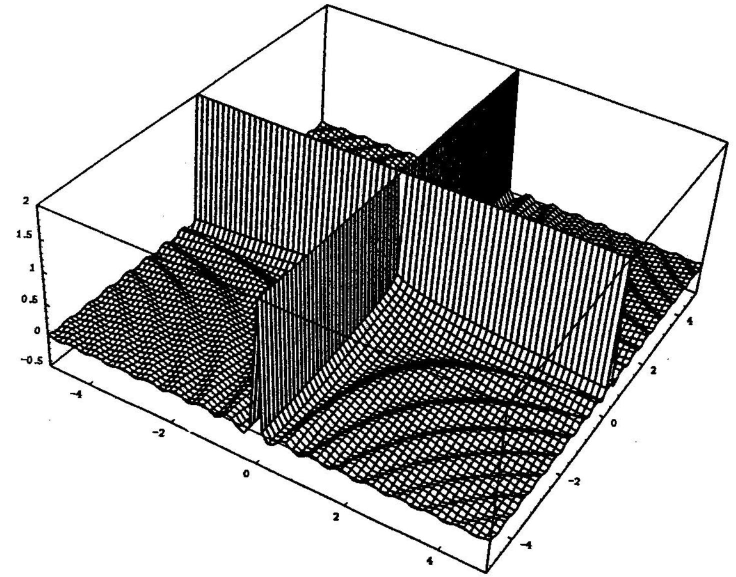

Fig. 1. Wigner quasiprobability for eigenstate $|\omega_+\rangle$ of the operator K_2 with even parity [normalized by means of the delta function $\langle\omega'_+|\omega_+\rangle = \delta(\omega_+ - \omega'_+)$].

normalizable for $|\zeta| < 1$. The nonunitary approach has the advantage that the states $|\beta_\pm\rangle$ can be also extended to $|\zeta| \geq 1$. They are even and odd cat states formed from squeezed coherent states instead of the usual coherent states. The eigenvalue problem for the operators (24) is completely solved by (25) or by linear combinations of $|\beta_+\rangle$ and $|\beta_-\rangle$ after simple transformations and we will not discuss it more.

5. Eigenstates of operator K_2

We consider the Hermitean operator $K_2 = -\frac{1}{4\hbar}(QP + PQ)$ corresponding to the unitary squeezing operators $\exp(iK_2x^2)$ as a representative for the class of unitary squeezing operators. First we show the way one easily obtains the solution of the eigenvalue problem of K_2 to the eigenvalues $\pm\frac{i}{4}$. The inverse operators Q^{-1} to Q and P^{-1} to P are not uniquely defined since the eigenvalue 0 is among the eigenvalues of Q and P but it is natural to define

$$Q^{-1} = \int_{-\infty}^{+\infty} dq \left(\mathcal{P}\frac{1}{q} + c\delta(q)\right) |q\rangle\langle q|, \quad P^{-1} = \int_{-\infty}^{+\infty} dp \left(\mathcal{P}\frac{1}{p} + c\delta(p)\right) |p\rangle\langle p|, \quad (26)$$

where $\mathcal{P}$ denotes the principle value and c are arbitrary constants determining the degree of nonuniqueness of Q^{-1} and P^{-1} and where $|q\rangle$ and $|p\rangle$ are the eigenstates of

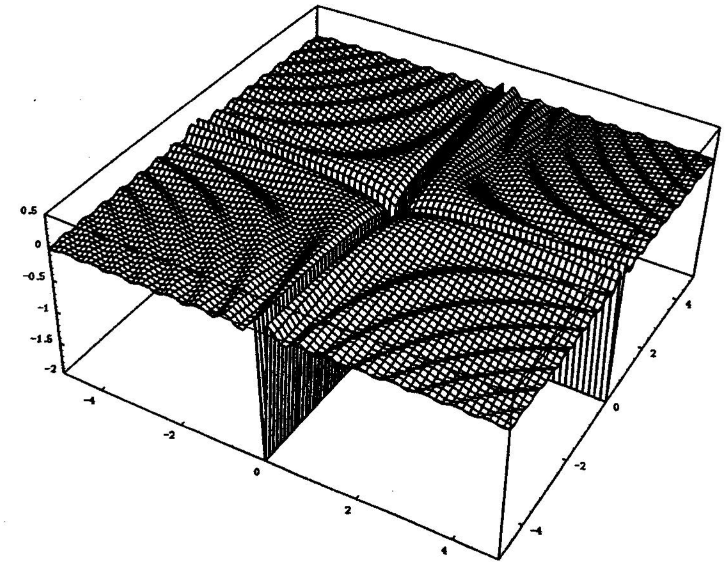

Fig. 2 Wigner quasiprobability for eigenstate $|\omega_-\rangle$ of the operator K_2 with odd parity [normalized by means of the delta function $\langle\omega'_-|\omega_-\rangle = \delta(\omega_- - \omega'_-)$].

Q and P in the usual way. From (26) it follows

$$\begin{aligned} Q^{-1}|p=0\rangle &= \frac{1}{\sqrt{2\hbar\pi}}\left\{\int_{-\infty}^{+\infty} dq\mathcal{P}\frac{1}{q}|q\rangle + c|q=0\rangle\right\}, \\ P^{-1}|q=0\rangle &= \frac{1}{\sqrt{2\hbar\pi}}\left\{\int_{-\infty}^{+\infty} dp\mathcal{P}\frac{1}{p}|p\rangle + c|p=0\rangle\right\}. \end{aligned} \quad (27)$$

Using this and the commutation relations (2) one finds

$$\begin{aligned} K_2Q^{-1}|p=0\rangle &= \left(-\frac{1}{2\hbar}PQ - \frac{i}{4}I\right)Q^{-1}|p=0\rangle = -\frac{i}{4}Q^{-1}|p=0\rangle, \\ K_2P^{-1}|q=0\rangle &= \left(-\frac{1}{2\hbar}QP + \frac{i}{4}I\right)P^{-1}|q=0\rangle = +\frac{i}{4}P^{-1}|q=0\rangle, \end{aligned} \quad (28)$$

or

$$\begin{aligned} K_2|q=0\rangle &= -\frac{i}{4}|q=0\rangle, \quad K_2\int_{-\infty}^{+\infty} dq\mathcal{P}\frac{1}{q}|q\rangle = -\frac{i}{4}\int_{-\infty}^{+\infty} dq\mathcal{P}\frac{1}{q}|q\rangle, \\ K_2|p=0\rangle &= +\frac{i}{4}|p=0\rangle, \quad K_2\int_{-\infty}^{+\infty} dp\mathcal{P}\frac{1}{p}|p\rangle = +\frac{i}{4}\int_{-\infty}^{+\infty} dp\mathcal{P}\frac{1}{p}|p\rangle. \end{aligned}$$

(29)

Now, one can use the root operators to K_2 explicitly given in (22) to displace the eigenvalues and to determine the corresponding eigenstates. For example, the operator $\left(\frac{Q^2}{2\hbar}\right)^{-\frac{1}{4}}$ displaces the eigenvalue from $+\frac{i}{4}$ to 0 and then the operator $\left(\frac{Q^2}{2\hbar}\right)^{-i\omega}$, which is a Hermitean operator for real ω displaces the eigenvalue from 0 to an arbitrary ω, i. e.

$$K_2|\omega_\pm\rangle = \omega|\omega_\pm\rangle, \quad |\omega_\pm\rangle = \exp\left\{-i\omega \ln\left(\frac{Q^2}{2\hbar}\right)\right\}|0_\pm\rangle_{\mathrm{norm}},$$
$$|0_+\rangle_{\mathrm{norm}} = \sqrt{\frac{\hbar}{|Q|}}|p=0\rangle, \quad |0_-\rangle_{\mathrm{norm}} = \frac{1}{i\pi}\sqrt{\frac{\hbar}{|Q|}}\int_{-\infty}^{+\infty} dp\mathcal{P}\frac{1}{p}|p\rangle, \tag{30}$$

where we have added normalization factors to keep normalization for real ω in the sense $\langle\omega_\pm|\omega'_\pm\rangle = \delta(\omega-\omega')$ but $\langle\omega_+|\omega'_-\rangle = 0$.

The states $|\omega_+\rangle$ have even and the states $|\omega_-\rangle$ odd parity. They possess the following position and momentum representations

$$\langle q|\omega_\pm\rangle = \frac{\Theta(q)\pm\Theta(-q)}{\sqrt{2\pi|q|}}\exp\left\{-i\omega\ln\left(\frac{q^2}{2\hbar}\right)\right\},$$
$$\langle p|\omega_\pm\rangle = \exp(i\varphi_\pm(\omega))\frac{\Theta(p)\pm\Theta(-p)}{\sqrt{2\pi|p|}}\exp\left\{+i\omega\ln\left(\frac{p^2}{2\hbar}\right)\right\}, \varphi_\pm(0) = 0, \tag{31}$$

where $\Theta(q)$ denotes Heaviside's step function $(\Theta(q) = 1$ for $q > 0)$, $\Theta(q) = 0$ for $q < 0$ and where $\varphi_\pm(\omega)$ are (little important) phases which we do not give here. It is interesting that the states $|\omega_\pm\rangle$ lead for real ω to the same "marginal" distributions

$$\langle q|\omega_\pm\rangle\langle\omega_\pm|q\rangle = \frac{1}{2\pi|q|}, \quad \langle p|\omega_\pm\rangle\langle\omega_\pm|p\rangle = \frac{1}{2\pi|p|}, \tag{32}$$

independent of ω and equal for even and odd eigenstates. This is, however, not the case for rotated coordinates (q,p). For the Bargmann representation of the states $|\omega_\pm\rangle$ we obtained

$$\langle\alpha;0|\omega_\pm\rangle = \frac{2^{i\omega}\Gamma(\frac{1}{2}-i2\omega)}{\sqrt{2\pi\sqrt{\pi}}}\left\{D_{-\frac{1}{2}+i2\omega}(-\sqrt{2}\alpha^*) \pm D_{-\frac{1}{2}+i2\omega}(\sqrt{2}\alpha^*)\right\}, \tag{33}$$

where $D_\nu(z)$ denotes the functions of the parabolic cylinder in the usual way and $|\alpha;0\rangle \equiv \exp(\frac{\alpha\alpha^*}{2})|\alpha\rangle$ are nonnormalized coherent states which analytically depend on α. From (33) one obtains a nonnormalized coherent-state quasiprobability $Q'(\alpha,\alpha^*)$ because the states $|\omega_\pm\rangle$ are normalized by means of the delta function. A corresponding nonnormalized Wigner quasiprobability has been obtained up to now only for the

eigenvalues 0 with the result (fig. 1 and fig. 2)

$$W'(q,p) = \frac{1}{2\hbar\pi}\left\{J_0\left(\frac{2pq}{\hbar}\right) \mp Y_0\left(\left|\frac{2pq}{\hbar}\right|\right)\right\}, \tag{34}$$

for $|0_\pm\rangle$ respectively, where $J_0(z)$ denotes the Bessel function and $Y_0(z)$ the Neumann function with index 0. This Wigner quasiprobability depends only on the product pq and is therefore constant on the hyperbolas $pq = \mathrm{const}$.

The operator $\exp(iK_3s^3)$ makes a rotation of the operators K_1 and K_2 according to (9)

$$\exp(iK_3s^3)(K_1,K_2)\exp(-iK_3s^3) = (K_1\cos(s^3) - K_2\sin(s^3), K_1\sin(s^3) + K_2\cos(s^3)). \tag{35}$$

In particular, one obtains the operator K_1 from K_2 by a rotation about an angle $s^3 = \frac{\pi}{2}$

$$K_1 = \exp(iK_3\frac{\pi}{2})K_2\exp(-iK_3\frac{\pi}{2}) = \exp(i\frac{\pi}{4}a^\dagger a)K_2\exp(-i\frac{\pi}{4}a^\dagger a). \tag{36}$$

Therefore, the eigenstates of K_1 can be obtained from the eigenstates of K_2 to the same eigenvalue by applying the operator $\exp(i\frac{\pi}{4}a^\dagger a)$ to the eigenstates of K_2.

6. Conclusion

The eigenvalue problem for squeezing–like operators in the single boson realization of the $SU(1,1)$–group was reduced to some standard problems and was explicitly solved for the Hermitean operator K_2. The position and momentum representations and the Bargmann representation were obtained. The number representation of these states leads to, in general, unknown polynomials and was not discussed. We obtained also the solution of the eigenvalue problem of some other quadratic boson operators as, for example, the right–hand and left–hand eigenstates of the operators $a^\dagger(a+\zeta a^\dagger)$ and $(a^\dagger+\zeta^* a)a$ but could not represent it here.

References

[1] A. Wünsche: *Ann. Phys. (Leipzig)* **1** (1992) 181;
[2] A. K. Roy, C. L. Mehta: *J. Mod. Opt.* **8** (1992) 1619;
[3] A. Wünsche: *Second International Workshop on Squeezed States and Uncertainty Relations* NASA Conference Publication 3219, Greenbelt, Maryland, (1993) p. 227;
[4] H. P. Yuen: *Phys. Rev. A* **13** (1976) 2226;
[5] X. Ma, W. Rhodes: *Il Nuovo Cimento* **104 B** (1989) 159;
[6] A. Orlowski: *Phys. Lett. A* **154** (1991) 319;
[7] C. F. Lo: *Phys. Rev. A* **42** (1990) 6752;
[8] C. F. Lo: *Quant. Opt.* **3** (1991) 333;
[9] C. G. Bollini, L. E. Oxman: *Phys. Rev. A* **47** (1993) 2339;
[10] G. S. Prakash, G. S. Agarwal: *Phys. Rev. A* **50** (1994) 4258;
[11] V. Bužek: *Phys. Rev. A* **39** (1989) 5432;

[12] V. Bužek: *J. Mod. Opt.* **37** (1990) 303;
[13] A. Vourdas: *Phys. Rev. A* **41** (1990) 1653;
[14] X. Ma, W. Rhodes: *Phys. Rev. A* **41** (1990) 4625;
[15] C. F. Lo, K. L. Liu: *Phys. Rev. A* **48** (1993) 3362;
[16] A. Perelomov: *Generalized Coherent States and Their Applications* (Springer, Berlin 1986);
[17] Y. S. Kim, M. E. Noz: *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore 1991);
[18] B. Wybourne: *Classical Groups for Physicist* (John Wiley, New York 1974);
[19] R. Gilmore: *Lie Groups, Lie Algebras and Some of Their Applications* (John Wiley, New York 1974);
[20] A. O. Barut, R. Raczka: *Theory of Group Representations and Applications* (PNW–Polish Scientific Publishers, Warszawa 1977);
[21] Z. Z. Xin, D. B. Wang, M. Hirayama, K. Matumoto: *Phys. Rev. A* **50** (1994) 2865;