acta physica slovaca vol. 45 No. 3, 387 – 390 June 1995

PROBABILITY DISTRIBUTIONS FOR THE PHASE DIFFERENCE[1]

A. Luis [2]
Department of Optics, Faculty of Natural Sciences, Palacký University, 17. listopadu 50, 77207 Olomouc, Czech Republic

L.L. Sánchez-Soto
Departamento de Optica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

Received 26 April 1995, accepted 10 May 1995

We study the quantum properties of the phase difference. We propose a general procedure to obtain the probability distribution for the phase difference for a broad class of phase approaches. Its properties are discussed.

1. Introduction

The main effort in the study of the phase in quantum optics has been devoted to the absolute phase. The failure of the polar decomposition of the amplitude operator for a one-mode field to give a unitary operator exponential of the phase has allowed the introduction of different and interesting quantum descriptions for this variable [1].

From a practical point of view, it seems that any observation of the phase must be relative to the phase of a reference system. Therefore, the most proper way to deal with the phase should be as a phase difference.

To take advantage of this fact, we can focus on the phase difference variable and try to define a phase-difference operator without any previous assumption about the description of the absolute phase. We have followed this procedure showing that the polar decomposition of a two-mode field allows the introduction of a unitary operator exponential of the phase difference [2].

Otherwise, we can also describe the phase difference in terms of previous approaches defining the absolute phases for the two modes. Here we follow this last approach noting the main coincidences and differences with the results of the polar decomposition.

[1] Presented at the **3rd central-european workshop on quantum optics**, Budmerice castle, Slovakia, 28 April - 1 May, 1995

[2] On leave of absence from: Departamento de Optica, Facultad de Ciencias Físicas, Universidad Complutense, Madrid, Spain

0323-0465/95 © Institute of Physics, SAS, Bratislava, Slovakia

2. Probability distributions for the phase difference

One route in the study of the phase difference starts from previous approaches defining probability distributions for the absolute phases ϕ_1 and ϕ_2 of the two modes involved. These probability distributions for the system state ρ can be defined in terms of an operator valued measure $\Delta_j(\phi_j)$ $j = 1, 2$ as

$$P(\phi_j) = \mathrm{tr}\,[\rho \Delta_j(\phi_j)] . \tag{1}$$

For most of these approaches, including the most widely used in recent years, this set of operators has the shifting property

$$e^{-i\phi_0 N_j} \Delta_j(\phi_j) e^{i\phi_0 N_j} = \Delta_j(\phi_j - \phi_0), \tag{2}$$

where N_j are the corresponding number operators. This property is verified for the sets $\Delta_j(\phi_j)$ that can be written in the number basis as [3]

$$\Delta_j(\phi_j) = \sum_{n_j, n_j'=0}^{\infty} G_{n_j, n_j'} e^{i(n_j - n_j')\phi_j} |n_j\rangle\langle n_j'| . \tag{3}$$

Here we will consider this general form for the description of the absolute phase. The coefficients $G_{n,n'}$ depend on the particular approach studied [4].

The joint probability distribution function for the absolute phases is

$$P(\phi_1, \phi_2) = \mathrm{tr}\,[\rho \Delta_1(\phi_1) \otimes \Delta_2(\phi_2)] . \tag{4}$$

Here we are interested in the phase difference, and our aim is to obtain its associated probability distribution from the knowledge of the corresponding one (4) for the absolute phases. Then we will study its main properties independent of the particular choice for $G_{n,n'}$. This can be done in many ways.

We can ask in first place for the probability distribution $\mathcal{P}(\phi_s, \phi_d)$ associated with the phase sum ϕ_s and phase difference ϕ_d variables

$$\phi_s = \phi_1 + \phi_2, \qquad \phi_d = \phi_1 - \phi_2, \tag{5}$$

and then integrate over the phase sum ϕ_s. One possible way to obtain $\mathcal{P}(\phi_s, \phi_d)$ is by means of a change of variables in $P(\phi_1, \phi_2)$. But it must be noted that, as far as we consider all the phase variables to be 2π-periodic, the transformation (5) is not bijective and to the phases (ϕ_1, ϕ_2) and $(\phi_1 + \pi, \phi_2 + \pi)$ correspond the same mod(2π) value of (ϕ_s, ϕ_d). According to this, it seems that the proper transformation law relating $\mathcal{P}(\phi_s, \phi_d)$ with $P(\phi_1, \phi_2)$ is obtained by a change of variables and adding the contributions from these two points, getting

$$\mathcal{P}(\phi_s, \phi_d) = \frac{1}{2}\left[P\left(\frac{\phi_s + \phi_d}{2}, \frac{\phi_s - \phi_d}{2}\right) + P\left(\frac{\phi_s + \phi_d}{2} + \pi, \frac{\phi_s - \phi_d}{2} + \pi\right)\right] . \tag{6}$$

In principle we do no have phase operators from which this transformation law could be derived. Nevertheless, the difficulties arising in the definition of the absolute phase are avoided in some approaches by an extension of the Hilbert space of the problem including negative number states. Then phase operators can be defined and it can be seen that the transformation law we are discussing is given by (6).

Another way to arrive at the transformation relating $\mathcal{P}(\phi_s, \phi_d)$ with $P(\phi_1, \phi_2)$ comes after imposing that, for any periodic function of the phase sum and difference, we must get the same mean value irrespectively of whether (ϕ_s, ϕ_d) or (ϕ_1, ϕ_2) variables are used. This means that we can impose

$$\int d\phi_s d\phi_d e^{ik\phi_s} e^{i\ell\phi_d} \mathcal{P}(\phi_s, \phi_d) = \int d\phi_1 d\phi_2 e^{ik(\phi_1 + \phi_2)} e^{i\ell(\phi_1 - \phi_2)} P(\phi_1, \phi_2), \tag{7}$$

for any integers k and ℓ. Since all these functions are periodic these equalities completely fix $\mathcal{P}(\phi_s, \phi_d)$ in terms of $P(\phi_1, \phi_2)$ giving the same result (6) as well.

After that, the probability distribution for the phase difference is obtained integrating over the phase sum variable ϕ_s. The result can be expressed again in terms of an operator valued measure $\Delta(\phi_d)$

$$\mathcal{P}(\phi_d) = \int d\phi_s \mathcal{P}(\phi_s, \phi_d) = \mathrm{tr}\,[\rho \Delta(\phi_d)] . \tag{8}$$

These operators $\Delta(\phi)$ (from now on we omit the subscript on ϕ_d) can be written as

$$\Delta(\phi) = \sum_{n=0}^{\infty} \Delta(n, \phi), \tag{9}$$

where

$$\Delta(n, \phi) = \sum_{n_1, n_1'=0}^{n} 2\pi G_{n_1, n_1'} G_{n-n_1, n-n_1'} e^{i(n_1 - n_1')\phi} |n_1, n - n_1\rangle\langle n_1', n - n_1'| . \tag{10}$$

We can see that these operators $\Delta(\phi)$ commute with the total photon number operator, and this is the first general property that has any phase approach in the form we are considering here. This could be expected from the shifting property (2) since the total photon number shifts just the phase sum, and therefore must not modify the phase difference. This also means that studying the properties of the phase difference we can focus on each subspace $\mathcal{H}_n$ with fixed total photon number n independently of the others. The decomposition (9) allows the introduction of a joint probability distribution function for the total number and the phase difference by

$$\mathcal{P}(n, \phi) = \mathrm{tr}\,[\rho \Delta(n, \phi)] . \tag{11}$$

But this property has also further consequences. The most general $\mathcal{P}(n, \phi)$ is, according to (11) and (10),

$$\mathcal{P}(n, \phi) = \sum_{k=-n}^{n} c_k e^{ik\phi}, \tag{12}$$

with $c_k = c^*_{-k}$. This means that $\mathcal{P}(n,\phi)$ depends just on $2n+1$ parameters and therefore can be completely fixed by its value on $2n+1$ properly chosen phase points $\phi_r^{(n)} = 2\pi r/(2n+1)$ for instance with r an integer running from $-n$ to n. Then we can express $\mathcal{P}(n,\phi)$ at any point as

$$\mathcal{P}(n,\phi) = \frac{1}{2n+1} \sum_{k,r=-n}^{n} \mathcal{P}(n,\phi_r^{(n)}) e^{ik(\phi-\phi_r^{(n)})}. \tag{13}$$

This is the last property that we discuss here. Since all the subspaces $\mathcal{H}_n$ have finite dimension, a continuous range of variation for the phase difference seems to contain more information than the strictly necessary to describe a variable in a finite dimensional space. We have shown that this is the case and the knowledge of the probability distribution function over a finite number of phase difference values is enough to completely characterize it within the subspace $\mathcal{H}_n$. It is worth noting that in an extended Hilbert space, including negative number states, the subspaces $\mathcal{H}_n$ are not finite-dimensional and therefore a continuous variation for ϕ is needed there.

3. Conclusions

We have shown that the commutation or compatibility of the phase difference with the total photon number is a requirement fulfilled by a broad class of phase approaches. But we can also note that this fact, directly or implicitly, demands a discrete character for the phase difference. The number of allowed values grows with the total photon number n, so discreteness is only relevant in the limit of small n and becomes a continuous for all practical considerations in the limit when one or the two modes are intense [5]. All these are points of coincidence with the conclusions obtained with the phase-difference operator arising from a polar decomposition for the two-mode case. The commutation with the total photon number is imposed there as the quantum traslation of the corresponding classical Poisson bracket and this leads to the discreteness of the phase difference. But despite the coincidences pointed out here, the phase-difference operator and the formalism discussed here are still very different approaches, since in the last one the phase difference is assumed to be continuous and so it is handled.

Acknowledgements This work was partially supported by a grant from the Ministerio Español de Educación y Ciencia.

References

[1] Special issue on quantum phase: *Phys. Scr.* **T 48** (1993) 1-144;
[2] A. Luis, L.L. Sánchez-Soto: *Phys. Rev. A* **48** (1993) 4702; L.L. Sánchez-Soto, A. Luis: *Opt. Commun.* **105** (1994) 84;
[3] U. Leonhardt, J. A. Vaccaro, B. Böhmer, H. Paul: *Phys. Rev. A* **51** (1995) 84;
[4] R. Tanaś, A. Miranowicz, T. S. Gantsog: *Phys. Scr.* **T 48** (1993) 53;
[5] A. Luis, L.L. Sánchez-Soto, R. Tanaś: *Phys. Rev. A* **51** (1995) 1634;