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NONLINEAR MEDIA AND COHERENT STATE IN FINITE DIMENSIONAL HILBERT SPACE[1]

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We discuss a system comprising a nonlinear medium of k-order driven by an external field. We assume that the system was initially in the vacuum state. We show that for the case of weak external field our system behaves the same way as that described in finite dimensional Hilbert space. Moreover, we perform a numerical experiment in which we simulate the dynamics of our system and compare the results with those of analytical attempts.

1. Introduction

Coherent state in infinite dimensional Hilbert was defined by Glauber [1] by application of a displacement operator $\hat{D}$ acting on the vacuum state $|0\rangle$. Recently, we observe great interest in the problem of coherent states defined in finite dimensional Hilbert space. For instance, Bužek *et al.* [2] discussed a coherent state analogous to that defined by Glauber, however the state proposed in [2] was defined in $(s+1)$ dimensional space. The same problem was a subject of the paper [3], where the analytical solutions for the coherent state was found.

In this paper, we propose a rather general model which, we believe, can lead to coherent state generation in finite dimensional space and, as we will show, corresponds to that discussed in [3]. The model combines the evolution of a nonlinear medium in a cavity and a weak external excitation. We show that for a sufficiently weak external field, resonance effects start to play a significant role, whereas nonresonant couplings become negligible. The nonlinear quantum evolution of the cavity field in the nonlinear medium is crucial for the preparation of a finite dimensional coherent state in such a system. The effectiveness of the preparation is, however, considerably diminished by the cavity losses. Nevertheless, it seems important to us that a cavity with a nonlinear

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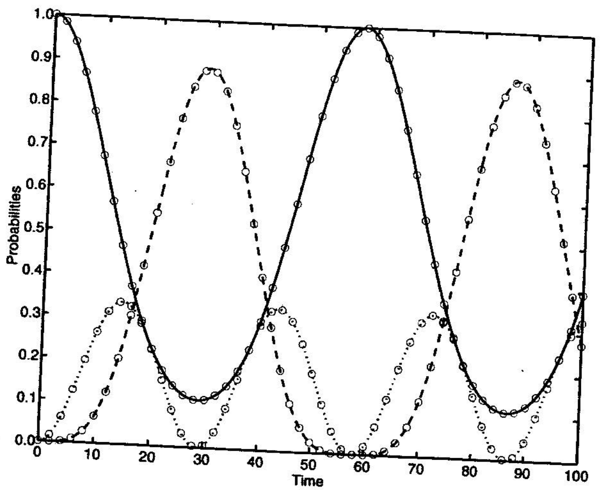

Fig.1. Analytical solutions for the probabilities of the vacuum (solid line), one-photon (dotted line) and two-photon (dashed line) states. The parameter $\epsilon = \pi/50$ (all parameters are measured in units of $\lambda_3 = 1$). Star marks correspond to the probabilities found in the numerical experiment.

medium (with a field initially in vacuum state) and a sufficiently weak external excitation, can lead with high accuracy to finite dimensional coherent state generation. For this situation we will derive analytical formulas for the probabilities corresponding to the Fock states we are interested in. Moreover, we will perform a numerical experiment in which we simulate the dynamics of our system and compare the results with those of an analytical attempt.

2. The model and an analytical solution.

Our system is governed by the following Hamiltonian (in the interaction picture):

$$\hat{H} = \frac{\lambda_k}{k} (\hat{a}^\dagger)^k \hat{a}^k + \epsilon(\hat{a}^\dagger + \hat{a}) , \quad (1)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, λ_k denotes the constant of k-th order nonlinearity, and ϵ corresponds to medium–external field interaction. We use units of $\hbar = 1$.

Let us express the wave-function in Fock basis:

$$|\Psi(t)\rangle = \sum_{j=0}^{\infty} a_j(t) \, |j\rangle \; . \quad (2)$$

This wave-function obeys a Schrödinger equation with Hamiltonian expressed by eqn. (1):

$$i\frac{d}{dt}|\Psi(t)\rangle = \left(\frac{\lambda_k}{k} (\hat{a}^\dagger)^k \hat{a}^k + \epsilon(\hat{a}^\dagger + \hat{a})\right) |\Psi(t)\rangle \; . \quad (3)$$

We assume here that the coupling is weak, i.e. $\epsilon \ll \lambda_k$. In consequence, we are able to treat our problem perturbatively. Applying the standard procedure to our wave-function (2) and the Hamiltonian (1) we obtain a set of equations for the probability amplitudes $a_j(t)$. They are of the form:

$$i\frac{d}{dt}a_j(t) \;=\; \frac{\lambda_k}{k}\left[j(j-1)...(j-k)\right] a_j(t) + \epsilon\left(\sqrt{j}a_{j-1}(t) + \sqrt{j+1}a_{j+1}(t)\right) \; , \quad (4)$$

where k denotes the order of the nonlinear process and j corresponds to the j-photon state. Obviously, one should keep in mind that for $j < 0$ we have $a_j = 0$. Although we see from (4) that the set of equations for a_j is infinite, the nonlinear process and weak external field cut some subspace of states out of all of the Fock states. In consequence, the dynamics of the physical process starts from the vacuum $|0\rangle$ and is restricted to the states $|m\rangle$, where $(m = 0, 1, 2, ..., k)$. The main point of our considerations is the fact that the part of the Hamiltonian (1) corresponding to the nonlinear medium evolution:

$$\hat{H}_{NL} \;=\; \frac{\lambda_k}{k} (\hat{a}^\dagger)^k \hat{a}^k \quad (5)$$

produces degenerate states (corresponding to $j = 0, 1, ..., k-1$). Moreover, the coupling is weak ($\epsilon \ll \lambda_k$) and we apply the standard rediagonalization procedure. This situation resembles that for k degenerate atomic levels coupled by the zero-frequency field, where this resonant interaction selects, from the whole set of levels, only those of them that lead to a closed system dynamics. Interaction with the remaining atomic levels can be treated as a negligible perturbation [4]. Thus, we can write the following equations of motion for the probability amplitudes:

$$i\frac{d}{dt}a_0(t) = \epsilon a_1(t), \qquad i\frac{d}{dt}a_1(t) = \epsilon(a_0(t) + \sqrt{2}a_2(t)),$$

$$\vdots \quad (6)$$

$$i\frac{d}{dt}a_{k-2}(t) = \epsilon(\sqrt{k-2}a_{k-3}(t) + \sqrt{k-1}a_{k-1}(t)), \quad i\frac{d}{dt}a_{k-1}(t) = \epsilon\sqrt{k-1}a_{k-2}(t).$$

We obtain the following solutions for the probability amplitudes for an arbitrary n-photon state. For instance, for $k = 3$ we get

$$a_0(t) = \tfrac{1}{3}\left(2 + \cos\left(\sqrt{3}\,\epsilon t\right)\right) \qquad a_1(t) = \frac{-i}{\sqrt{3}} \sin\left(\sqrt{3}\,\epsilon t\right),$$

$$a_2(t) = \tfrac{\sqrt{2}}{3}\left(\cos(\sqrt{3}\,\epsilon t) - 1\right) \qquad a_3(t) = \frac{-2i\epsilon}{3\lambda\sqrt{6}}\left(\cos(\sqrt{3}\,\epsilon t) - 1\right),$$

$$a_4(t) = \tfrac{i\epsilon^2}{9\sqrt{6}\lambda^2}\left(\cos(\sqrt{6}\,\epsilon t) - 1\right) . \quad (7)$$

We see that for a sufficiently weak external field the time-evolution is restricted to a very narrow subset of n-photon states. The probabilities corresponding to the states for higher n are proportional to $\epsilon^2, \epsilon^4, \epsilon^6, \dots$. Therefore, we can treat our model as spanned on a finite-dimensional space, as discussed in [2,3] (for the case shown here we are restricted to the three-dimensional space). Moreover, our results are in agreement with those of Miranowicz *et al.* derived for the coherent state in finite-dimensional Hilbert space [3]. This agreement is achieved for any order of nonlinearity k and the value of k determines the dimension of the space.

3. The numerical experiment.

To verify these results we shall now perform a numerical experiment and compare its results with those based on our formulas. This will be done similarly as in the paper [5].

The history of our system is governed by the unitary evolution operator $\hat{U}(t)$ defined as follows:

$$\hat{U}(t) = \exp(-i\hat{H}t) \ . \tag{8}$$

Hence, the wave-function $|\Psi(t)\rangle$ for arbitrary time t can be expressed as:

$$|\Psi(t)\rangle = \hat{U}(t)\,|0\rangle \, . \tag{9}$$

Of course, both of them, the wave function $|\Psi(t)\rangle$ and the operator $\hat{U}$, are expressed in the Fock-state basis. For our experiment we chose the nonlinear process for $k = 3$. Thus, Fig.1 shows the time dependence of the probabilities for the subsequent n-photon states obtained from eqns.(7) and from the numerical experiment. It is visible that the dynamics of our system is restricted to the three n-photon states: $|0\rangle$, $|1\rangle$, and $|2\rangle$. Moreover, the influence of the higher Fock states $((|3\rangle, |4\rangle, ...)$ is negligible. For instance, for the parameters corresponding to Fig.1, $|a_3(t)|^2 \sim 10^{-3}$. However, the most important seems to be the fact that the results of our experiment show very good agreement with those of the analytical attempt. So we conclude that our system produces the coherent state defined in the three-dimensional Hilbert space, described by the eqns.(7) and discussed in [2,3].

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