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RECONSTRUCTING PHOTON STATISTICS FROM HOMODYNE EXPERIMENTS[1]

T. Kiss[2], U. Leonhardt, U. Herzog
Arbeitsgruppe "Nichtklassische Strahlung" der Max–Planck–Gesellschaft an der Humboldt–Universität zu Berlin Rudower Chaussee 5, 12484 Berlin Germany

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In the tomographical reconstruction of the density matrix a set of pattern functions has to be averaged with respect to the measured data. We discuss the evaluation of the pattern functions in the Fock basis and give an algorithm to calculate them. The effect of the inefficient detectors can be compensated by numerical deconvolution, which we separate from the measuring and reconstructing processes. The compensation is possible in general only if the losses do not exceed the critical value of 50%.

1. Introduction

Photon statistics plays an important role in describing the properties of light. The photon–number distribution of a single mode of the radiation field can show interesting features for nonclassical light, such as the Schleich–Wheeler oscillations of squeezed states [1]. The precise measurement of the photon statistics, however, is a nontrivial task [2]. Inefficient detectors and other losses attenuate the signal and beyond that cause extra noise, smearing out the subtle details.

In recent experiments homodyne tomography was used to reconstruct the Wigner function [3]. The tomographycal scheme can also be applied directly to the density matrix [4,5]. We discuss the possibility of reconstructing the photon statistics in this scheme. The density matrix elements in a given basis are obtained by averaging a set of pattern functions with respect to the measured quadrature distributions [5]. One way to calculate the pattern functions in the Fock basis is the use of recursion relations, which are, unfortunately, instable for the large photon numbers. Another, nonrecursive method can avoid the direct evaluation of the pattern functions. Here the measured data

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[2]Permanent address: Research Laboratory for Crystal Physics, Hungarian Academy of Sciences, P.O. Box 132, H-1502 Budapest, Hungary

are integrated with simple weight functions and a linear combination of such integrals yields the matrix element.

The effect of losses can be compensated physically by preamplification [6] or numerically by deconvolution. In Ref.[5] the deconvolution was achieved by using more peaked pattern functions. We treat here the compensation separately, as a second step after the reconstruction. We show that the generalized Bernoulli transformation, describing the effect of the inefficiencies, can be inverted [7]. The deconvolution process is always convergent if the overall efficiency is larger than the critical value $\frac{1}{2}$. For smaller efficiences, however, the statistical errors are amplified and therefore the compensation is possible only in some special cases.

2. Density matrix reconstruction in the Fock basis

In a homodyne experiment the measured quantities are rotated quadratures x_θ where the rotation angle θ is defined by the phase difference of the signal and the local oscillator. According to the tomographycal scheme a set of pattern functions [5] $F_{nm}(x_\theta,\theta)$ is averaged with respect to the measured distributions in order to get the matrix elements

$$\langle n|\hat{\rho}|m\rangle = \int_0^\pi \int_{-\infty}^{\infty} F_{nm}(x_\theta,\theta) w_\theta(x_\theta) dx_\theta d\theta \,. \tag{1}$$

The phase dependence of the pattern functions is trivial

$$F_{nm}(x_\theta,\theta) = \exp[i(n-m)\theta] f_{nm}(x_\theta) \,, \tag{2}$$

where the one variable $f_{nm}(x_\theta)$ functions are defined by

$$f_{nm}(x_\theta) = \frac{1}{2\pi} \int \langle n| \exp[i\zeta(\hat{x} - x_\theta)] |m\rangle |\zeta| d\zeta \,. \tag{3}$$

The evaluation of the $f_{nm}(x_\theta)$ functions, however, leads to numerical difficulties. The recursive method is instable in certain regions because the difference of large numbers occurs in it. We define here a method, where the pattern functions have not to be evaluated explicitly. The integral in (3) can be expanded and after some calculation $f_{nm}(x)$ can be expressed as a linear combination of simple weight functions

$$f_{nm}(x) = \sum_{l=0}^{\infty} a_{nml} z_{2l+p_{nm}}(x) \,, \quad z_k(x) = \mathcal{N} x^k \exp(-x^2) \,. \tag{4}$$

Here $\mathcal{N}$ stands for the normalization factor $\mathcal{N} = \left([\frac{k}{2}]!\right)^{-1}$ and $p_{nm} = 0$ or 1 if $n-m$ is even or odd, respectively. For the a_{nml} coefficients a closed form can be found

$$a_{nml} = -\frac{2}{\pi} 2^{p_{nm}/2} (n!m!)^{\frac{1}{2}} \sum_{\nu=0}^{n} \frac{(2\nu+m-n+1)!}{\nu!(n-\nu)!(m-n+\nu)! \prod_{\mu}^{\nu+1+[\frac{m-n}{2}]} 2l \mp 1 - 2\mu} \,. \tag{5}$$

This expansion offers an elegant way for the averaging. Exchanging the summation and the integration, one can first calculate the integrals with a number of weight functions (introducing a k_{max} cut-off) and then the linear combination of the integrals with the a_{nml} coefficients yields the matrix elements.

3. Compensating the effect of losses

The effect of inefficient detectors and other losses can be taken into account with a simple model. A fictitous beam splitter is placed in front of an homodyne apparatus with ideal detectors and its transparence is set to be equal to the overall detector efficiency η. The signal density matrix $\hat{\rho}_{sig}$ is transformed and mixed with the vacuum $\hat{\rho}_0$ entering the unused port. The properties of the resulting $\hat{\rho}_{meas}$ density matrix are then actually measured.

Several mathematically equivalent models exist to describe the beam-splitter transformation [8]. Using normally ordered products a simple rule connects the signal and measured values

$$\langle \hat{a}^{\dagger n}_{meas} \hat{a}^m_{meas} \rangle = \eta^{\frac{n+m}{2}} \langle \hat{a}^{\dagger n}_{sig} \hat{a}^m_{sig} \rangle \,. \tag{6}$$

The equations are invariant to the simultaneous exchange of the indices $sig \to meas$ and $\eta \to \eta^{-1}$. This suggests that the inversion of the transformation can be achieved by substituting η by η^{-1}. Another way for the mathematical description is to utilize the fact that beam-splitters are also models for damping processes [8]. We introduce a formal dissipation process with the master equation

$$\frac{d\hat{\rho}}{dt} = \frac{1}{2} \left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} \right) \,. \tag{7}$$

The formally introduced time t is related to the transmittance or with other words to the efficiency by

$$\eta = e^{-t} \,. \tag{8}$$

Running the process backwards we get from $\hat{\rho}_{meas}$ the original $\hat{\rho}_{sig}$. This means again the exchange of η by η^{-1}.

Both methods suggest the simple recipe for the compensation: calculate the transformation for the density matrix elements in a given basis and then replace η by η^{-1}. In Fock basis this yields the *inversion of the generalized Bernoulli transformation*

$$\langle n_1 | \hat{\rho}_{\text{sig}} | n_1' \rangle = \eta^{-\frac{1}{2}(n_1+n_1')} \sum_{k=0}^{\infty} \langle n_1 + k | \hat{\rho}_{\text{meas}} | n_1' + k \rangle \times \left[\binom{n_1+k}{n_1} \binom{n_1'+k}{n_1'} \right]^{\frac{1}{2}} \left(1 - \frac{1}{\eta} \right)^k . \tag{9}$$

The existence of such an inversion formula may be surprizing, in particular if we take into account that dissipation is regarded as an irreversible process. The losses, however, have a statistically well-defined character and therefore they can be compensated provided that the process converges. Analyzing the structure of the inversion (9) we see that the different diagonals are transformed separately. If one would like to know only the photon statistics, represented by the main diagonals, the knowledge of the off-diagonals is not necessary.

We discuss now three different cases. First, the inversion always converges if η exceeds the critical value of $\frac{1}{2}$ as in this case the power series is a decreasing one. The statistical uncertainty in the high photon numbers does not effect the result of the inversion. Second, in the interesting special case when the original density matrix was finite the compensation works for all efficiences. The sums are finite, the statistical error can be diminished by increasing the number of measurements. In the third and worst case, when η is below the critical value and the matrix is infinite, the compensation procedure, in general, cannot be carried out. Taking the thermal distribution as an example we find that the inversion converges provided the efficiency is above the value $\eta > \frac{1}{2}(1 - \frac{1}{\bar{n}})$, where $\bar{n}$ denotes the mean photon number of the signal distribution. This condition is always fulfilled for $\bar{n} < 1$. The finiteness of the statistical errors needs a stronger condition, and it can be proved that the critical value for this is $\eta_{\rm cr} = \left(\frac{1}{\bar{n}} + 2\right)^{-1}$. We note that there are also other errors e.g. the experimental uncertainty of η itself, which are amplified the if $\eta < \frac{1}{2}$ adding extra noise to the result.

To summarize, we have shown that homodyne tomography is an appropriate tool for the indirect measurement of the photon statistics. The losses in the measuring process can be compensated by numerical deconvolution, provided the overall efficiency exceeds the critical value $\frac{1}{2}$. It is interesting to note that $\frac{1}{2}$ is the critical attenuation [9] in a damping process, where the Wigner function of an initially pure m-photon state becomes entirely positive.

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