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WIGNER FUNCTION DESCRIPTION OF A GENERALIZED TWO-DIMENSIONAL OSCILLATOR[1]

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The evolution of the optical Schrödinger-cat state in a coupled two-dimensional oscillator is presented by means of the Wigner distribution function. It is shown how the quantum interference collapses and revivals during the exchange between the modes.

Recently much attention has been paid to the study of the Hamiltonian

$$H = \frac{1}{2}\sum_{n=1,2}\left[\frac{p_n^2}{m} + m\tilde{\omega}^2 q_n^2\right] + \Omega(q_1p_2 - q_2p_1). \tag{1}$$

This Hamiltonian can describe the motion of an electron in a constant magnetic field provided $\omega = \Omega$, the optical directional coupler [1], a vibrating and rotating molecule with two degrees of freedom, radiation field in a cavity with moving mirror [2]. The accidental degeneracy of the Hamiltonian (1) was investigated in [3]. Nother's theorem was applied to get the symmetry Lie algebra of the system.

The systems mentioned above can be considered as two-dimensional harmonic oscillators with a coupling. The coupling mixes the initial states. It is an interesting question how one of the field mode affects the other one during the time evolution. It is convenient to choose such an initial state which is a direct product of a classical and nonclassical state. Testing in some way the classicality/nonclassicality of the emerging states we can gain some inside into the nature of the coupling.

The coherent states have features as classical as possible. On the other hand even the most simple discrete superpositions of coherent states can realize quantum fields with nonclassical properties due to the quantum interference [4]. The most simple nonclassical states are the even and odd optical Schrödinger-cat states

$$|\,\alpha,\pm\rangle = \mathcal{N}_\pm(|\,\alpha\rangle \pm |\,-\alpha\rangle). \tag{2}$$

In this paper we investigate the "interaction" of a Schrödinger-cat state and the vacuum state. To this end we derive the time dependent characteristic function and

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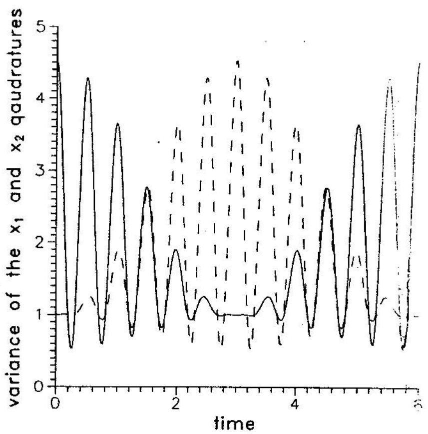

Fig.1. The variance of the field quadrature operators $x_i = a_i + a_i^\dagger, i = 1,2$. The solid line corresponds to the 1st mode the dashed to the 2nd. The input state is the direct product of the even Schrödinger-cat state and the vacuum state. The periodiacal collapse and revival of the states in the modes can be seen clearly.

Wigner distribution of the field. The variances of the field quadratures are determined by means of the derivatives of the characteristic function. The plots of the Wigner distributions visualize the evolution of the initial states.

To find the time dependent characteristic function we introduce the photon creation $a^\dagger$ and annihilation a operators in the usual manner: $q_n = \sqrt{\hbar/2m\tilde{\omega}}(a_n + a_n^\dagger)$ and $p_n = i\sqrt{\hbar m\tilde{\omega}/2}(a_n^\dagger - a_n)$. The Hamiltonian (1) reads

$$H = \hbar\omega a_1^\dagger a_1 + \hbar\omega a_2^\dagger a_2 + i\hbar\Omega(a_1^\dagger a_2 - a_1 a_2^\dagger) \tag{3}$$

in the new variables, where $\omega = \sqrt{\tilde{\omega}^2 + \Omega^2}$. The Heisenberg equation of motion of the annihilation operators are

$$\frac{da_1}{dt} = -i\omega a_1 - \Omega a_2,$$
$$\frac{da_2}{dt} = -i\omega a_2 + \Omega a_1. \tag{4}$$

The solution of these equations can be written in the form

$$a_1(t) = \{a_1(0)\cos(\Omega t) - a_2(0)\sin(\Omega t)\}e^{-i\omega t},$$
$$a_2(t) = \{a_2(0)\cos(\Omega t) + a_1(0)\sin(\Omega t)\}e^{-i\omega t}, \tag{5}$$

where $a_n(0)$ denote the annihilation operator at $t = 0$.

The characteristic function of a two-mode field is defined by [5]

$$\chi(\eta,\gamma) = \mathrm{Tr}\{\varrho e^{\eta a_1^\dagger(t)} e^{-\eta^* a_1(t)} e^{\gamma a_2^\dagger(t)} e^{-\gamma^* a_2(t)}\}, \tag{6}$$

where we show explicitly the time dependence of the photon operators. Let us suppose that a Schrödinger-cat state is the input of the 1st mode and the vacuum state of the 2nd. Substituting (2) and (5) into (6) we find the characteristic function

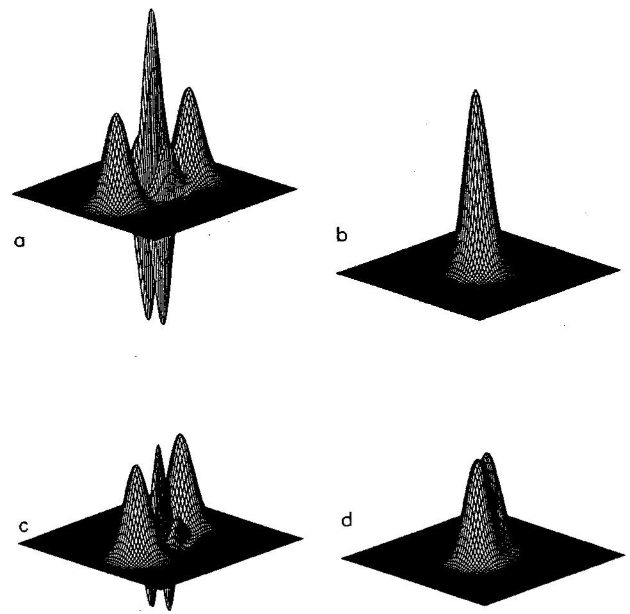

Fig. 2. The Wigner distribution function associated with the initial state $(1/\mathcal{N})(|\alpha\rangle + |-\alpha\rangle) \otimes |vac\rangle$. Picture a and b show the contours of the Wigner functions at $t = 0$ in the mode 1 and 2 correspondingly. Picture a and b show the contours of the Wigner functions at $t = 0.5$ in the mode 1 and 2 correspondingly. It can be seen that the interference pattern of the input state in the mode 1 deteriorates and the Schrödinger-cat state 'flows' to the mode 2.

$$\chi(\eta,\gamma) = \frac{1}{\mathcal{N}}(e^{\eta m^* - \eta^* m + \gamma n^* - \gamma^* n} + e^{-\eta m^* + \eta^* m - \gamma n^* + \gamma^* n} +$$
$$e^{i\phi - 2|\alpha|^2} e^{\eta m^* + \eta^* m + \gamma n^* + \gamma^* n} + e^{-i\phi - 2|\alpha|^2} e^{-\eta m^* - \eta^* m - \gamma n^* - \gamma^* n})$$
$$m = \alpha\cos(\Omega t)e^{-i\omega t}, n = \alpha\sin(\Omega t)e^{-i\omega t} \tag{7}$$

To characterize the states in the modes we calculate the variance of the field quadrature operators $x_i = a + a^\dagger$ and $y_i = i(a^\dagger - a)$. The variances can be expressed in terms of the mean values of the normally ordered product of the field creation and annihilation operators. One can readily find these quantities by means of the characteristic function [5]

$$\langle a_1^{\dagger m} a_1^n \rangle = \frac{\partial^{m+n}}{\partial\eta^m \partial(-\eta^{*n})}\chi(\eta,\gamma)|_{\eta=\gamma=0}. \tag{8}$$

To calculate the mean values in the other mode one have to take the derivatives with respect to γ.

In fig. 1 we show the variances of the position operators. The solid line corresponds to the 1st mode while the dashed line to the 2nd. Here $\alpha = 2$, $\omega = 2\pi/1$, $\Omega = 2\pi/12$. It is clearly seen that after a quarter of the period $T = 2\pi/\Omega$ the initial states of the modes will be exactly exchanged. Between the two limiting time moments the states of the modes will be the mixture of the vacuum and the Schrödinger-cat state. The plots can be considered as the visibility of the quantum interference fringes between the coherent states of the Schrödinger-cat state. At $t = 0$ the visibility is maximal in the 1st mode manifesting the presence of quantum interference. At the same time there is no interference in the other mode since the input of that mode is the vacuum. During the time evolution of the system the visibility in the 1st mode disappears but became larger in the other mode. At $t = 2\pi/\Omega$ the visibility is maximal in the 2nd mode and 0 in the 1st mode. Then this process takes place in the opposite direction.

The Wigner function associated with a state is suitable to visualize the deterioration of quantum interference of the state. The Wigner distributions of the modes 1 and 2 are

$$W_1(x) = \frac{2}{\pi N}(e^{-2|x-m|^2} + e^{-2|x+m|^2} + e^{i\phi-2|\alpha|^2}e^{-2(|x|^2-m^*x+mx^*-|m|^2)} + e^{-i\phi-2|\alpha|^2}e^{-2(|x|^2+m^*x-mx^*-|m|^2)},$$
$$W_2(y) = \frac{2}{\pi N}(e^{-2|y-n|^2} + e^{-2|y+n|^2} + e^{i\phi-2|\alpha|^2}e^{-2(|y|^2-n^*y+ny^*-|n|^2)} + e^{-i\phi-2|\alpha|^2}e^{-2(|y|^2+n^*y-ny^*-|n|^2)},$$
$$x = x_1 + ix_2, y = y_1 + iy_2, \tag{9}$$

where x_i and y_i are the canonical variables associated with the modes (here the indices 1 and 2 refer to the position and momentum rather than the modes).

Figure 2.a and 2.b shows the contour of the Wigner functions at $t = 0$, (the system is the same as in fig. 1). In figure 2.c and 2.d we show the Wigner functions at an intermediate time moment. It can be seen clearly, that the interference fringes deteriorates in the 1st mode and the state in the 2nd mode becomes quadrature squeezed.

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