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QUASIPROBABILITY DENSITY OPERATORS AND GENERALIZED PARITY[1]

A. Czirják[2], M.G. Benedict[3]
Department of Theoretical Physics, Attila József University,
6701 Szeged POB 428, Hungary

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By shifting the parity operator in phase space, we obtain a class of operators, the Wigner operators, whose expectation value is equal to the Wigner function. By introducing a trace-class extension of this operator and its integral representation we find, that the Q– and P–functions are also expectation values of certain generalized parity operators.

1. Introduction

It is well known, that the Wigner function defined as

$$W(q,p) = \frac{1}{\pi\hbar}\int \mathrm{d}x \langle q - x \mid \hat{\rho} \mid q + x\rangle e^{2ipx/\hbar}, \tag{1}$$

is suitable to represent the state of the system in classical phase space if we calculate the expectation value of a symmetrically ordered operator [1], [2], [3].

Looking at the definition of the Wigner function, we can notice, that its value in the origin is proportional to the expectation value of the parity operator $\hat{P}_0$:

$$W(0,0) = \frac{1}{\pi\hbar}\int \mathrm{d}x \langle -x \mid \hat{\rho} \mid x\rangle = \frac{1}{\pi\hbar}\int \mathrm{d}x \langle x \mid \hat{P}_0\hat{\rho} \mid x\rangle = \frac{1}{\pi\hbar}\mathrm{Tr}(\hat{P}_0\hat{\rho}), \tag{2}$$

and that the proportionality factor consists of universal constants.

In the following we shall show, that the value of the Wigner function at any point of the phase space can be related to the expectation value of an operator in a similar manner, which therefore can be given the name Wigner operator. We note that in [4] an experiment has been described to measure the parity of an optical field mode i. e. the value of the Wigner function in one special point, namely in the origin.

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[2]E-mail address: czirjak@sol.cc.u-szeged.hu
[3]E-mail address: benedict@physx.u-szeged.hu

2. The shifted parity and the Wigner function

The Wigner operator is actually a class of operators, depending on the arguments q and p. It is the shifted parity operator, which reflects states with respect to the phase space point specified by its arguments. To realise this transformation we shift the state to the origin, reflect it by the parity and shift it back again to its initial position. That is, we apply the $\hat{D}(q,p)\hat{P}_0\hat{D}^{-1}(q,p)$ operator, where $\hat{D}(q,p) = e^{i(\hat{q}p-\hat{p}q)/\hbar}$ is the displacement operator. In order to get precise correspondence with the Wigner function as stated in the Introduction, we define the Wigner operator including the factor $\frac{1}{\pi\hbar}$:

$$\hat{W}(q,p) = \frac{1}{\pi\hbar}\hat{D}(q,p)\hat{P}_0\hat{D}^{-1}(q,p). \tag{3}$$

Statement: In any state the expectation value of the Wigner operator corresponding to any fixed point in phase space is identical to the value of the Wigner function of that particular state in that specified point.

Proof: Let us represent the physical state by the $\hat{\rho}$ density operator. Then

$$\langle\hat{W}(q_0,p_0)\rangle_\rho = \frac{1}{\pi\hbar}\int \mathrm{d}x \int \mathrm{d}y \langle x \mid \hat{\rho} \mid y\rangle\langle y \mid \hat{D}(q_0,p_0)\hat{P}_0\hat{D}^{-1}(q_0,p_0) \mid x\rangle.$$

The inverse of the displacement operator effects the elements of the coordinate basis as

$$\hat{D}^{-1}(q_0,p_0) \mid x\rangle = e^{\frac{i}{2}q_0p_0/\hbar - ip_0x/\hbar}\mid x - q_0\rangle, \tag{4}$$

and by adjoining a similar formula, one can obtain the expression for $\langle y|\hat{D}(q_0,p_0)$. Since $\hat{P}_0|x-q_0\rangle = |q_0 - x\rangle$ and $\langle y - q_0|q_0 - x\rangle = \delta(y-(2q_0-x))$ we arrive at

$$\langle\hat{W}(q_0,p_0)\rangle_\rho = \frac{1}{\pi\hbar}\int \mathrm{d}x\langle x \mid \hat{\rho} \mid 2q_0 - x\rangle e^{2ip_0(q_0-x)/\hbar} = W(q_0,p_0). \ \square \tag{5}$$

In the following instead of $\hat{q}$ and $\hat{p}$ we shall use the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, and the corresponding complex coordinates α and α^*. Then the displacement operator takes the form

$$\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}, \tag{6}$$

(the third formula is its normally ordered form) and we can write the Wigner operator as

$$\hat{W}(\alpha) = \frac{1}{\pi\hbar}\hat{D}(\alpha)\hat{P}_0\hat{D}^{-1}(\alpha). \tag{7}$$

We note that using the Wigner operator it is simple to verify the known fact that the Wigner function cannot be measured in the sense that its values at different points correspond to incompatible physical quantities. To see this, the commutator of two Wigner operators at different values of the argument has to be calculated. This equals zero if and only if the arguments are identical.

Any complete set of vectors, which have a definite parity with respect of the phase space point α, is an eigenstate basis of the Wigner operator $\hat{W}(\alpha)$. One such possibility is the set of the displaced number states:

$$\mid n,\alpha\rangle \equiv \hat{D}(\alpha) \mid n\rangle. \tag{8}$$

As it can be easily verified with the help of definition (7), these states are the eigenstates of the corresponding Wigner operator $\hat{W}(\alpha)$ with the eigenvalues $\frac{1}{\pi\hbar}$ or $\frac{-1}{\pi\hbar}$ according to whether n is even or odd, respectively.

In the following we shall need the expansion of the displaced number states in the number state basis. Using (6) we can write $\mid n,\alpha\rangle$ as follows:

$$\mid n,\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\frac{\sqrt{n!(n-j+k)!}}{j!k!(n-j)!}\alpha^k(-\alpha^*)^j \mid n-j+k\rangle. \tag{9}$$

The scalar product $\langle m \mid n,\alpha\rangle$ can be expressed with the associated Laguerre polynomials $L_n^{(a)}(x)$ [5]:

$$\langle m \mid n,\alpha\rangle = \begin{cases} e^{-|\alpha|^2/2}(-\alpha^*)^{n-m}\sqrt{\frac{m!}{n!}}L_m^{(n-m)}(|\alpha|^2) & \text{if } m < n, \\ e^{-|\alpha|^2/2}\alpha^{m-n}\sqrt{\frac{n!}{m!}}L_n^{(m-n)}(|\alpha|^2) & \text{if } m \geq n. \end{cases} \tag{10}$$

3. Trace-class extension and integral representation

The parity and therefore the Wigner operator are not trace-class operators:

$$\mathrm{Tr}(\hat{W}(\alpha)) = \frac{1}{\pi\hbar}\mathrm{Tr}(\hat{D}(\alpha)\hat{P}_0\hat{D}(-\alpha)) = \frac{1}{\pi\hbar}\mathrm{Tr}(\hat{P}_0) = \frac{1}{\pi\hbar}\sum_{n=0}^{\infty}(-1)^n, \tag{11}$$

which is clearly not convergent. We can however introduce trace-class operators, which can be brought arbitrarly close to the parity operator by changing a parameter.

One of the simplest possibilities to generalize $\sum_{n=0}^{\infty}(-1)^n$ to a convergent series is the substitution of -1 by a real number λ, with $|\lambda| < 1$. Then the sum of the series $\sum_{n=0}^{\infty}\lambda^n$ obtained this way is $\frac{1}{1-\lambda}$ and this tends to $\frac{1}{2}$ if λ goes to -1. The sum of the generalized series will not depend on λ, if we divide it by its sum. We include also a factor $\frac{1}{2}$ to be in accord with the limit mentioned above. Thus we would like to find some operators depending on the parameter λ which have the trace $\frac{1-\lambda}{2}\sum_{n=0}^{\infty}\lambda^n$. A possible choice is the operator $\hat{P}(\lambda)$:

$$\hat{P}(\lambda) \mid n\rangle = \frac{1-\lambda}{2}\lambda^n \mid n\rangle \tag{12}$$

defined by its effect on the elements of the number state basis.

With the help of $\hat{P}(\lambda)$ it is straightforward to generalize the Wigner operator as

$$\hat{W}(\alpha,\lambda) = \frac{1}{\pi\hbar}\hat{D}(\alpha)\hat{P}(\lambda)\hat{D}(-\alpha). \tag{13}$$

In the following we calculate the expectation value of the generalized Wigner operator in the state described by the density operator $\hat{\rho}$. Since the displaced number states (8) are also the eigenstates of the generalized Wigner operator, we expand the trace on the basis formed by them:

$$\langle\hat{W}(\alpha,\lambda)\rangle_\rho=\sum_{n=0}^{\infty}\langle n,\alpha\mid\hat{\rho}\hat{W}(\alpha,\lambda)\mid n,\alpha\rangle=\frac{1-\lambda}{2\pi\hbar}\sum_{n=0}^{\infty}\langle n,\alpha\mid\hat{\rho}\mid n,\alpha\rangle\lambda^n. \tag{14}$$

Due to our extension introduced in (13) the operator $\hat{W}(\alpha,\lambda)$ is also Hilbert-Schmidt. Therefore we can use the known fact that the displacement operators form a basis in the space of Hilbert-Schmidt operators (similarly to the Fourier–basis of the square integrable functions) [6], with the inversion formulae

$$\hat{F}=\frac{1}{\pi}\int d^2\xi\; f(\xi)\hat{D}(\xi),\qquad f(\xi)=\mathrm{Tr}(\hat{F}\hat{D}^{-1}(\xi)) \tag{15}$$

where f is square-integrable and its "Fourier–transform" $\hat{F}$ is Hilbert-Schmidt.

If we apply (15) to the generalized Wigner operator and expand the trace on the displaced number states (8), we find that

$$w(\xi,\alpha,\lambda)\equiv\mathrm{Tr}(\hat{D}^{-1}(\xi)\hat{W}(\alpha,\lambda))=\frac{1}{2\pi\hbar}e^{\alpha\xi^*-\alpha^*\xi}e^{\frac{1}{2}|\xi|^2\frac{\lambda+1}{\lambda-1}}. \tag{16}$$

Using (15) we can write the integral representation $\hat{W}(\alpha,\lambda)$ as

$$\hat{W}(\alpha,\lambda)=\frac{1}{2\pi^2\hbar}\int d^2\xi\;\hat{D}(\xi)e^{\frac{1}{2}\frac{\lambda+1}{\lambda-1}|\xi|^2+\alpha\xi^*-\alpha^*\xi}. \tag{17}$$

Taking the limit $\lambda\to-1+0$ in (17), the first term of the exponent vanishes, i.e. the Wigner operator is essentially the Fourier–transform of the displacement operators.

4. Quasiprobability density operators

By a suitable choice of the parameter λ we can introduce operators, which have similar relation with the P– and Q–function as the Wigner operator with the Wigner function. Thus the generalized Wigner operator can be thought as a quasiprobability density operator.

The P– and Q–functions represent the state of the system over the phase space in the case of a normally and antinormally ordered operator, respectively. They are the Fourier–transforms of the

$$\chi_N(\xi)=\mathrm{Tr}(\hat{\rho}e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}}),\qquad \chi_A(\xi)=\mathrm{Tr}(\hat{\rho}e^{-\xi^*\hat{a}}e^{\xi\hat{a}^\dagger}) \tag{18}$$

characteristic functions for normal and antinormal ordering, respectively [7].

Let us substitute $\lambda=0$ in (14), then there remains only the 0th term in the sum, all the others vanish:

$$\langle\hat{W}(\alpha,\lambda=0)\rangle_\rho=\frac{1}{2\pi\hbar}\langle 0\mid\hat{D}(-\alpha)\hat{\rho}\hat{D}(\alpha)\mid 0\rangle=\frac{1}{2\pi\hbar}\langle\alpha\mid\hat{\rho}\mid\alpha\rangle\equiv\frac{1}{2\hbar}Q(\alpha). \tag{19}$$

Thus, the expectation value of the generalized Wigner operator at $\lambda=0$ is proportional to the Q–function.

It is well known, that the P–function of a pure state is singular in general, and it does not exist as an ordinary function [8]. Thus we cannot expect, that in the convergence interval of (14) for λ we get the P–function. However, if we let $\lambda\to-\infty$, we obtain apart from the factor $\frac{1}{2\hbar}$ the P–function of the corresponding state. To see this, we evaluate the expectation value of the generalized Wigner operator using its integral representation (17). Taking into account (6) and (18), it is easy to show, that

$$\langle\hat{W}(\alpha,\lambda)\rangle_\rho=\frac{1}{2\pi^2\hbar}\int d^2\xi\;\chi_N(\xi)e^{\frac{1}{\lambda-1}|\xi|^2+\alpha\xi^*-\alpha^*\xi}. \tag{20}$$

If we let now $\lambda\to-\infty$, the first term of the exponent vanishes as $\lambda\to-\infty$, so there remains only the Fourier–transform of the characteristic function for normal ordering, which is the P–function by definition.

Finally we illustrate the above connections by calculating the expectation value of the generalized Wigner operator in the coherent state.

Let the state of an optical mode be the coherent state $\mid\beta\rangle$. In this case the density operator is $\hat{\rho}_\beta=\mid\beta\rangle\langle\beta\mid$, so the matrix element to be calculated in (14) is the following:

$$\langle n,\alpha\mid\hat{\rho}_\beta\mid n,\alpha\rangle=|\langle\beta\mid n,\alpha\rangle|^2=e^{-|\alpha-\beta|^2}\frac{|\alpha-\beta|^{2n}}{n!}. \tag{21}$$

After substitution into (14) we get:

$$\langle\hat{W}(\alpha,\lambda)\rangle_\beta=\frac{1-\lambda}{2\pi\hbar}e^{|\alpha-\beta|^2(\lambda-1)}. \tag{22}$$

It is readily seen, that in the limits of λ at 0 and -1 the $\langle\hat{W}(\alpha,\lambda)\rangle_\beta$ is the well known Q– and Wigner function of the coherent state, respectively, while as $\lambda\to-\infty$, $\langle\hat{W}(\alpha,\lambda)\rangle_\beta$ tends to $\delta(\alpha-\beta)$ which is the P–function of the coherent state multiplied by $\frac{1}{2\hbar}$.

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