acta physica slovaca vol. 45 No. 3, 323 – 330 June 1995

DISSIPATION AND AMPLIFICATION OF GENERALIZED GAUSSIAN STATES[1]

Gerhard Adam[2]
Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria

Received 27 April, accepted 10 May 1995

We study the damping and amplification of a single–mode radiation field. We define a generating function of this field and find its time–development in the case of damping and amplification. We apply these results to three different fields, i.e., to the Gaussian state field (GSF) (a field with Gaussian Wigner function) and to two generalizations of this GSF. The final result shows that the time–dependent solution for the density matrix elements is given by the same function as initially, with the only difference that the field parameters become time–dependent.

1. Master equations and their solutions

The dynamics of single–mode radiation field, interacting with a zero–temperature reservoir in the Born–Markov approximation can be described by the following master equation [1]

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H_0, \rho] + \frac{\gamma}{2}(2a\rho a^+ - a^+ a\rho - \rho a^+ a)\,, \tag{1}$$

where $H_0 = \hbar\omega(a^+ a + \frac{1}{2})$, ω is the resonant radiation frequency, and a and a^+ are the annihilation and creation operators and γ is the damping constant. Analogously, the master equation for the perfect amplifier is given by

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H_0, \rho] + \frac{\gamma}{2}(2a^+ \rho a - aa^+ \rho - \rho aa^+)\,. \tag{2}$$

The corresponding density–matrix elements in the Fock basis satisfy, in the case of damping, the differential equation

$$\begin{aligned}\frac{\partial}{\partial t}\langle m|\,\rho(t)\,|n\rangle &= -i\omega(m-n)\,\langle m|\,\rho(t)\,|n\rangle - \frac{\gamma}{2}(m+n)\,\langle m|\,\rho(t)\,|n\rangle \\ &\quad +\gamma\sqrt{(n+1)(m+1)}\,\langle m+1|\,\rho(t)\,|n+1\rangle\,.\end{aligned} \tag{3}$$

---

[1]Presented at the **3rd central-european workshop on quantum optics**, Budmerice castle, Slovakia, 28 April - 1 May, 1995

[2]E-mail address: gadam@ecxph.tuwien.ac.at

The analogous equation in the case of amplification reads as

$$\frac{\partial}{\partial t}\langle m|\hat{\rho}(t)|n\rangle = -i\omega(m-n)\langle m|\rho(t)|n\rangle - \frac{\gamma}{2}(m+n+2)\langle m|\rho(t)|n\rangle + \gamma\sqrt{nm}\,\langle m-1|\rho(t)|n-1\rangle\,. \qquad (4)$$

The above equations can be solved by using different methods and the solutions are well-known [2-5]. We would like to give a simplified treatment being similar to that of Arnoldus [6]. From eqs. (3) and (4), after some calculations, we obtain decoupled equations for normally and antinormally ordered moments in the case of damping and amplification, respectively

$$\frac{d}{dt}m^N_{k,l}(t) = [i\omega(k-l) - \frac{\gamma}{2}(k+l)]m^N_{k,l}(t)\,, \qquad (5)$$

$$\frac{d}{dt}m^A_{k,l}(t) = [i\omega(k-l) + \frac{\gamma}{2}(k+l)]m^A_{k,l}(t)\,, \qquad (6)$$

where the normally and antinormally ordered moments are given by

$$m^N_{k,l}(t) \equiv \langle (a^+)^k a^l\rangle(t) = Tr[(a^+)^k a^l \rho(t)] = \sum_{h=0}^{\infty}\frac{\sqrt{(k+h)!(l+h)!}}{h!}\langle l+h|\rho(t)|k+h\rangle\,, \qquad (7)$$

$$m^A_{k,l}(t) \equiv \langle a^l (a^+)^k\rangle(t) = Tr[a^l (a^+)^k \rho(t)] = \sum_{h=0}^{\infty}\frac{(k+l+h)!}{\sqrt{(k+h)!(l+h)!}}\langle l+h|\rho(t)|k+h\rangle\,. \qquad (8)$$

In both equations the moments for different k and l evolve independently. The solution for the damping is

$$m^N_{k,l}(t) = \Gamma^k_N \Gamma^{*l}_N m^N_{k,l}(0)\,, \qquad (9)$$

with

$$\Gamma_N(t) \equiv \exp\left[(i\omega - \tfrac{\gamma}{2})\,t\right]\,, \qquad (10)$$

and that for the amplification is

$$m^A_{k,l}(t) = \Gamma^k_A \Gamma^{*l}_A m^A_{k,l}(0)\,, \qquad (11)$$

with

$$\Gamma_A(t) \equiv \exp\left[(i\omega + \tfrac{\gamma}{2})\,t\right]\,. \qquad (12)$$

It is convenient to express the density matrix elements by using the generating function. By using the generating function [7]

$$G_s(\lambda_1,\lambda_2,\lambda_0;t) = Tr[\rho(t)\{\exp(\lambda_1 a^+ + \lambda_2 a - \lambda_0 a^+ a)\}_s] \qquad (13)$$

the density matrix elements can be expressed in terms of

$$\langle m|\rho(t)|n\rangle = Tr(\rho(t)|n\rangle\langle m|) = \frac{1}{\sqrt{m!\,n!}}\left(\frac{\partial}{\partial\lambda_1}\right)^n\left(\frac{\partial}{\partial\lambda_2}\right)^m G_N(\lambda_1,\lambda_2,\lambda_0=1;t)|_{\lambda_1=\lambda_2=0} \qquad (14)$$

and the ordered moments of annihilation and creation operators in terms of

$$m^s_{k,l}(t) = \langle\{(a^+)^k a^l\}_s\rangle(t) = Tr[\rho(t)\{(a^+)^k a^l\}_s] = \left(\frac{\partial}{\partial\lambda_1}\right)^k\left(\frac{\partial}{\partial\lambda_2}\right)^l G_s(\lambda_1,\lambda_2,\lambda_0=0;t)|_{\lambda_1=\lambda_2=0}\,. \qquad (15)$$

The subscripts $s=1$ (or N), $s=-1$ (or A) and $s=0$ (or W) in eqs. (13)–(15) denote the operator ordering: normal, antinormal and symmetrical (Weyl). $\{(a^+)^k a^l\}_W$ denotes the expectation value of the symmetrized form of the operator $(a^+)^k a^l$ [8]. This product of k creation operators and l annihilation operators can be ordered in $(k+l)!/(k!l!)$ ways. The symmetrically (Weyl) ordered product of these operators, denoted by $\{(a^+)^k a^l\}_W$, is just the average of all of these differently ordered products. Eq. (14) has been obtained by making use of the relation (see e.g. [1])

$$|n\rangle\langle m| = \lim_{\lambda_0\to 1}\frac{1}{\sqrt{m!\,n!}}\sum_{h=0}^{\infty}\frac{1}{h!}(-\lambda_0)^h (a^+)^{h+n} a^{h+m} \qquad (16)$$

and employing eq. (13).

From eqs. (13)–(15) we obtain directly the Taylor series expansion of the generating function

$$G_s(\lambda_1,\lambda_2,\lambda_0;t) = \sum_{j_1,j_2,j_0=0}^{\infty}\frac{\lambda_1^{j_1}\lambda_2^{j_2}(-\lambda_0)^{j_0}}{j_1!j_2!j_0!}m^s_{j_1+j_0,j_2+j_0}(t) \qquad (17)$$

or

$$G_N(\lambda_1,\lambda_2,\lambda_0;t) = \sum_{j_1,j_2,j_0=0}^{\infty}\frac{\lambda_1^{j_1}\lambda_2^{j_2}(1-\lambda_0)^{j_0}}{j_1!j_2!j_0!} \times\sqrt{(j_1+j_0)!\,(j_2+j_0)!}\,\langle j_2+j_0|\rho(t)|j_1+j_0\rangle\,. \qquad (18)$$

By employing eqs. (17) and (9) the relation for the damping follows:

$$G_N(\lambda_1,\lambda_2,\lambda_0;t) = G_N(\lambda_1',\lambda_2',\lambda_0';0)\,, \qquad (19)$$

$$\lambda_1'(t) = \Gamma_N(t)\lambda_1,\; \lambda_2'(t) = \Gamma^*_N(t)\lambda_2,\; \lambda_0'(t) = |\Gamma_N(t)|^2\lambda_0\,. \qquad (20)$$

Analogously, by using eq. (11) the relation for the amplification follows:

$$G_A(\lambda_1,\lambda_2,\lambda_0;t) = G_A(\lambda_1',\lambda_2',\lambda_0';0)\,, \qquad (21)$$

$$\lambda_1'(t) = \Gamma_A(t)\lambda_1,\; \lambda_2'(t) = \Gamma^*_A(t)\lambda_2,\; \lambda_0'(t) = |\Gamma_A(t)|^2\lambda_0\,. \qquad (22)$$

The results for the matrix elements in the case of damping read as

$$\langle m|\rho(t)|n\rangle = \Gamma^n_N(t)\Gamma^{*m}_N(t)\sum_{h=0}^{\infty}\frac{1}{h!}\sqrt{\frac{(m+h)!(n+h)!}{m!n!}} \times\left(1-|\Gamma_N(t)|^2\right)^h\langle m+h|\rho(0)|n+h\rangle\,, \qquad (23)$$

where we used eq. (19) along with eq. (14).

The treatment in the amplification case is similar, with the only difference that eq. (21) should be rewritten by using the normally ordered generating function G_N. For generating functions of the different order N and A the following relationship holds

$$G_N(\lambda_1,\lambda_2,\lambda_0;t) = \frac{1}{1-\lambda_0}\exp\left[-\lambda_1\lambda_2(1-\lambda_0)^{-1}\right] G_A(\Lambda_1,\Lambda_2,\Lambda_0;t)\,, \tag{24}$$

where

$$\Lambda_i = \frac{\lambda_i}{1-\lambda_0}, \quad i=0,1,2\,. \tag{25}$$

The above equation can be obtained quite analogously as in paper [9], in which the relationship between G_s and G_W has been found. By inserting eq. (21) into eq. (24) we obtain

$$G_N(\lambda_1,\lambda_2,\lambda_0;t) = \frac{1}{1-\lambda_0}\exp\left[-\lambda_1\lambda_2(1-\lambda_0)^{-1}\right] G_A(\Lambda_1',\Lambda_2',\Lambda_0';0)\,, \tag{26}$$

with

$$\Lambda_1' = \Gamma_A(t)\Lambda_1, \quad \Lambda_2' = \Gamma_A^*(t)\Lambda_2, \quad \Lambda_0' = |\Gamma_A(t)|^2\Lambda_0\,. \tag{27}$$

By using eq. (24) we can express $G_A(0)$ in eq. (26) by

$$G_A(\Lambda_1',\Lambda_2',\Lambda_0';0) = \frac{1}{1-\lambda_0'}\exp\left[\lambda_1'\lambda_2'(1-\lambda_0')^{-1}\right] G_N(\lambda_1',\lambda_2',\lambda_0';0)\,, \tag{28}$$

where λ_i' is defined by

$$\frac{\lambda_i'}{1-\lambda_0'} = \Lambda_i', \quad i=0,1,2\,. \tag{29}$$

Finally, we get

$$G_N(\lambda_1,\lambda_2\lambda_0;t) = \frac{1}{E}\exp\left[\lambda_1\lambda_2\left(|\Gamma_A(t)|^2-1\right)E^{-1}\right] G_N(\lambda_1',\lambda_2',\lambda_0';0)\,, \tag{30}$$

with

$$\lambda_1'(t) = \frac{\Gamma_A(t)}{E}\lambda_1,\ \lambda_2'(t) = \frac{\Gamma_A^*(t)}{E}\lambda_2,\ \lambda_0'(t) = \frac{|\Gamma_A(t)|^2}{E}\lambda_0,\ E\equiv 1-\left(1-|\Gamma_A(t)|^2\right)\lambda_0\,. \tag{31}$$

In obtaining eq. (31) we made use of eqs. (25), (27) and (29). By applying the Taylor expansion (18) on both sides of eq. (30) (and additionally expanding the exponantial function on the rhs. of eq. (30) as well), and using eq. (14) we obtain the known solutions for the density matrix elements in the case of the amplification

$$\begin{aligned}\langle m|\,\rho(t)\,|n\rangle \;&=\; \Gamma_A^{-m-1}(t)\Gamma_A^{*-n-1}(t)\sum_{h=0}^{\min(m,n)}\frac{1}{h!}\sqrt{\frac{m!n!}{(m-h)!(n-h)!}}\\ &\times\left(|\Gamma_A(t)|^2-1\right)^h\langle m-h|\,\rho(0)\,|n-h\rangle\,.\end{aligned} \tag{32}$$

2. Initial fields

We apply the results in the preceding section to different initial fields: The Gaussian state field (GSF) (i.e. single–mode radiation field with Gaussian Wigner function) and two different generalisations of the GSF. The initial fields will be defined in the following.

We can write the initial density operator of the GSF in the form [9]

$$\begin{aligned}\rho(0) \;&=\; \left(\tau^2-4|\mu_A|^2-\tfrac14\right)^{-\frac12}\exp\left\{-\frac{1}{\sqrt{\tau^2-4|\mu_A|^2}}\ln\left(\frac{\sqrt{\tau^2-4|\mu_A|^2}+\frac12}{\sqrt{\tau^2-4|\mu_A|^2}-\frac12}\right)\right.\\ &\times\left.\left[\mu_A(a-\alpha_0)^2+\mu_A^*(a^+-\alpha_0^*)^2+\tau(a^+-\alpha_0^*)(a-\alpha_0)+\frac{\tau}{2}\right]\right\}.\end{aligned} \tag{33}$$

The independent parameters of the GSF are given by the lower–order moments of annihilation and creation operators a and a^+ at the initial time

$$\langle a\rangle(0) = \alpha_0, \quad \langle a^2\rangle(0) = -2\mu_A^*+\alpha_0^2, \quad \langle a^+a\rangle(0) = \tau-\tfrac12+|\alpha_0|^2\,. \tag{34}$$

The initial states of the generalisation of the GSF are defined by the density operators [9]

$$\bar\rho(0) = \bar{\mathcal N}a^L\rho(0)(a^+)^L\,, \tag{35}$$

$$\tilde\rho(0) = \tilde{\mathcal N}(a^+)^M\rho(0)a^M\,, \tag{36}$$

where $\rho(0)$ is the initial density operator of the GSF and $\bar{\mathcal N}$ and $\tilde{\mathcal N}$ are the normalization constants.

The generating function of the initial GSF

$$G_s(\lambda_1,\lambda_2,\lambda_0;t=0) = \Phi(s,\lambda_0)\exp\left[y_1\lambda_1+y_2\lambda_2-\tfrac12\sum_{i,j=1}^{2}\lambda_i c_{ij}\lambda_j\right], \tag{37}$$

can be expressed as a sum over the Hermitian polynomials of two variables [9]:

$$G_s(\lambda_1,\lambda_2,\lambda_0;0) = \Phi(s,\lambda_0)\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{\lambda_1^{m_1}\lambda_2^{m_2}}{m_1!\,m_2!}H^{(c)}_{m_1,m_2}(x_1,x_2)\,. \tag{38}$$

The function $\Phi(s,\lambda_0)$ and the variables $c=(c_{ij})$, y_1, y_2, x_1 and x_2 are functions of the field parameters α_0,μ_A,τ, the ordering parameter s and λ_0.

By applying the derivatives (14) and (15) to the generating function (38) we obtain for the matrix elements and the ordered moments of the GSF the following results [9]:

$$\langle m\,|\rho(0)|\,n\rangle = \frac{1}{\sqrt{m!n!}}\Phi_0 H^{(\tilde c)}_{n,m}(\tilde x_1,\tilde x_2)\,, \tag{39}$$

$$\Phi_0 \equiv \frac{1}{\sqrt{(\tau+\frac12)^2-4|\mu_A|^2}}\exp\left[-\frac{\mu_A\alpha_0^2+\mu_A^*\alpha_0^{*2}+(\tau+\frac12)|\alpha_0|^2}{(\tau+\frac12)^2-4|\mu_A|^2}\right], \tag{40}$$

$$\begin{aligned} m^s_{k,l}(0) &= \langle\{(a^+)^k a^l\}_s\rangle(0) = H^{(c(s))}_{k,l}(x_1(s), x_2(s)) \\ &= \sum_{v=0}^{\left[\frac{k}{2}\right]}\sum_{w=0}^{\left[\frac{l}{2}\right]}\sum_{h=0}^{h_m} \frac{k!l!(-1)^{v+w}(\alpha_0^*)^{k-h-2v}(\mu_A)^v[\tau-\frac{s}{2}]^h(\mu_A^*)^w\alpha_0^{l-h-2w}}{(k-h-2v)!v!h!w!(l-h-2w)!}, \\ h_m &= \min(k-2v, l-2w). \end{aligned} \tag{41}$$

By using the moments of the GSF $m^s_{k,l}(0)$, the moments of the generalisation of the GSF (35) and (36) can be written as [9]

$$\bar{m}^N_{k,l}(0) = \frac{m^N_{k+L,l+L}(0)}{m^N_{L,L}(0)}, \tag{42}$$

$$\tilde{m}^A_{k,l}(0) = \frac{m^A_{k+M,l+M}(0)}{m^A_{M,M}(0)}. \tag{43}$$

Analogously, by using the density matrix elements of GSF the density matrix elements of the generalisation of the GSF [cf. (35) and (36)] take the form

$$\langle m|\,\bar{\rho}(0)\,|n\rangle = \sqrt{\frac{(m+L)!(n+L)!}{m!n!}}\,\frac{\langle m+L|\,\rho(0)\,|n+L\rangle}{m^N_{L,L}(0)}, \tag{44}$$

$$\begin{aligned} \langle m|\,\tilde{\rho}(0)\,|n\rangle &= \sqrt{\frac{m!n!}{(m-M)!(n-M)!}}\,\frac{\langle m-M|\,\rho(0)\,|n-M\rangle}{m^A_{M,M}(0)}, \quad m,n \ge M, \\ \langle m|\,\tilde{\rho}(0)\,|n\rangle &= 0, \quad m<M \ \text{ or } \ n<M. \end{aligned} \tag{45}$$

3. Dissipation in the case of different GSF's

The initial moments of the GSF (given by (41)) inserted into the solution for the moments (9) yield the simple result

$$m^N_{k,l}(t) = m^N_{k,l}(0)\Big|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{46}$$

where the time-independent parameter α_0, μ_A, τ appearing in eq. (41) are replaced by the corresponding time-dependent parameters

$$\alpha_0(t) = \Gamma^*_N(t)\alpha_0,\ \mu_A(t) = \Gamma^2_N(t)\mu_A,\ \tau(t) = |\Gamma_N(t)|^2\left(\tau-\tfrac{1}{2}\right)+\tfrac{1}{2}. \tag{47}$$

Analogously, the time-dependent density matrix-elements of the GSF can be written in the following form

$$\langle m|\,\rho(t)\,|n\rangle = \langle m|\,\rho(0)\,|n\rangle\Big|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}. \tag{48}$$

Analogously, for the generalisation of the GSF $\bar{\rho}$ it follows that

$$\bar{m}^N_{k,l}(t) = \frac{m^N_{k+L,l+L}(0)}{m^N_{L,L}(0)}\Bigg|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{49}$$

$$\langle m|\,\bar{\rho}(t)\,|n\rangle = \sqrt{\frac{(m+L)!(n+L)!}{m!n!}}\,\frac{\langle m+L|\,\rho(0)\,|n+L\rangle}{m^N_{L,L}(0)}\Bigg|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{50}$$

where the time-dependent parameters are again given by eqs. (47) and $\rho(0)$ is the density operator of the GSF.

4. Amplification in the case of different GSF's

In the case of the amplification by using eqs. (41) and (11) the time-dependent moments of the GSF can be obtained:

$$m^A_{k,l}(t) = m^A_{k,l}(0)\Big|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{51}$$

where

$$\alpha_0(t) = \Gamma^*_A(t)\alpha_0,\ \mu_A(t) = \Gamma^2_A(t)\mu_A,\ \tau(t) = |\Gamma_A(t)|^2\left(\tau+\tfrac{1}{2}\right)-\tfrac{1}{2} \tag{52}$$

have to be used. Further, the time-dependent matrix-elements of the GSF can be written in the following form

$$\langle m|\,\rho(t)\,|n\rangle = \langle m|\,\rho(0)\,|n\rangle\Big|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}. \tag{53}$$

Analogously, in the case of the amplification we get for the generalisation of the GSF $\tilde{\rho}$ the time-dependent moments and matrix-elements

$$\tilde{m}^A_{k,l}(t) = \frac{m^A_{k+M,l+M}(0)}{m^A_{M,M}(0)}\Bigg|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{54}$$

$$\langle m|\,\tilde{\rho}(t)\,|n\rangle = \sqrt{\frac{m!n!}{(m-M)!(n-M)!}}\,\frac{\langle m-M|\,\rho(0)\,|n-M\rangle}{m^A_{M,M}(0)}\Bigg|_{\substack{\alpha_0 \to \alpha_0(t) \\ \mu_A \to \mu_A(t) \\ \tau \to \tau(t)}}, \tag{55}$$

where the time-dependent parameters are given by eq. (53) and $\rho(0)$ is the density operator of the GSF.

5. Conclusions

We studied the dissipation and amplification of a single–mode radiation field. In both cases, dissipation and amplification, the time–development of the corresponding moments (normally or antinormally ordered) can be described by appropriate exponential functions. By using these results the time development of the generating function has been calculated for any initial field state for both above mentioned cases. These results have been applied for the description of the dynamics of the density matrix elements and photon distributions for the GSF and its two generalizations. The final result shows that the time–dependent solution for the density matrix elements is given by the same function as initially, with the only difference that the field parameters become time–dependent.

Acknowledgements This work was supported by the East–West Program of the Austrian Academy of Sciences under the contract No. 45.367/1-IV/6a/94 of the Österreichisches Bundesministerium für Wissenschaft und Forschung. In this connection I would like to thank Professor J. Seke for reading the manuscript and many useful comments.

References

[1] W.H. Louisell: *Quantum Statistical Properties of Radiation* (John Wiley, New York, 1973);

[2] P. Marian, T.A. Marian: *Phys. Rev. A* **47** (1993) 4487;

[3] S.M. Barnett, P.L. Knight: *Phys. Rev. A* **33** (1986) 2444;
S.J.D. Phoenix: *Phys. Rev. A* **41** (1990) 5132;
M. S. Kim: *J. Mod. Opt.* **7** (1993) 1331;

[4] V.Bužek, C.H. Keitel, P.L. Knight: *Phys. Rev. A* **51** (1995) 2575;
V.Bužek, M.S. Kim, Min Gyu Kim: *J. of Korean Phys. Soc.* **28** (1995) 123;

[5] V. Peřinova, A. Lukš, P. Szlachetka: *J. Mod. Opt.* **36** (1989) 1435;

[6] H. F. Arnoldus: *J. Mod. Opt.* **41** (1994) 503;

[7] J. Peřina: *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (D. Reidel Publishing Company, Dordrecht/Boston/Lancaster, 1984);

[8] G.S. Agarwal, E. Wolf: *Phys. Ref. D* **2** (1970) 2161; *ibid* 2187; *ibid* 2206;

[9] G. Adam: *Phys. Lett. A* **171** (1992) 66;
G. Adam: *J. Mod. Opt.*, in print;