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RECONSTRUCTION OF THE QUANTUM STATE OF AN N-MODE LIGHT FIELD BY MULTIPORT HOMODYNING[1]

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In any multiport homodyne detection the characteristic functions of the (joint) count distributions in the various output channels are proportional to the characteristic function of the quantum state of the multimode signal field in the input channels, provided that the local oscillators are sufficiently strong. To obtain the characteristic function of the quantum state of an N-mode signal for all values of its complex N arguments from the characteristic function of a joint count distribution, a one-to-one correspondence between $2N$ real variables in the two functions is required. This may be achieved, for example, by a succession of phase-controlled measurements of N-fold joint count distributions or by measuring a $2N$-fold joint count distribution in an extended scheme where N input channels are unused.

1. Introduction

Information on phase-sensitive light properties can be obtained by means of homodyne detection, where in the simplest case a signal-field mode whose properties are desired to be observed and a strong local-oscillator mode are combined by a beam splitter. Detecting the superimposed light, field strengths of the signal-field mode can be measured [1,2]. The effect of detection efficiencies has been analyzed [3] and various detection schemes have been considered [4]. A detailed quantum analysis of balanced homodyne four-port detection has been given with special emphasis on the properties of the local oscillator, the detection efficiencies, and the relation between the measured difference-count distributions and the field-strength distributions of the signal-field mode [5,6].

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In particular, the feasibility of reconstructing the quantum state of a signal mode from the data measured in homodyne detection has been of increasing interest [7-11]. Recently, experimental reconstruction of the quantum state of a radiation-field mode has been performed by using optical homodyne tomography [12]. The Wigner function of the signal mode has been reconstructed from the difference-count distributions measured in balanced homodyne four-port detection using inverse Radon transformation, and the density matrix of the field has been obtained as the Fourier transform of the Wigner function.

As we will see below, in any multiport homodyne detection the characteristic functions of the joint count distributions in the various output channels become proportional to the characteristic function of the quantum state of the multimode signal field in the input channels when the local oscillators are sufficiently strong. Hence, by appropriate parameter variation, the characteristic function of the quantum state of a multimode light field can directly be obtained from the characteristic functions of the joint count distributions measured in multiport homodyning.

2. The quantum state of light in terms of its characteristic function

Let us consider a (correlated) N-mode radiation field, with the creation and destruction operators $\hat{a}_j^\dagger$ and $\hat{a}_j$, respectively,

$$[\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}, \quad j, j' = 1, 2, \ldots, N. \tag{1}$$

Using standard quantum mechanics, the state of the field is described in terms of the density operator $\hat{\varrho}$. Equivalently, the quantum state can be described by its characteristic function [13]

$$\Phi(\{\alpha_j\}) = \mathrm{Tr}\{\hat{\varrho}\hat{D}(\{\alpha_j\})\} = \mathrm{Tr}\left\{\hat{\varrho}\exp\left[\sum_{j=1}^{N}\left(\alpha_j\hat{a}_j^\dagger - \alpha_j^*\hat{a}_j\right)\right]\right\}. \tag{2}$$

It is well known that $\Phi(\{\alpha_j\})$ and $\hat{\varrho}$ are uniquely related to each other, i.e., $\Phi(\{\alpha_j\})$ yields all knowable information on the quantum state of the N-mode radiation field (see, e.g., [13]). In particular, knowing the characteristic function $\Phi(\{\alpha_j\})$, the density operator in any representation can be obtained.

2.1. Coherent-state representations

Using the (non-orthogonal and overcomplete) coherent states

$$|\{\alpha_j\}\rangle = \prod_{j=1}^{N}|\alpha_j\rangle, \quad \hat{a}_j|\alpha_j\rangle = \alpha_j|\alpha_j\rangle, \tag{3}$$

an R representation can be defined by means of the operator expansion [14]

$$\hat{\varrho} = \frac{1}{\pi^{2N}}\int\{d^2\alpha_j\}\{d^2\beta_j\}\,R(\{\alpha_j^*\},\{\beta_j\})\,\exp\left[-\tfrac{1}{2}\sum_{j=1}^{N}(|\alpha_j|^2+|\beta_j|^2)\right]|\{\alpha_j\}\rangle\langle\{\beta_j\}|, \tag{4}$$

where

$$R(\{\alpha_j^*\},\{\beta_j\}) = \langle\{\alpha_j\}|\hat{\varrho}|\{\beta_j\}\rangle\exp\left[\tfrac{1}{2}\sum_{j=1}^{N}(|\alpha_j|^2+|\beta_j|^2)\right] \tag{5}$$

is related to the characteristic function $\Phi(\{\alpha_j\})$ as

$$R(\{\alpha_j^*\},\{\beta_j\}) = \exp\left[\tfrac{1}{2}\sum_{j=1}^{N}(|\alpha_j|^2+|\beta_j|^2+\beta_j^*\alpha_j-\beta_j\alpha_j^*)\right]$$
$$\times\frac{1}{\pi^N}\int\{d^2\gamma_j\}\,\Phi(\{\beta_j-\alpha_j+\gamma_j\})\exp\left\{-\tfrac{1}{2}\sum_{j=1}^{N}[|\gamma_j|^2-(\alpha_j+\beta_j)\gamma_j^*+(\alpha_j^*+\beta_j^*)\gamma_j]\right\}. \tag{6}$$

The coherent states are frequently used to define representations in terms of pseudodistributions that are formally similar to classical probability distributions. Expanding the density operator as [15]

$$\hat{\varrho} = \pi^N\int\{d^2\alpha_j\}\,P(\{\alpha_j\};s)\,\hat{\delta}(\{\alpha_j-\hat{a}_j\};-s), \tag{7}$$

where

$$\hat{\delta}(\{\alpha_j-\hat{a}_j\};s) = \frac{1}{\pi^{2N}}\int\{d^2\beta_j\}\,\hat{D}(\{\beta_j\})\exp\left[\sum_{j=1}^{N}(\tfrac{1}{2}s|\beta_j|^2+\alpha_j\beta_j^*-\alpha_j^*\beta_j)\right], \tag{8}$$

s-parametrized pseudodistributions

$$P(\{\alpha_j\};s) = \frac{1}{\pi^{2N}}\int\{d^2\beta_j\}\,\Phi(\{\beta_j\})\exp\left[\sum_{j=1}^{N}(\tfrac{1}{2}s|\beta_j|^2+\alpha_j\beta_j^*-\alpha_j^*\beta_j)\right] \tag{9}$$

are introduced [$P(\{\alpha_j\};1)\equiv P(\{\alpha_j\})$, P function; $P(\{\alpha_j\};0)\equiv W(\{\alpha_j\})$, Wigner function; $P(\{\alpha_j\};-1)=\pi^{-1}\langle\{\alpha_j\}|\hat{\varrho}|\{\alpha_j\}\rangle\equiv Q(\{\alpha_j\})$, Q function]. For $s>-1$ the distributions $P(\{\alpha_j\};s)$ do not necessarily exist as positive functions. Moreover, for $s>0$ they are not necessarily well-behaved. In particular, the P function of nonclassical light may be extremely singular.

The problems that may arise from singular behavior of the P function can be avoided by using generalized P representations [13,16], such as

$$\hat{\varrho} = \int\{d^2\alpha_j\}\{d^2\beta_j\}\,P(\{\alpha_j\},\{\beta_j\})\,\frac{|\{\alpha_j\}\rangle\langle\{\beta_j^*\}|}{\langle\{\beta_j^*\}|\{\alpha_j\}\rangle}. \tag{10}$$

Here, the pseudodistribution

$$P(\{\alpha_j\},\{\beta_j\}) = \frac{1}{(4\pi)^N}\exp\left[-\tfrac{1}{4}\sum_{j=1}^{N}|\alpha_j-\beta_j^*|^2\right]P[\{\tfrac{1}{2}(\alpha_j+\beta_j^*)\};-1] \tag{11}$$

always exists as a well-behaved positive function. Using Eq. (9), with $s=-1$, $P(\{\alpha_j\},\{\beta_j\})$ can easily be expressed in terms of the characteristic function $\Phi(\{\alpha_j\})$.

2.2 Orthonormal-basis representations

In the photon-number basis,

$$|\{n_j\}\rangle = \prod_{j=1}^{N} |n_j\rangle, \quad \hat{a}_j^\dagger \hat{a}_j\, |n_j\rangle = n_j\, |n_j\rangle, \tag{12}$$

the density operator is given by

$$\hat{\varrho} = \sum_{\{n_j\}} \sum_{\{m_j\}} \langle\{n_j\}|\hat{\varrho}|\{m_j\}\rangle\, |\{n_j\}\rangle\langle\{m_j\}| \tag{13}$$

Expressing the photon-number states in terms of the coherent states, $\langle\{n_j\}|\hat{\varrho}|\{m_j\}\rangle$ can be related to the characteristic function $\Phi(\{\alpha_j\})$ as

$$\langle\{n_j\}|\hat{\varrho}|\{m_j\}\rangle = \int \{\mathrm{d}^2\beta_j\}\, \Phi(\{\beta_j\})\; K(\{n_j\},\{m_j\},\{\beta_j\}), \tag{14}$$

where

$$K(\{n_j\},\{m_j\},\{\beta_j\}) = \frac{1}{\pi^N} \exp\!\Big(-\tfrac{1}{2}\sum_{j=1}^{N} |\beta_j|^2 \Big)$$
$$\times \prod_{j=1}^{N} \frac{1}{\sqrt{n_j! m_j!}} \frac{\partial^{n_j}}{\partial(\alpha_j^*)^{n_j}} \frac{\partial^{m_j}}{\partial\alpha_j^{m_j}} \exp\!\big(|\alpha_j|^2 + \alpha_j\,\beta_j^* - \alpha_j^*\,\beta_j\big)\Big|_{\alpha_j=\alpha_j^*=0} . \tag{15}$$

Finally, let us represent $\hat{\varrho}$ in the basis of phase-parametrized field-strengths [17]

$$\hat{F} = \sum_{j=1}^{N} \hat{F}_j(\varphi_{F_j}), \tag{16}$$

where

$$\hat{F}_j(\varphi_{F_j}) = F_j\,\hat{a}_j + F_j^*\,\hat{a}_j^\dagger, \quad F_j = |F_j|\,e^{i\varphi_{F_j}}. \tag{17}$$

Introducing the eigenkets of $\hat{F}$,

$$|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\rangle = \prod_{j=1}^{N} |\mathcal{F}_j,\varphi_{F_j}\rangle, \tag{18}$$

where the $|\mathcal{F}_j,\varphi_{F_j}\rangle$ are the eigenkets of the single-mode operators $\hat{F}_j(\varphi_{F_j})$,

$$\hat{F}_j(\varphi_{F_j})\,|\mathcal{F}_j,\varphi_{F_j}\rangle = \mathcal{F}_j\,|\mathcal{F}_j,\varphi_{F_j}\rangle, \tag{19}$$

$\hat{\varrho}$ may be represented in the form

$$\hat{\varrho} = \int \{\mathrm{d}\mathcal{F}_j\}\{\mathrm{d}\mathcal{F}'_j\}\, \langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}|\hat{\varrho}|\{\mathcal{F}'_j\},\{\varphi_{F_j}\}\rangle\, |\{\mathcal{F}'_j\},\{\varphi_{F_j}\}\rangle\langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}|, \tag{20}$$

where the density matrix elements can be obtained from the characteristic function $\Phi(\{\alpha_j\})$ as [9]

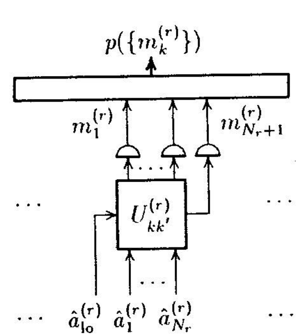
$$\langle\{\mathcal{F}_j^{(1)}+\mathcal{F}_j^{(2)}\},\{\varphi_{F_j}\}|\hat{\varrho}|\{\mathcal{F}_j^{(1)}-\mathcal{F}_j^{(2)}\},\{\varphi_{F_j}\}\rangle$$
$$= \frac{1}{(2\pi)^N} \int \{\mathrm{d}y_j\}\, \Phi\Big(\Big\{ -\frac{\mathcal{F}_j^{(2)} F_j^*}{|F_j|^2} + i y_j\, F_j^* \Big\}\Big) \exp\!\Big(-i \sum_{j=1}^{N} y_j \mathcal{F}_j^{(1)} \Big). \tag{21}$$

Note that the density-matrix elements of an N-mode field in a field-strength basis can already be obtained from the characteristic function $\Phi(\{\alpha_j\})$ by a single Fourier integral per mode, which favors the use of this representation. Further, it should be pointed out that when the field-strength distributions

$$p(\{\mathcal{F}_j\},\{\varphi_{F_j}\}) = \langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}|\hat{\varrho}|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\rangle \tag{22}$$

are known for all values of the phases φ_{F_j} within π intervals, then all knowable information on the quantum state of the field is available [18].

3. Multiport homodyne detection



The characteristic function $\Phi(\{\alpha_j\})$ of the quantum state of an N-mode signal field can directly be obtained from the characteristic functions of the count distributions measured in multiport homodyning. Let us suppose that the N modes of the field under consideration can be separated from each other so that each mode can be used as an input-signal mode in a multiport linear device. In particular, to detect a field that consists of modes of different frequencies ω_r ($r=1,\ldots,R$), we assume that the set of modes can be subdivided into R groups, $N=\sum_{r=1}^{R} N_r$, where the N_r modes belonging to the rth group are equal in frequency. We further assume that the coherent reference field consists of R modes of frequencies ω_r ($r = 1,\ldots,R$), so that each group of the signal-field modes can be assigned to a local oscillator (whose frequency is equal to the frequency of the modes of the group). This implies a detection scheme, where a linear lossless $2(N+R)$-port apparatus that consists of R subdevices is used (cf. the figure). The N_r signal modes of the rth group and the associated local-oscillator mode are combined by the rth subdevice to give N_r+1 output modes. The $N+R$ output modes of all the subdevices fall on photodetectors and the overall $(N+R)$-joint count distribution

$$p(\{m_k^{(r)}\}) \equiv p\Big(m_1^{(1)},\ldots,m_{N_1+1}^{(1)},\ldots,m_1^{(R)},\ldots,m_{N_R+1}^{(R)}\Big) \tag{23}$$

can be measured.

Since the rth subdevice is a linear lossless $2(N_r+1)$-port apparatus that transforms N_r+1 input modes into N_r+1 output modes, we may write

$$\hat{b}_k^{(r)} = \sum_{k'=1}^{N_r+1} U_{kk'}^{(r)} \hat{a}_{k'}^{(r)} \quad (r = 1,\ldots,R), \tag{24}$$

where $\hat{a}_k^{(r)}$ and $\hat{b}_k^{(r)}$ are the photon destruction operators of the input and output modes, respectively, and $U_{kk'}^{(r)}$ are unitary matrices,

$$\sum_{k''=1}^{N_r+1} U_{kk''}^{(r)} (U_{k'k''}^{(r)})^* = \sum_{k''=1}^{N_r+1} U_{k''k}^{(r)} (U_{k''k'}^{(r)})^* = \delta_{kk'} \tag{25}$$

($U_{k,k'}^{(r)} = |U_{k,k'}^{(r)}| \exp(i\varphi_{kk'}^{(r)}) \neq 0$). Note that any discrete finite-dimensional unitary matrix can be constructed in the laboratory using devices, such as beam splitters, phase shifters, and mirrors [19].

3.1 The characteristic functions of the joint count distributions

From photodetection theory [20] it is well known that the characteristic function of the joint count distribution $p(\{m_k^{(r)}\})$,

$$\Omega(\{x_k^{(r)}\}) = \sum_{\{m_k^{(r)}\}} p(\{m_k^{(r)}\}) \exp\left(i \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} x_k^{(r)} m_k^{(r)} \right), \tag{26}$$

is given by

$$\Omega(\{x_k^{(r)}\}) = \left\langle : \exp\left[\sum_{r=1}^{R} \sum_{k=1}^{N_r+1} \left(e^{ix_k^{(r)}} - 1 \right) \eta_k^{(r)} \hat{n}_k^{(r)} \right] : \right\rangle. \tag{27}$$

Here $\hat{n}_k^{(r)}$ are the photon-number operators of the output modes,

$$\hat{n}_k^{(r)} = (\hat{b}_k^{(r)})^\dagger \hat{b}_k^{(r)}, \tag{28}$$

and $\eta_k^{(r)}$ are the detection efficiencies. The notation : : indicates normal order.

We now assume, that the (N_r+1)th input mode in each subdevice $(r = 1,\ldots,R)$ is prepared in a coherent state $|\alpha^{(r)}\rangle$ $(\alpha^{(r)} = |\alpha^{(r)}| e^{i\varphi_o^{(r)}})$ with large value of $|\alpha^{(r)}|$ (strong local oscillators). To obtain the asymptotic behavior of the joint count distribution for large values of $|\alpha^{(r)}|$, it is convenient to introduce scaled counts

$$M_k^{(r)} = \frac{m_k^{(r)} - \eta_k^{(r)} |U_{k\,N_r+1}^{(r)}|^2 |\alpha^{(r)}|^2}{\eta_k^{(r)} |\alpha^{(r)}|} \tag{29}$$

($k = 1,\ldots,N_r+1$, $r = 1,\ldots,R$) and to consider the joint scaled count distribution $p_{\rm sc}(\{M_k^{(r)}\}) = p(\{\eta_k^{(r)}|\alpha^{(r)}|M_k^{(r)} + \eta_k^{(r)}|U_{k\,N_r+1}^{(r)}|^2|\alpha^{(r)}|^2\})$. Its characteristic function $\Omega_{\rm sc}(\{x_k^{(r)}\})$ is easily proved to be related to $\Omega(\{x_k^{(r)}\})$ as

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \Omega\left(\left\{ \frac{x_k^{(r)}}{\eta_k^{(r)}|\alpha^{(r)}|} \right\}\right) \exp\left\{ -i \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} |U_{k\,N_r+1}^{(r)}|^2 |\alpha^{(r)}| x_k^{(r)} \right\}. \tag{30}$$

Combining Eqs. (30) and (27) and following [4], we now expand in the resulting expression for $\Omega_{\rm sc}(\{x_k^{(r)}\})$ the exponential $\exp[ix_k^{(r)}/\eta_k^{(r)}|\alpha^{(r)}|]$ into a power series $(|x_k^{(r)}|/(\eta_k^{(r)}|\alpha^{(r)}|) \ll 1)$ and omit terms of higher than second order in $|x_k^{(r)}|/(\eta_k^{(r)}|\alpha_r|)$. After some calculation we obtain [21], on keeping only the leading terms,

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \exp\left\{ -\tfrac{1}{2} \sum_{r=1}^{R} \sum_{k',k''=1}^{N_r+1} \left[\sum_{k=1}^{N_r+1} \frac{U_{kk'}^{(r)} (U_{kk''}^{(r)})^*}{\eta_k^{(r)}} \right] z_{k'}^{(r)} (z_{k''}^{(r)})^* \right\}$$
$$\times \exp\left[\tfrac{1}{2} \sum_{r=1}^{R} \sum_{n=1}^{N_r} \left| z_n^{(r)} \right|^2 \right] \Phi\left(\{ ie^{i\varphi_o^{(r)}} z_n^{(r)} \} \right), \tag{31}$$

where

$$z_k^{(r)} = \sum_{k'=1}^{N_r+1} U_{k'\,N_r+1}^{(r)} (U_{k'k}^{(r)})^* x_{k'}^{(r)}. \tag{32}$$

In practice balanced homodyning is frequently preferred, because in this way the classical excess noise of the local oscillators may be eliminated. Introducing N scaled difference counts

$$D_{lk_r}^{(r)} = M_l^{(r)} - \frac{|U_{l\,N_r+1}^{(r)}|^2}{|U_{k_r\,N_r+1}^{(r)}|^2} M_{k_r}^{(r)} \tag{33}$$

($l = 1,\ldots,k_r-1, k_r+1,\ldots,N_r+1$), the characteristic function $\Omega_{\rm sdc}(\{x_l^{(r)}\})$ of the N-fold joint scaled difference-count distribution $p_{\rm sdc}(\{D_{lk_r}^{(r)}\})$ can simply be obtained by specification of Eqs. (31) and (32) [21]:

$$\Omega_{\rm sdc}(\{x_l^{(r)}\}) = \Omega_{\rm sc}(\{x_k^{(r)}\}) \quad \text{for} \quad z_{N_r+1}^{(r)} = 0. \tag{34}$$

We find that in any case the characteristic function of the count distribution is proportional to the characteristic function of the quantum state of the N-mode signal field. In particular, assuming (for simplicity) equal detection efficiencies $(\eta_k^{(r)} = \eta_{k'}^{(r')} \equiv \eta)$, from Eqs. (34) and (31) [together with the relations (25)] we see that

$$\Omega_{\rm sdc}(\{x_l^{(r)}\}) = \exp\left[-\frac{1-\eta}{2\eta} \sum_{r=1}^{R} \sum_{n=1}^{N_r} \left| z_n^{(r)} \right|^2 \right] \Phi\left(\{ ie^{i\varphi_o^{(r)}} z_n^{(r)} \} \right), \tag{35}$$

which reveals that in the case of perfect detection the characteristic function of the N-fold joint scaled difference-count distribution is, for appropriately chosen arguments, nothing else than the characteristic function of the quantum state of the N-mode signal field.

3.2 Reconstruction of the characteristic function of the quantum state of an N-mode signal field

To obtain the characteristic function of the quantum state of the signal field, we may invert Eqs. (35) [or (31)] and (32), on recalling the relations (25),

$$\Phi(\{\beta_n^{(r)}\}) = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left|\beta_n^{(r)}\right|^2\right]\Omega_{\mathrm{sdc}}(\{x_l^{(r)}\}), \tag{36}$$

$$x_l^{(r)} = \sum_{n=1}^{N_r}\left|\frac{U_{ln}^{(r)}}{U_{l\,N_r+1}^{(r)}}\beta_n^{(r)}\right|\sin\left(\varphi_{ln}^{(r)}-\varphi_{l\,N_r+1}^{(r)}+\varphi_{\beta_n}^{(r)}-\varphi_\alpha^{(r)}\right), \tag{37}$$

where, owing to the real quantities $x_l^{(r)}$, the allowed values of $\beta_n^{(r)}=|\beta_n^{(r)}|\exp(i\varphi_{\beta_n}^{(r)})= iz_n^{(r)}\exp(i\varphi_\alpha^{(r)})$ are restricted to those satisfying the conditions

$$\sum_{n=1}^{N_r}|U_{ln}^{(r)}\beta_n^{(r)}|\cos\left(\varphi_{ln}^{(r)}-\varphi_{l\,N_r+1}^{(r)}+\varphi_{\beta_n}^{(r)}-\varphi_\alpha^{(r)}\right)=0 \tag{38}$$

($l=1,\ldots,k_r-1,k_r+1,\ldots,N_r+1$; $r=1,\ldots,R$). In particular, the N equations (38) reveal that for given $N-1$ ratios $|\beta_n^{(r)}|/|\beta_{n'}^{(r)}|$ the N phases $\varphi_{\beta_n}^{(r)}$ cannot be chosen freely. To obtain $\Phi(\{\beta_n^{(r)}\})$ for arbitrary arguments, the $\beta_n^{(r)}$ must of course be allowed to attain arbitrary complex values. This can be achieved by appropriately varying the transformation matrices $U_{kk'}^{(r)}$, which implies (successive) measurement of a set of count distributions, for example, by using phase shifters in the apparatus and varying N phase parameters from measurement to measurement.

To give a simple example, let us assume that

$$U_{kk'}^{(r)}=\tilde{U}_{kk'}^{(r)}\exp\left(i\varphi_{k'}^{(r)}\right)\quad\left(\tilde{U}_{kk'}^{(r)}\ \text{real};\ \varphi_{N_r+1}^{(r)}=0\right). \tag{39}$$

and introduce the notation $\Omega_{\mathrm{sc}}(\{x_k^{(r)}\},\{\varphi_n^{(r)}\})$ to explicitly indicate the dependence on the $\varphi_n^{(r)}$ of the measured distributions and their characteristic functions. Making use of Eqs. (36) - (38), we easily obtain

$$\Phi(\{\beta_n^{(r)}\}) = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{N_r}|\beta_n^{(r)}|^2\right]$$
$$\times\,\Omega_{\mathrm{sdc}}\left(\left\{x_l^{(r)}=\sum_{n=1}^{N_r}\frac{\tilde{U}_{ln}^{(r)}}{\tilde{U}_{l\,N_r+1}^{(r)}}|\beta_n^{(r)}|\right\},\left\{\varphi_n^{(r)}=\pm\frac{\pi}{2}+\varphi_\alpha^{(r)}-\varphi_{\beta_n}^{(r)}\right\}\right), \tag{40}$$

Hence, phase-controlled measurement of the N-fold joint difference-count distributions for all values of the N phase parameters $\varphi_n^{(r)}$ within π intervals yields all knowable information on the quantum state of an N-mode signal field.

The $2N$-fold manifold of data necessary for a reconstruction of the quantum state of an N-mode signal field may also be obtained including into the detection scheme unused input ports, which of course introduce additional noise. For example, the characteristic function of the quantum state of the signal field can be reconstructed from the $2N$-fold joint difference-count distribution measured in a $2(2N+R)$-port detection scheme, with N input ports being unused (i.e., N_r input ports in the rth subdevice). Straightforward application of Eq. (36) to this case yields ($N_r\to 2N_r$)

$$\Phi(\{\beta_{n_s}^{(r)}\}) = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{2N_r}\left|\beta_n^{(r)}\right|^2\right]\exp\left[\tfrac{1}{2}\sum_{r=1}^{R}\sum_{n_v=N_r+1}^{2N_r}\left|\beta_{n_v}^{(r)}\right|^2\right]\Omega_{\mathrm{sdc}}(\{x_l^{(r)}\}) \tag{41}$$

($l=1,\ldots,k_r-1,k_r+1,\ldots,2N_r+1$; $n=1,\ldots,2N_r$; $n_s^{(r)}=1,\ldots,N_r$, signal inputs; $n_v^{(r)}=N_r+1,\ldots,2N_r$, vacuum inputs), and Eqs. (37) and (38) now read as

$$x_l^{(r)} = \sum_{n=1}^{2N_r}\left|\frac{U_{ln}^{(r)}}{U_{l\,2N_r+1}^{(r)}}\beta_n^{(r)}\right|\sin\left(\varphi_{ln}^{(r)}-\varphi_{l\,2N_r+1}^{(r)}+\varphi_{\beta_n}^{(r)}-\varphi_\alpha^{(r)}\right), \tag{42}$$

$$\sum_{n_s=1}^{N_r}|U_{ln_s}^{(r)}\beta_{n_s}^{(r)}|\cos\left(\varphi_{ln_s}^{(r)}-\varphi_{l\,2N_r+1}^{(r)}+\varphi_{\beta_{n_s}}^{(r)}-\varphi_\alpha^{(r)}\right)$$
$$=-\sum_{n_v=N_r+1}^{2N_r}|U_{ln_v}^{(r)}\beta_{n_v}^{(r)}|\cos\left(\varphi_{ln_v}^{(r)}-\varphi_{l\,2N_r+1}^{(r)}+\varphi_{\beta_{n_v}}^{(r)}-\varphi_\alpha^{(r)}\right). \tag{43}$$

We see that the values of the $|\beta_{n_s}^{(r)}|$ and $\varphi_{\beta_{n_s}}^{(r)}$ can indeed be chosen freely. The values of the $|\beta_{n_v}^{(r)}|$ and $\varphi_{\beta_{n_v}}^{(r)}$ are then determined by the conditions (43). Hence, any set of N complex values of the $\beta_{n_s}^{(r)}$ can be associated with $2N$ real values of the $x_l^{(r)}$. In other words, there is a one-to-one correspondence between the $2N$-fold joint difference-count distribution and the quantum state of an N-mode signal field in a detection scheme with N unused input channels. Note that other than vacuum reference fields could be used.

When the detection efficiencies are less than unity, the inverse Gaussians in Eqs. (36) and (41) obviously correspond to a deconvolution in the Fourier space, and a very careful consideration of the experimental inaccuracies is required.

4. Concluding remarks

Although at least one local oscillator per group of signal-field modes of equal frequencies is needed to reconstruct the quantum state of a multimode signal field, in practice it may be advantageous to increase the number of local oscillators. For example, introducing more than R subdevices of the type described, the dimensions of the transformation matrices $U^{(r)}_{kk'}$ are reduced. In particular, using N appropriate four-port subdevices and mixing in each subdevice a signal mode with a local-oscillator mode, the N-fold joint difference-count distributions measured is closely related to the joint field-strength distribution of the N-mode signal field.

A decomposition of the detection scheme into more than R subschemes may also be useful when the reconstruction of the quantum state of a signal field is desired to be performed through unused input ports as described in Sec. 3.2. For example, using N appropriate eight-port subdevices and mixing in each subdevice a signal mode with two vacuum modes and a local-oscillator mode, the $2N$-fold joint difference-count distribution measured is closely related to the Q function of the N-mode signal field.

Finally, it should be pointed out that the theory also applies when the numbers of local oscillators assigned to each subdevice is increased (and the dimensions of the transformation matrices $U^{(r)}_{kk'}$ are increased as well). Following Sec. 3., the characteristic functions of the joint count distributions and the quantum state of the signal field can be related to each other in a very similar way as is given there.

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