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ATTRACTOR STATES IN DISSIPATIVE QUANTUM OPTICAL SYSTEMS[1]

J. Steinbach[2], B.M. Garraway, P.L. Knight
Optics Section, The Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2BZ, United Kingdom.

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We show how the newly-developed quantum jump method for describing the dissipative evolution of individual quantum systems in an open environment can be employed to describe atomic fluorescence. This approach leads naturally to the idea of "attractor states" to which systems tend to evolve.

1. Master Equation and Quantum Jump Approaches to Dissipation

The traditional way of treating dissipative coupling between a small system and a large reservoir employs a linear Liouville master equation for the system reduced density operator [1]. The two major assumptions in this approach are first a Born Approximation and second a Markov Approximation. Having traced out the states of the reservoir the evolution of the system reduced density operator is governed by the master equation

$$\frac{d}{dt}\rho_S = \mathcal{L}\rho_S = \frac{i}{\hbar}[\rho_S, H_S] + \mathcal{L}_{\text{relax}}(\rho_S)\,, \qquad (1)$$

where

$$\mathcal{L}_{\text{relax}}(\rho_S) = -\frac{1}{2}\sum_m (C_m^\dagger C_m \rho_S + \rho_S C_m^\dagger C_m) + \sum_m C_m \rho_S C_m^\dagger\,, \qquad (2)$$

is a general form of relaxation Liouvillian. The Lindblad operators C_m [2] represent the effect of coupling the system to the reservoir.

In recent years an alternative way to describe the evolution of dissipative systems has been developed [3, 4, 5, 6, 9]. In contrast to the density operator master equation treatment of an ensemble, the dynamics of the dissipative system is described by a state vector. The need to employ a density operator to describe the dynamics of a dissipative system arises from the ignorance about the traced-out reservoir states. The density operator formalism allows for the classical probabilities incurred by this ignorance. It

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[2] Present Address: Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany.

is possible to remove these classical probabilities and describe the system with a state vector provided we have maximum information about the system. This information can be acquired by different measurement schemes [7] each of which leads to a different stochastic evolution of a state vector which is conditioned by the specific measurement. When a decay quantum is detected, the smooth evolution of the system is interrupted by a quantum jump as a result of the information gain from such a detection event [8].

A simple dissipative model system is the two-level system coupled to a continuum of field modes in the vacuum state. We can write a master equation (eq. (1)) with the relaxation operator

$$\mathcal{L}_{\text{relax}}(\rho_S) = -\frac{\gamma}{2}(\sigma^+\sigma^-\rho_S + \rho_S\sigma^+\sigma^-) + \gamma\sigma^-\rho_S\sigma^+ . \tag{3}$$

The system consists of a stable ground state $|g\rangle$ and an excited state $|e\rangle$ with lifetime γ^{-1} and with transition frequency ω_0. The decay constant γ describes the coupling to the vacuum. Because the reservoir is at zero temperature there is only one Lindblad operator $C = \sqrt{\gamma}\sigma^-$. The essence of the quantum jump method is the following. If the state of the system at time t is $|\Psi(t)\rangle$ there are two possible states at time $t+\delta t$:

(1) If a quantum jump occurs, the Lindblad operator projects $|\Psi(t)\rangle \longrightarrow |\Psi(t+\delta t)\rangle = \frac{1}{\mathcal{N}}\sigma^-|\Psi(t)\rangle$, with normalization $\mathcal{N} = (\langle\Psi(t)|\sigma^+\sigma^-|\Psi(t)\rangle)^{1/2}$. The probability δp for such a quantum jump to occur during δt is

$$\delta p = \gamma\delta t\,\langle\Psi(t)|\sigma^+\sigma^-|\Psi(t)\rangle . \tag{4}$$

(2) If there is no jump we have evolution with the non-Hermitian Hamiltonian $H_{\text{eff}} = H_S - i\hbar\gamma\sigma^+\sigma^-/2$. Because the evolution is non-unitary the state vector has to be renormalized, as

$$|\Psi(t)\rangle \longrightarrow |\Psi(t+\delta t)\rangle = \frac{1}{\mathcal{N}}\exp(-\frac{i}{\hbar}H_{\text{eff}}\delta t)|\Psi(t)\rangle , \tag{5}$$

with $\mathcal{N} = (\langle\Psi(t)|\exp(\frac{i}{\hbar}H^\dagger_{\text{eff}}\delta t)\exp(-\frac{i}{\hbar}H_{\text{eff}}\delta t)|\Psi(t)\rangle)^{1/2}$.

For sufficiently small δt, i.e. $|\lambda_j|\delta t/\hbar \ll 1$ (where λ_j are the eigenvalues of H_{eff}), this can be performed to first order in δt : $|\Psi(t)\rangle \longrightarrow |\Psi(t+\delta t)\rangle = \frac{1}{\mathcal{N}}(1-\frac{i}{\hbar}H_{\text{eff}}\delta t)|\Psi(t)\rangle$, where $\mathcal{N} = (\langle\Psi(t)|(1+\frac{i}{\hbar}H^\dagger_{\text{eff}}\delta t)(1-\frac{i}{\hbar}H_{\text{eff}}\delta t)|\Psi(t)\rangle)^{1/2} = (1-\gamma\delta t\,\langle\Psi(t)|\sigma^+\sigma^-|\Psi(t)\rangle)^{1/2} = (1-\delta p)^{1/2}$.

(3) Time is divided into discrete steps δt. At every timestep the probability for a quantum jump δp is evaluated and compared to a random number ε, uniformly distributed between 0 and 1. If δp is smaller than ε no jump occurs and the wave function is evolved with the non-Hermitian Hamiltonian (eq. (5)). On the other hand, if δp is bigger than ε a quantum jump occurs and the state of the system at time $t+\delta t$ is represented by the state vector projected with the Lindblad operator. Repeating this step, each time drawing a new random number ε, leads to a state vector evolution.

(4) To complete the procedure the state vector evolution has to be repeated many times before the average over a large number of evolutions describes an ensemble of dissipative systems, a result that satisfies the underlying master equation.

We have described elsewhere [10] how the first-order Euler time steps associated with the "unravelling" of the master equation above may be replaced by a higher-order unravelling of much greater accuracy.

As a result of the continuous detection, there are only two possible outcomes for the state vector at time $t+\delta t$ assuming the state of the system is known at time t :

$$|\Psi(t)\rangle \begin{cases} \nearrow |\Psi(t+\delta t)\rangle_{\text{no-jump}} \\ \searrow |\Psi(t+\delta t)\rangle_{\text{jump}} \end{cases} \tag{6}$$

The result for the density operator after the timestep δt is a mixed state, given by

$$\begin{aligned}\rho_S(t+\delta t) \quad &= \quad p_{\text{no-jump}}|\Psi(t+\delta t)\rangle_{\text{no-jump}}\langle\Psi(t+\delta t)|_{\text{no-jump}} \\ &\quad + p_{\text{jump}}|\Psi(t+\delta t)\rangle_{\text{jump}}\langle\Psi(t+\delta t)|_{\text{jump}} ,\end{aligned} \tag{7}$$

where the state vectors are normalized. The two coefficients p_{jump} and $p_{\text{no-jump}}$ are classical probabilities corresponding to the probability for detecting and not detecting the decay of a quantum respectively.

When observing the decay of a quantum the mixture reduces to a pure state. The new state vector then assigned to the small system is

$$|\Psi(t+\delta t)\rangle_{\text{jump}} = \frac{\sigma^-|\Psi(t)\rangle}{(\langle\Psi(t)|\sigma^+\sigma^-|\Psi(t)\rangle)^{1/2}} , \tag{8}$$

which can be intuitively understood as immediately after the decay of a quantum the two-level system has lost all of its energy and must be in the ground state. However, when no quantum is detected the mixture is reduced to the alternative state $|\Psi(t+\delta t)\rangle_{\text{no-jump}}$.

Integrating the master equation with a first-order Euler step results in a conventional density operator evolving from $\rho_S(t) = |\Psi(t)\rangle\langle\Psi(t)|$ to

$$\begin{aligned}\rho_S(t+\delta t) &= \rho_S(t) + \delta t\left(\frac{i}{\hbar}[\rho_S(t), H_S] + \mathcal{L}_{\text{relax}}(\rho_S)\right)\\ &= |\Psi(t)\rangle\langle\Psi(t)| + \frac{i}{\hbar}\delta t\,[|\Psi(t)\rangle\langle\Psi(t)|, H_S]\\ &\quad - \frac{\gamma}{2}\delta t\,\sigma^+\sigma^-|\Psi(t)\rangle\langle\Psi(t)|\\ &\quad - \frac{\gamma}{2}\delta t\,|\Psi(t)\rangle\langle\Psi(t)|\sigma^+\sigma^-\\ &\quad + \gamma\delta t\,\sigma^-|\Psi(t)\rangle\langle\Psi(t)|\sigma^+ .\end{aligned} \tag{9}$$

Indeed the two parts of the result that we expect from continuous detection (eq. (7)) can be identified with the different parts of the result given by evolution with the master equation (eq. (9)). In eq. (7) the term that corresponds to the case of a jump can be identified with the last term in the density operator (eq. (9)) that results from integrating the master equation with an Euler step. The coefficient of this term is then the probability for a quantum jump (denoted by δp in eq. (4)) to occur during the timestep δt :

$$p_{\text{jump}} = \gamma\delta t\,\langle\Psi(t)|\sigma^+\sigma^-|\Psi(t)\rangle . \tag{10}$$

The remaining terms in eq. (9) can be recovered by evolving the state vector with the non-Hermitian Hamiltonian (eq. (5))

$$H_{\text{eff}} = H_S - \frac{i\hbar}{2}\gamma\sigma^+\sigma^- . \tag{11}$$

The probability for no quantum jump to be detected during the time interval δt is given by the normalization of the state vector evolved with the non-Hermitian Hamiltonian:

$$p_{\text{no-jump}} = \langle\Psi(t)|\exp(\frac{i}{\hbar}H^{\dagger}_{\text{eff}}\delta t)\exp(-\frac{i}{\hbar}H_{\text{eff}}\delta t)|\Psi(t)\rangle = \mathcal{N}^2 . \tag{12}$$

The significance of the non-Hermitian evolution part in the Monte-Carlo state vector method is its connection to the gain of information from a no-detection result [3]. The non-Hermitian or no-detection evolution causes the state vector of the system slightly to rotate towards the ground state which reflects the gain of information from a no-detection result. Because no quantum was lost from the system it becomes more likely to have been in the ground state and hence the rotation. While no quantum jump occurs this continuously reduces the probability δp for a quantum jump. Once a photon is detected the system stops evolving in time. By setting up a differential equation for the probability for no quantum jump to occur up to time t, one finds that the probability for no quantum jump to occur at all is the absolute square of the ground state coefficient in the initial state as expected.

Spontaneous emission decay from a two-level system is easy to simulate using the procedure above and compare to an analytical solution. The master equation in a rotating frame reads:

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}_S &= \mathcal{L}_{\text{relax}}(\tilde{\rho}_S)\\ &= -\frac{\gamma}{2}(\sigma^+\sigma^-\tilde{\rho}_S + \tilde{\rho}_S\sigma^+\sigma^-) + \gamma\,\sigma^-\tilde{\rho}_S\sigma^+ .\end{aligned} \tag{13}$$

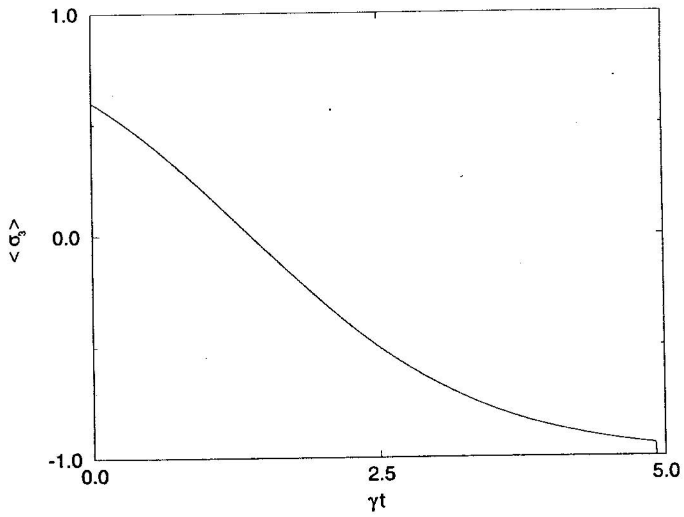

Figure 1: Expectation value of inversion $\langle\sigma_3\rangle$ for the time evolution of a single state vector in spontaneous emission from a two-level system. The initial state of the system is $|\Psi(0)\rangle = (1/\sqrt{5})\,(2|e\rangle + |g\rangle)$. ($\gamma = 0.5$).

The non-Hermitian evolution is governed by the effective Hamiltonian

$$H_{\text{eff}} = -\frac{i\hbar}{2}\gamma\,\sigma^+\sigma^- . \tag{14}$$

Fig 1.shows an example for the conditioned evolution of the inversion $\langle\sigma_3\rangle$ of a single state vector starting from the initial state $|\Psi(0)\rangle = (1/\sqrt{5})\,(2|e\rangle + |g\rangle)$. The rotation towards the ground state mentioned above can clearly be seen. Just before $\gamma t = 5$ a quantum is detected so that immediately afterwards the system must be in its ground state. The inversion "jumps" to a value of $\langle\sigma_3\rangle = -1$ and the system ceases to evolve. The result of an ensemble average using 10000 state vector evolutions is shown in Fig. 2 together with an analytical solution, showing that the Monte-Carlo state vector procedure gives the same result as the master equation.

2. Resonance Fluorescence from a Two-Level Atom

It is simple to extend the spontaneous emission problem to simulate resonance fluorescence. We add an interaction piece H_I to the system Hamiltonian

$$H_I = \hbar\,(ga\sigma^+ + g^* a^{\dagger}\sigma^-) \tag{15}$$

which describes the interaction of the driving laser field of frequency Ω with the two-level atom in dipole and rotating wave approximation [1]. The operators $a^{\dagger}$ and a are

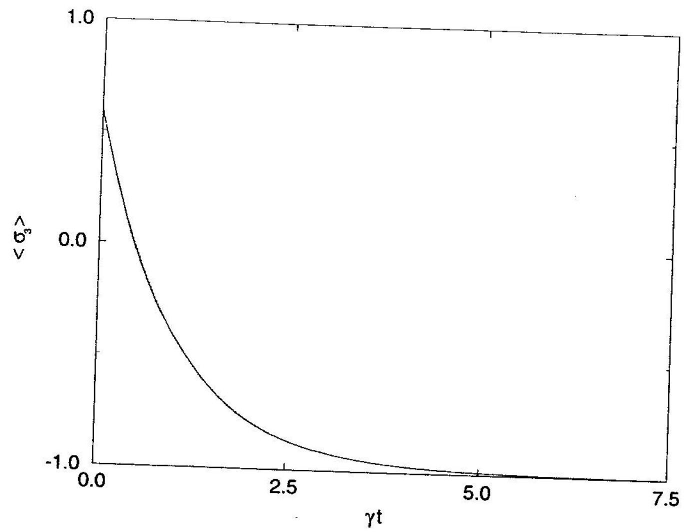

Figure 2: Energy decay from a two-level system via spontaneous emission. The value of inversion $\langle\sigma_3\rangle$ in the two-level system decays exponentially. The solid line shows an average over 10000 single evolutions. It is indistinguishable from the analytical result of an ensemble treatment (dashed line). Parameters are the same as for the single evolution shown in Fig. 1.

the usual creation and annihilation operators for the single mode driving field. The coupling constant $g = -\wp\mathcal{E}_0/2\hbar$ is half the vacuum Rabi frequency and incorporates the amplitude per photon $\mathcal{E}_0$ of the driving field and the atomic dipole matrix element $\wp$. The system is described in the rotating frame by

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S &= \frac{i}{\hbar}[\tilde{\rho}_S, \tilde{H}_S] + \mathcal{L}_{\text{relax}}(\tilde{\rho}_S)\,, \\ \tilde{H}_S &= \hbar\Delta\sigma^+\sigma^- + \hbar(ga\sigma^+ + g^* a^\dagger\sigma^-), \\ \mathcal{L}_{\text{relax}}(\tilde{\rho}_S) &= -\frac{\gamma}{2}(\sigma^+\sigma^-\tilde{\rho}_S + \tilde{\rho}_S\sigma^+\sigma^-) + \gamma\sigma^-\tilde{\rho}_S\sigma^+, \end{aligned} \tag{16}$$

where $\Delta = \omega_0 - \Omega$ is the detuning. If the driving field is treated classically the operators a and $a^\dagger$ become c-numbers, and the driving field amplitude E_0 is then included in the new coupling constant $\tilde{g} = -\wp E_0/2\hbar$.

The two distinct elements of the Monte-Carlo state vector method are firstly the smooth evolution under the influence of the non-Hermitian Hamiltonian

$$H_{\text{eff}} = \hbar\Delta\sigma^+\sigma^- + \hbar(\tilde{g}\sigma^+ + \tilde{g}^*\sigma^-) - \frac{i\hbar}{2}\gamma\sigma^+\sigma^-\,, \tag{17}$$

and secondly the stochastic influence of projections with the Lindblad operator $C = \sqrt{\gamma}\sigma^-$.

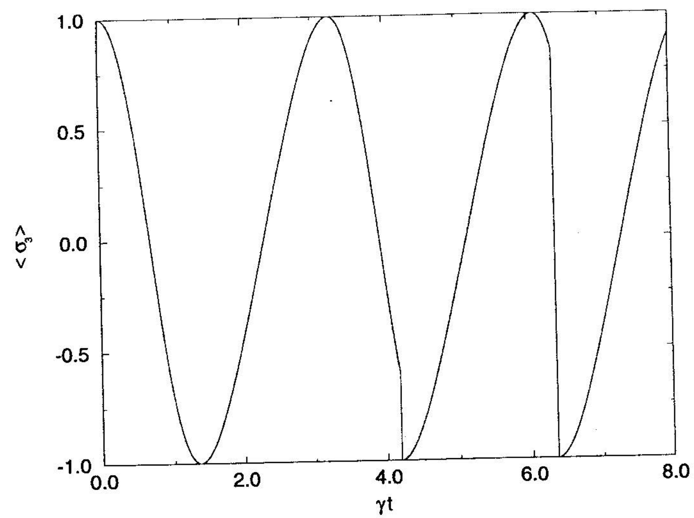

Figure 3: Monte-Carlo state vector evolution of a single two-level system showing the inversion $\langle\sigma_3\rangle$. The coherent driving leads to Rabi oscillations which are interrupted by spontaneous emission events. The system starts in the excited atomic state $|\Psi(0)\rangle = |e\rangle$. ($\Delta = 0$, $\tilde{g}/\gamma = 1$, $\gamma\delta t = 0.0008$).

A typical single trajectory evolution is shown in Fig. 3. Rabi oscillations in the inversion of the two-level system are interrupted by sudden "jumps" to an inversion value of $\langle\sigma_3\rangle = -1$, which lead to a phase shift in the Rabi oscillations. The jumps mark the random interruption of the smooth evolution by projections to the ground state. Fig 4. shows an average over many such evolutions together with an analytical solution of the master equation. Although continuous renormalization of the system state vector prevents the amplitude of the Rabi oscillations in Fig 3. from shrinking, the averaging process together with the stochastic sequence of phase shifts result in the stationary state. This underlines the significance of the projections included in the quantum jump method [11]. The analytical solution shown in Fig. 4 was obtained by solving eq. (16) for the case of zero detuning. In the atomic basis $|e\rangle, |g\rangle$, the density matrix of the two-level system is

$$\tilde{\rho}(t) = \begin{pmatrix} \tilde{\rho}_{11}(t) & \tilde{\rho}_{12}(t) \\ \tilde{\rho}_{21}(t) & \tilde{\rho}_{22}(t) \end{pmatrix} \tag{18}$$

and for the case of zero detuning $\tilde{H}_S$ becomes

$$\tilde{H}_S = \hbar \begin{pmatrix} 0 & \tilde{g} \\ \tilde{g}^* & 0 \end{pmatrix}. \tag{19}$$

The solution to eq. (16) for an arbitrary initial state $|\Psi(0)\rangle = \alpha|e\rangle + \beta|g\rangle$ in the param-

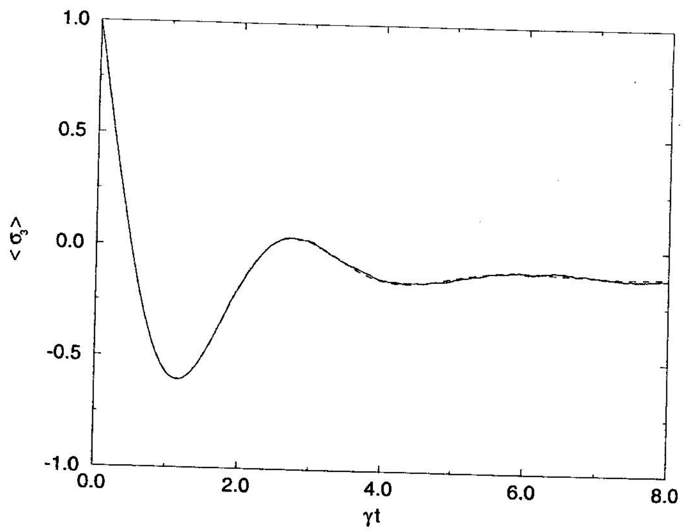

Figure 4: The single two-level atom evolves towards a stationary state. The master equation result (dashed line) is recovered when averaging over many single evolutions (Fig 3.) The sample average shown includes 10000 state vector evolutions (solid line). The visible difference between the Monte-Carlo state vector and the analytical result will decrease when increasing the number of realizations included in the sample. Parameters are the same as in Fig 3.

eter regime $\gamma < 8|\tilde{g}|$ then is:

$$\tilde{\rho}_{11}(t) = c_0 + e^{-3\gamma t/4}\left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right), \tag{20}$$

where $\omega = \sqrt{4|\tilde{g}|^2 - \gamma^2/16}$, and $c_0 = 2|\tilde{g}|^2/(4|\tilde{g}|^2 + \gamma^2/2)$, $c_1 = (-2\,\Im(\alpha\beta^*\tilde{g}^*) - \gamma(|\alpha|^2 + 3c_0)/4)/\omega$, $c_2 = |\alpha|^2 - c_0$, so that

$$\tilde{\rho}_{22}(t) = 1 - \tilde{\rho}_{11}(t), \tag{21}$$

$$\tilde{\rho}_{12}(t) = \alpha\beta^* e^{-\gamma t/2} + i\,\tilde{g}\left(c_3 + c_4 e^{-\gamma t/2} + e^{-3\gamma t/4}\left(c_5 \sin(\omega t) + c_6 \cos(\omega t)\right)\right), \tag{22}$$

$$\tilde{\rho}_{21}(t) = \tilde{\rho}^*_{12}(t), \tag{23}$$

where $c_3 = -\gamma/(4|\tilde{g}|^2 + \gamma^2/2)$, $c_4 = -\Im(\alpha\beta^*\tilde{g}^*)/|\tilde{g}|^2$, $c_5 = (2|\alpha|^2 - 1 + \Im(\alpha\beta^*\tilde{g}^*)\gamma/4|\tilde{g}|^2 + 3\gamma^2/(16|\tilde{g}|^2 + 2\gamma^2))/\omega$, and $c_6 = \Im(\alpha\beta^*\tilde{g}^*)/|\tilde{g}|^2 + \gamma/(4|g|^2 + \gamma^2/2)$.

A measurement which determines whether the atom is in the excited or in the ground state can yield either $|e\rangle$ or $|g\rangle$. The result changes from experiment to experiment. Although quantum mechanics predicts that $|e\rangle$ and $|g\rangle$ occur with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively, there remains an indeterminacy when performing a single measurement. These measurement fluctuations have been observed and called quantum projection noise in an experiment with single trapped ions [12]. In a purely damped system such as a two-level atom coupled to a dissipative reservoir in the vacuum state, the no-detection result leads to a continuous rotation of the state vector towards the ground state (Fig 1.) As a result, the probability $|\beta|^2$ for finding the system in its ground state increases and the quantum projection noise is reduced. This is true for the no-detection evolution of any initial state except for the excited state, $|\Psi(0)\rangle = |e\rangle$. In this particular case the state of the system remains unchanged. There is no indeterminacy and hence no information gain from a no-detection result. Because, apart from the excited state, all initial states evolve towards the ground state we will call the ground state an "attractor state" for the no-detection evolution. Even though the atom is not detected in the ground state it is more and more likely that a measurement yields the ground state. The probability to find the system in the ground state will become unity after an infinite time without the detection of a decay quantum.

The additional coherent driving field in a resonance fluorescence experiment can alter the state of the atom without the detection of decay quanta. The atom coherently exchanges photons with the driving field. Despite this fact the feature of an attractor state in the no-detection evolution is retained for the case of weak driving ($|\tilde{g}| < \gamma/4$) (Fig. 5). However, for strong driving ($|\tilde{g}| > \gamma/4$) the no-detection evolution changes from this relaxation to an oscillatory behaviour (Fig. 6). Both the specific superposition of $|e\rangle$ and $|g\rangle$ in the attractor state, and the switch between the weak and the strong driving regime are determined by the relative values of the decay rate γ and the coupling constant $\tilde{g}$.

The existence of an attractor state depends on whether there is a state of the system for which it is most likely not to detect decay quanta. For the purely damped system this state is obviously the ground state. If this system is in the ground state the probability for the no-detection result is always one. As stated in eq. (12), the general probability for no quantum jump to occur is given by the normalization of the state vector when it evolves with the non-Hermitian Hamiltonian H_{eff}. When determining this normalization for the states of the system which are eigenstates of the effective Hamiltonian it becomes clear why an attractor state only exists in the regime of weak driving. For the case of zero detuning, the effective Hamiltonian H_{eff} which governs the non-Hermitian evolution is

$$H_{\text{eff}} = \hbar(\tilde{g}\sigma^+ + \tilde{g}^*\sigma^-) - \frac{i\hbar}{2}\gamma\sigma^+\sigma^- . \tag{24}$$

The solutions to the eigenvector equation

$$H_{\text{eff}}|\Phi\rangle_j = \lambda_j|\Phi\rangle_j \tag{25}$$

depend on the relative size of the coupling constant $\tilde{g}$ and the decay rate γ.

In the weak driving regime ($|\tilde{g}| < \gamma/4$) these solutions are

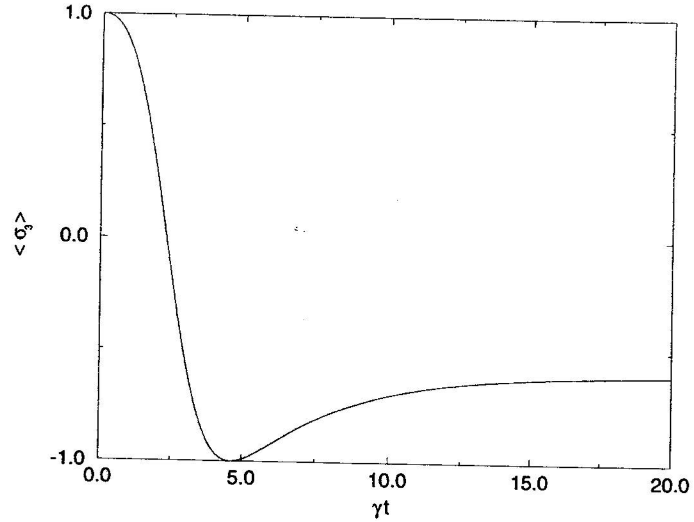

Figure 5: Single evolution of the two-level system inversion $\langle\sigma_3\rangle$ in the absence of detection events (no-detection evolution) in the weak driving regime ($|\tilde{g}| < \gamma/4$). The system starts in the excited atomic state $|\Psi(0)\rangle = |e\rangle$ and evolves towards a specific state, the attractor state. ($\Delta = 0$, $\tilde{g}/\gamma = 0.2$).

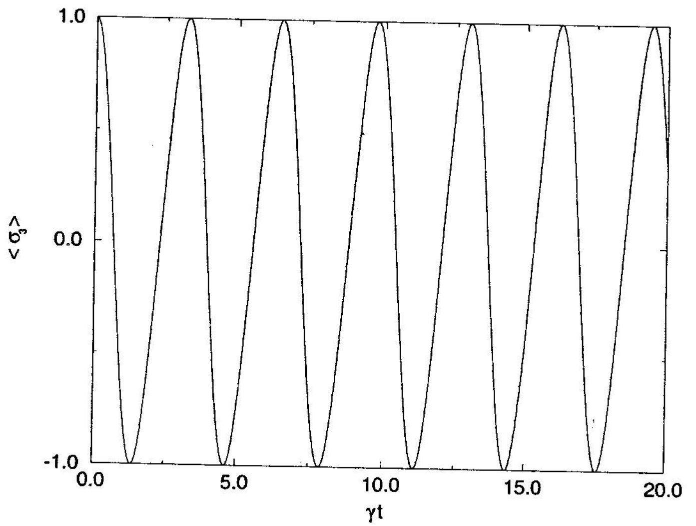

Figure 6: The same evolution as in Fig. 5 but in the strong driving regime ($|\tilde{g}| > \gamma/4$). There is no constant attracting state. ($\Delta = 0$, $\tilde{g}/\gamma = 1$).

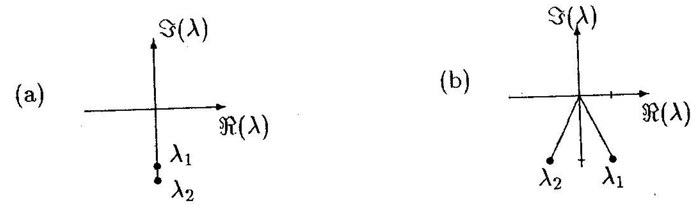

Figure 7: Position of the two eigenvalues λ_1, λ_2 of the non-Hermitian Hamiltonian $H_{\rm eff}$ (24) depicted in the complex plane. (a) shows λ_1, λ_2 in the weak and (b) in the strong driving regime. For weak driving ($|\tilde{g}| < \gamma/4$) the imaginary parts of λ_1 and λ_2 are different whereas for strong driving ($|\tilde{g}| > \gamma/4$) they become equal.

$$\lambda_1 = -i\frac{\hbar}{2}(\frac{\gamma}{2} - r) \qquad |\Phi\rangle_1 = \frac{1}{\mathcal{N}_1}\left(-i(\frac{\gamma}{4} - \frac{1}{2}r)|e\rangle + \tilde{g}^*|g\rangle\right),$$

$$\lambda_2 = -i\frac{\hbar}{2}(\frac{\gamma}{2} + r) \qquad |\Phi\rangle_2 = \frac{1}{\mathcal{N}_2}\left(-i(\frac{\gamma}{4} + \frac{1}{2}r)|e\rangle + \tilde{g}^*|g\rangle\right),$$

where

$$r = \sqrt{\gamma^2/4 - 4|\tilde{g}|^2} \tag{26}$$

and $\mathcal{N}_1$, $\mathcal{N}_2$ are the normalizations for the states $|\Phi\rangle_1$, $|\Phi\rangle_2$.

In the strong driving regime ($|\tilde{g}| > \gamma/4$) the solutions are

$$\lambda_1 = \frac{\hbar}{2}(R - i\frac{\gamma}{2}) \qquad |\Phi\rangle_1 = \frac{1}{\mathcal{N}_1}\left((\frac{1}{2}R - i\frac{\gamma}{4})|e\rangle + \tilde{g}^*|g\rangle\right),$$

$$\lambda_2 = -\frac{\hbar}{2}(R + i\frac{\gamma}{2}) \qquad |\Phi\rangle_2 = \frac{1}{\mathcal{N}_2}\left(-(\frac{1}{2}R + i\frac{\gamma}{4})|e\rangle + \tilde{g}^*|g\rangle\right);$$

where

$$R = \sqrt{4|\tilde{g}|^2 - \gamma^2/4} \tag{27}$$

and the normalizations $\mathcal{N}_1$, $\mathcal{N}_2$ are changed appropriately. The positions of the eigenvalues λ_1, λ_2 are indicated schematically in Fig. in the complex plane. When the system is in one of these eigenstates, $|\Psi\rangle = |\Phi\rangle_j$ ($j = 1, 2$), the no-detection evolution becomes

$$\begin{aligned}|\Psi(0)\rangle = |\Phi\rangle_j \longrightarrow |\Psi(t)\rangle &= \frac{1}{\mathcal{N}_j}\exp(-\frac{i}{\hbar}H_{\rm eff}t)|\Phi\rangle_j \\ &= \frac{1}{\mathcal{N}_j}\exp(-\frac{i}{\hbar}\lambda_j t)|\Phi\rangle_j,\end{aligned} \tag{28}$$

and the normalization $\mathcal{N}_j$ is

$$\mathcal{N}_j = \exp(\Im(\lambda_j)t/\hbar). \tag{29}$$

As the imaginary part of the eigenvalue $\Im(\lambda_j)$ is negative, the eigenvalue with the smallest absolute imaginary part $|\Im(\lambda_j)|$ is the one with the largest normalization $\mathcal{N}_j$. In the weak driving regime (eq. (26)) this eigenvalue is λ_1 and the corresponding eigenstate $|\Phi\rangle_1$ is the state with the largest normalization. So according to eq. (12), $|\Phi\rangle_1$ is the state for which a quantum jump is most unlikely. Apart from $|\Phi\rangle_2$ any initial state will evolve towards $|\Phi\rangle_1$ as long as this is not interrupted by a quantum jump. The reason for this is that any other state than $|\Phi\rangle_2$ must be a superposition of both states $|\Phi\rangle_1$ and $|\Phi\rangle_2$ which although they are not orthogonal form a basis since they are linearly independent. An arbitrary state evolves from

$$|\Psi\rangle = c_1|\Phi\rangle_1 + c_2|\Phi\rangle_2 \tag{30}$$

to

$$|\Psi(t)\rangle = \frac{1}{\mathcal{N}}\left(c_1 \exp(-\frac{i}{\hbar}\lambda_1 t)|\Phi\rangle_1 + c_2 \exp(-\frac{i}{\hbar}\lambda_2 t)|\Phi\rangle_2\right). \tag{31}$$

where $\mathcal{N}$ is the necessary normalization. We see that because λ_1 has a smaller absolute imaginary part than λ_2 ($\Im(\lambda_1) < \Im(\lambda_2)$) the component $|\Phi\rangle_1$ is weighted more than $|\Phi\rangle_2$ as long as $c_1 \neq 0$. So any initial state with $c_1 \neq 0$ will eventually evolve into the state $|\Phi\rangle_1$. For example, with the specific parameters in Fig. 5 this is

$$|\Phi\rangle_1 = (1/\sqrt{5})\,(-i|e\rangle + 2|g\rangle) \tag{32}$$

and there is a final value of inversion $\langle\sigma_3\rangle = -0.6$, corresponding to the stationary value in Fig. 5.

As the ratio $|\tilde{g}|/\gamma$ becomes close to 1/4 (from below) the imaginary parts of λ_1 and λ_2 become more and more equal. Finally, for values $|\tilde{g}| > \gamma/4$ the imaginary part of the eigenvalues λ_1 and λ_2 are equal. There is no specific state for which it is more likely that no quanta will be detected. In this regime it is impossible to associate a no-detection result with a specific state.

For a better physical understanding, it is helpful to separate the competing influences of driving field and dissipative coupling. A coherently driven two-level system is usually described with the picture of a Bloch vector $\vec{R}$.

$$\rho_S = R_1\sigma_1 + R_2\sigma_2 + R_3\sigma_3, \quad \vec{R} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}, \tag{33}$$

where σ_1, σ_2 and σ_3 are the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{34}$$

For the case of zero detuning, the Bloch equations are

$$\dot{\vec{R}} = \vec{R} \times \vec{S}, \quad \vec{S} = \begin{pmatrix} 2|\tilde{g}| \\ 0 \\ 0 \end{pmatrix}, \tag{35}$$

describing the precession of the Bloch vector around the vector $\vec{S}$ with a frequency $\nu_p = |\tilde{g}|/\pi$. The norm of the Bloch vector remains unchanged throughout this precession so that it moves on a sphere (the Bloch sphere). It takes a time $T_p = \pi/|\tilde{g}|$ for an atom initially in the excited state to coherently emit and absorb a photon from the driving field. If on the other hand the system is not driven but coupled to a vacuum reservoir then the no-detection evolution is governed by

$$H_{\rm eff} = \frac{1}{2}\hbar\omega_0\sigma_3 - \frac{i\hbar}{2}\gamma\sigma^+\sigma^- \tag{36}$$

and the system evolves over a timestep δt :

$$\begin{aligned} \alpha|e\rangle + \beta|g\rangle \quad \longrightarrow \quad & \alpha \exp(-\gamma\delta t/2)\exp(-i\omega_0\delta t/2)|e\rangle \\ & + \beta \exp(i\omega_0\delta t/2)|g\rangle. \end{aligned} \tag{37}$$

The information from a no-detection result is gained on a timescale $T_d = 2/\gamma$. If on this timescale the atom can exchange one photon with the driving field, i.e. if $T_d > T_p$ then there is no specific state of the system towards which it evolves as a result of no-detection. The vanishing of the attractor state beyond a specific ratio of $|\tilde{g}|/\gamma$ is abrupt. However, the possibility of measuring the atom in the ground state by not observing any decay quantum for an infinite time is lost for any driving $|\tilde{g}| > 0$. The reason is that although there is an attractor state in the weak driving regime, this state contains an indeterminacy with respect to measurements of excited or ground state.

We have discussed how dissipation leads to the idea of "attractor states" in atomic fluorescence. Whilst we have illustrated this with the simplest atomic system with just two levels, the concept is of use in describing multilevel transitions. We will describe elsewhere how attractor states govern the quantum random telegraph signal characteristic of three-level systems involving transitions either to a metastable, or a rapidly decaying, state.

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