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HAMILTONIAN FORMULATION OF DYNAMICAL SYSTEMS; FIRST-ORDER LAGRANGIANS AND CANONICAL VARIABLES[1]

Christofer Cronström[2]
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

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A review of the Hamiltonian formulation of dynamical systems with a finite number of degrees of freedom is given. The exceptional case in which the dynamics is defined by an action principle with a general first-order Lagrangian is considered in detail. In this case the usual procedure for obtaining a canonical Hamiltonian formulation fails. A canonical Hamiltonial formulation nevertheless exists under fairly general conditions, as shown by an explicit construction of the canonical variables and the Hamiltonian. This construction uses only elements of linear algebra and the theory of partial differential equations. As an illustrations, the formalism is applied to the quantization of a real self-dual field in 1 + 1 dimensions.

1. Introduction

The problem of obtaining a canonical Hamiltonian formulation of any given dynamical system is always of intrinsic interest, and also of interest in view of a possible quantization of the system in question. This is true regardless of whether one uses path-integral methods or canonical operator methods in the quantization procedure.

Here we will first discuss the general problem of obtaining a canonical Hamiltonian formulation of an "arbitrary" dynamical system with a finite number of degrees of freedom, and then consider the particular case in which the dynamics of the system is specified by a so-called first-order Lagrangian, i.e. a Lagrangian linear in the "velocities". In this case the standard construction of the Hamiltonian fails, but a Hamiltonian formulation is nevertheless possible under fairly general circumstances, as will be shown below.

The recent interest in systems described in terms of first-order Lagrangians was sparked by a paper of Floreanini and Jackiw [1] which deals with the quantization of a real self-dual (bosonic) field in a two-dimensional space-time. This paper gave rise to a lot of animated discussion in the literaure [2,3].

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[2] Permanent address: Department of Theoretical Physics, P. O. Box 9, FIN-00014 University of Helsinky, Finland, e-mail: cronstrom@phcu.helsinki.fi.

The whole issue was clarified later in a paper by Faddeev and Jackiw [4], who referred to classical theorems of Darboux (see e.g. Abraham and Marsden [5] for a proof of the relevant theorems) for the existence of the canonical variables in the case of systems defined by the first-order Lagrangians.

However, when dealing with specific models, it is frequently not enough to know that the model under consideration has a canonical structure; one really has to construct a set of canonical variables explicitly. Here I give an account of a recent constructive procedure [6] for obtaining canonical variables for a general first-order Lagrangian describing a system with a finite number of degrees of freedom, with variables which classically are commuting. The generalization to an infinite number of degrees of freedom or to classically anticommuting variables presents no essential difficulties in specific cases.

Recently there appeared a set of lectures by Jackiw [7] in which the construction of canonical variables in the case of a general first-order Lagrangian was carried out in a fairly explicit manner using the relevant Darboux theorem. This constructive procedure differs in many details from the one presented here, and may be regarded as complementary to the procedure presented in this paper, although the basic mathematical problems addressed in both approaches are the same. The lectures by Jackiw are warmly recommended to earnest students of the subject under discussion.

The "Hamiltonianization" of any system can trivially be obtained if one is willing to enlarge the state space of the system at will. This circumstance will also be discussed below.

In the last section the formalism developed here will be applied to the field-theory model involving a self-dual field 1 + 1 dimension, which was referred to above.

2. The Standard Route from Lagrangian to a Hamiltonian

Here I consider briefly the usual reduction of a Lagrangian system to a canonical Hamiltonian system. It is assumed that the dynamics of the system is defined by an action principle involving a general Lagrangian L,

$$S = \int dt L(q, \dot{q}). \tag{1}$$

Here the vectors q and $\dot{q}$ designate the coordinates and velocities, respectively, of the system,

$$q = (q^1, q^2, \ldots, q^N),\ \dot{q} = (\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^N). \tag{2}$$

One then defines the momenta p canonically conjugate to the coordinates q as follows,

$$p_i = \frac{\partial L}{\partial \dot{q}^i},\ i = 1, \ldots, N. \tag{3}$$

The Hamiltonian H is given by the following expression,

$$H(p, q) = \sum_{i=1}^{N} p_i \dot{q}^i - L \tag{4}$$

As the notation on the left hand side of Eq. (4) indicates, one assumes that the velocities $\dot{q}^i$ occuring in this equation can be expressed in the terms of the coordinates q and momenta p. That is, one assumes that one can solve the equations (3) for the velocities $\dot{q}$,

$$\dot{q}^i = \dot{q}^i(p, q) \tag{5}$$

This requires that the following condition should hold true,

$$det \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \neq 0. \tag{6}$$

If the condition (6) is not in force, one frequently uses the term "singular" to characterize the corresponding Lagrangian. If the Lagrangian is singular, in the sense just explained, then one cannot pass from Lagrangian to a Hamiltonian formalism in the straightforward manner described above. This is for instance the case if the Lagrangian is of first-order, i.e. linear in the "velocities" $\dot{q}$, because then the determinant (6) vanished identically. We consider this case in detail later on; for the time being assume that condition (6) is in force.

Then, taking into account the impliccit relation (5), one obtaines the following results using Eqns. (3) and (4),

$$\frac{\partial H}{\partial q^i} + \frac{\partial L}{\partial q^i} = 0 \tag{7}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \tag{8}$$

The equations (3), (4) and (7), (8) reveal the basic symmetry between the Lagrangian and the Hamiltonian formulation; the equations in question are invariant under the simultaneous interchanges

$$L \rightleftharpoons H,\ p \rightleftharpoons \dot{q}. \tag{9}$$

Under the assumption (6) the Lagrangian and Hamiltonian formulations are completely equivalent; given Hamiltonian $H(p, q)$ on can construct the corresponding Lagrangian L using Eq. (4). This of course requires that Eq. (8) can be solved for the momentum variables p_i,

$$p_i = p_i(q, \dot{q}) \tag{10}$$

i.e. that

$$det \left(\frac{\partial^2 H}{\partial p_i \partial p_j} \right) \neq 0. \tag{11}$$

However the validity of Eq. (11) is not an independent assumption; the determinants in Eqns. (6) and (11) are reciprocal numbers and therefore the conditions (11) is a consequence of the assumption (6).

3. Enlarged Phase-Space Hamiltonians

It is not true that every conceivable dynamical system is canonical Hamiltonian system, yet if one is willing to enlarge the phase space of the system at will, then one can, superficially at least, obtain a canonical formulation for any system, as pointed out e.g. by Arzanykh [8].

Consider the following system of ordinary differential equations,

$$\frac{dx^k}{dt} = Z^k(t, x^1, \ldots, x^n),\ k = 1, \ldots, n \tag{12}$$

where the Z^k are given functions of the arguments indicated. Introduce a set of new variables y_k and form the function

$$H = \sum_{k=1}^{n} y_k Z^k(t, x^1, \ldots, x^n) \tag{13}$$

so that the equations (12) can be written as follows,

$$\frac{dx^k}{dt} = \frac{\partial H}{\partial y_k},\ k = 1, \ldots, n \tag{14}$$

Determining the additional variables y_k by the following equations

$$\frac{dy_k}{dt} = -\frac{\partial H}{\partial x^k},\ k = 1, \ldots, n \tag{15}$$

one obtains a Hamiltonian system with H as a Hamiltonian and the variables x and y as canonical coordinates and momenta, respectively. However, it is clear that this is a rather artificial solution to the problem of whether a given system of equations is equivalent to a Hamiltonian system. It should also be noted that the Hamiltonian given by Eq. (13) is singular in the sense that the determinantal condition (11) is violated. Thus, one cannot without further analysis construct a variational principle involving a Lagrangian using the Hamiltonian (13) as a starting point.

4. First-order Lagrangians, Equations of Motion and Constraints

We now begin the discussion of our main subject, a system defined by a first-order Lagrangian, with a finite number of degrees of freedom. The state of the system is then described by some time-dependent N-component object $\xi(t) = (\xi^1(t), \ldots, \xi^N(t))$, say, in configuration space. The configuration space is thus an N-dimensional space, where N is a positive even or odd integer. The dynamics of the system is defined by the following action S,

$$S = \int dt \left[\sum_{A=1}^{N} \xi^A F_A(\xi) - G(\xi) \right]. \tag{16}$$

The integrand in Eq. (16) defines a first-order Lagrangian $\mathcal{L}^{(1)}$, which we write as a Lagrangian one-form,

$$dt\mathcal{L}^{(1)} = \sum_{A=1}^{N} d\xi^A F_A(\xi) - dtG(\xi). \tag{17}$$

The functions $F_A(\xi)$ and $G(\xi)$ occuring in Eqs. (16) and (17) above are given functions, which are assumed to be smooth enough, for the usual variational procedure connected with the action S to make sense. The variational equations obtained from Eq. (16) are the following

$$\sum_{B=1}^{N} M_{AB}\dot{\xi}^B - \frac{\partial G}{\partial \xi^A} = 0, \tag{18}$$

where

$$M_{AB}(\xi) := \frac{\partial F_B}{\partial \xi^A} - \frac{\partial F_A}{\partial \xi^B}. \tag{19}$$

In the terminology of classical mechanics [9] the equations of motion (18) are invariantly related to the one-form (17); it should be noted that the quantities $F_A(\xi)$ do not enter directly in the equations of motion (18) but only through the curl $M_{AB}(\xi)$ (19), which can be taken to define a two-form invariantly related to the one-form (17). Rather than using the language of forms, we consider the quantity $M_{AB}(\xi)$ as an antisymmetric tensor quantity; the crucial property of this quantity besides the antisymmetry,

$$M_{AB}(\xi) = -M_{BA}(\xi) \tag{20}$$

is the (Bianchi) identity,

$$\partial_A M_{BC}(\xi) + \partial_B M_{CA}(\xi) + \partial_C M_{AB}(\xi) = 0, \tag{21}$$

which follows straightforwardly from the definition (19).

The main question is now simply whether Eqs. (18) are canonical Hamiltonian equations, albeit perhaps in disguise, or whether they contain a subset which is Hamiltonian. As mentioned in the Introduction, this question has been answered by the affirmative under fairly general circumstances by Faddeev and Jackiw [4], who refer to the classical Darboux theorem [5] for the existence of canonical coordinates and momenta related to the system described by the action (16). In the following two sections we provide a constructive procedure for obtaining the canonical variables in question, analysing at the same time in detail the conditions under which the canonical structure actually emerges. We first consider the important special case in which quantity M_{AB} is *independent* of the variables $\xi(t)$.

5. The Case of a Constant M_{AB}

It is well known [10] that, by making an appropriate basis transformation, any antisymmetric $N \times N$ matrix (M_{AB}) can be transformed into the following normal form:

$$\begin{pmatrix} 0 & \lambda_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 0 & \lambda_n & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -\lambda_n & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{22}$$

where the (positive) quantities $\lambda_\alpha, \alpha = 1, \ldots, n$ are the square roots of those characteristic values of the matrix $(-M^2_{AB})$, which are different from zero. The number n of non-zero characteristic values λ^2_α is detemined by the rank $2n$ of the matrix (M_{AB}).

It is convenient to relate the normal form (22) to the following linear equations,

$$\sum_{B=1}^{N} M_{AB} x_{\alpha B} = -\lambda_\alpha y_{\alpha A}, \ \ \alpha = 1, \ldots, n \tag{23}$$

$$\sum_{B=1}^{N} M_{AB} y_{\alpha B} = +\lambda_\alpha x_{\alpha A}, \ \ \alpha = 1, \ldots, n \tag{24}$$

and

$$\sum_{B=1}^{N} M_{AB} z_{\beta B} = 0, \ \ \beta = 1, \ldots, N-2n \tag{25}$$

with the understanding that Eq. (25) is empty if the rank of the matrix (M_{AB}) is N (so that $N = 2n$), in which case the matrix M is *regular*,

$$det(M_{AB}) \neq 0. \tag{26}$$

Since the quantity M_{AB} (by assumption, in this Section) is independent of the state-space variable $\xi(t)$, the eigenvalue quantities λ_α as well as the vectors $x_{\alpha A}, y_{\alpha A}, \alpha = 1, \ldots, n$ and $z_{\beta A}, \beta = 1, \ldots, N-2n$ are independent of $\xi(t)$ as well, i.e. simply constant quantities.

In the general case Eqns. (25) may be nonempty, so that at least one non-trivial solution $z_{\beta B}$ exists. We ortho-normalize the vectors $x_{\alpha A}, y_{\alpha A}$ and $z_{\beta B}$, properly,

$$(x_\alpha, x_\beta) = \delta_{\alpha\beta}, (y_\alpha, y_\beta) = \delta_{\alpha\beta}, (z_\alpha, z_\beta) = \delta_{\alpha\beta}, \tag{27}$$

where the inner product (u, v) of any vectorlike quantities u_A and v_A is defined as follows,

$$(u, v) \equiv \sum_{A=1}^{N} u_A v_A. \tag{28}$$

The equations (23), (24) and (25) imply the following orthogonality relations,

$$\begin{aligned} (x_\alpha, y_\beta) &= 0, \alpha, \beta = 1, \ldots, n; \\ (x_\alpha, z_\beta) &= (y_\alpha, z_\beta) = 0, \alpha = 1, \ldots, n, \beta = 1, \ldots, N-2n \end{aligned} \tag{29}$$

We then introduce the following sets of new variables,

$$p_\alpha = \sum_{A=1}^{N} (\lambda_\alpha)^{\frac{1}{2}} \xi^A x_{\alpha A}, \ \ \alpha = 1, \ldots, n \tag{30}$$

$$q_\alpha = \sum_{A=1}^{N} (\lambda_\alpha)^{\frac{1}{2}} \xi^A y_{\alpha A}, \ \ \alpha = 1, \ldots, n \tag{31}$$

and

$$r_\beta = \sum_{A=1}^{N} \xi^A z_{\beta A}, \ \ \beta = 1, \ldots, N-2n \tag{32}$$

As the notation above indicates, and as will be demonstrated below, the quantities p_α and q_α play the role of (a set of) canonical momenta and coordinates, respectively, for the system of equations (18) with a ξ-independent M_{AB}, whereas the quantities r_β are related to constraits in this system.

Using the definitions (30), (31) and (32) one readily obtains the following result,

$$\frac{\partial G}{\partial \xi^A} = \sum_{\alpha=1}^{n} (\lambda_\alpha)^{\frac{1}{2}} \frac{\partial G}{\partial p_\alpha} x_{\alpha A} + \sum_{\alpha=1}^{n} (\lambda_\alpha)^{\frac{1}{2}} \frac{\partial G}{\partial q_\alpha} y_{\alpha A} + \sum_{\beta=1}^{N-2n} \frac{\partial G}{\partial r_\beta} z_{\beta A} \tag{33}$$

Contracting the equations of motion (18) with a solution $z_{\beta A}$ to the zero-eigenvalue equation (25) one obtains, using the orthogonality conditions (27), (29),

$$\sum_{A=1}^{N} z_{\beta A} \frac{\partial G}{\partial \xi^A} \equiv \frac{\partial G}{\partial r_\beta} = 0, \ \ \beta = 1, \ldots, N-2n \tag{34}$$

Thus, the existence of non-trivial solution $z_{\beta A}$ to the zero-eigenvalue equation (25) implies the constraints (34). These constraints are $N-2n$ in number, and leave thus in principle $2n$ genuine unconstrained dynamical variables to be determined by the equations of motion (18).

From Eq. (30) and the relation (24) follows that

$$\dot{p}_\alpha = \sum_{A,B} (\lambda_\alpha)^{-\frac{1}{2}} \dot{\xi}^A M_{AB} y_{\alpha B} = -\sum_{B} (\lambda_\alpha)^{-\frac{1}{2}} \frac{\partial G}{\partial \xi^B} y_{\alpha B} \tag{35}$$

where the last equality follows using the equations of motion (18). Finaly, using the result (33) and the orthogonality conditions (27), (29) one obtains,

$$\dot{p}_\alpha = -\frac{\partial G}{\partial q_\alpha} \tag{36}$$

Similary, one obtaines straightforwardly,

$$\dot{q}_\alpha = +\frac{\partial G}{\partial p_\alpha} \tag{37}$$

We have thus demonstrated that the original equations of motion (18) imply a set of constraints, namely the relations (34) (if the matrix M_{AB} has eigenvectors corresponding to zero eigenvalues), and the proper equations of motion (36) and (37), which are indeed in the canonical form, with the variables p_α and q_α acting as canonical momenta and coordinates, respectively. The canonical variables (30) and (31) introduced above (as well as the residual variables r_β) are of course not the only possible variables one can choose for the purpose of showing that the original equations of motion (18) can be put in the form of Hamiltons equations supplemented with constraints.

In the next section we show how to construct canonical variables when the quantity M_{AB} is an arbitrary (sufficiently smooth) function of the configuration space variables ξ.

6. The Canonical Variables in the General Case

6.1 The Constraints

As we have seen above, the existence of non-trivial solutions z_β to Eqs. (25) is related to constraints among the variables ξ^A which satisfy the equations (18). In the general case such solutions depend on ξ,

$$\sum_{A=1}^{N} M_{AB}(\xi) z_\beta^B(\xi) = 0,\ \beta = 1, \ldots, N-2n \tag{38}$$

The equation (38) certainly has at least one non-trivial solution if N is an odd integer, but may also have non-trivial solutions if N is even, in which case there must be an even number of such solutions.

Contracting the equations of motion (18) with a solution z_β of Eq. (38), one obtains

$$\sum_{A=1}^{N} z_\beta^A(\xi) \frac{\partial G(\xi)}{\partial \xi^A} = 0,\ \beta = 1, \ldots, N-2n. \tag{39}$$

But Eqs. (39) are genuine constraints; they cannot be a subset of a (unconstrained) Hamiltonian system, since they contain no derivatives with respect to time. In principle the constraints (39) can be solved at least locally in some appropriate region. This means that there exists a set of coordinates u_i, say, $i = 1, \ldots, 2n$, such that

$$\xi^A = \xi^A(u^1, \ldots, u^{2n}),\ A = 1, \ldots, N. \tag{40}$$

Needless to say, the mapping (40) is assumed to be sufficiently smooth. The proper equations of motion are then obtained from (18) by using expression

$$\dot{\xi}^B = \sum_{k=1}^{2n} \frac{\partial \xi^B}{\partial u^k} \dot{u}^k \tag{41}$$

and by multiplying Eq. (18) with $\partial \xi^A / \partial u^j$, then by summing over A,

$$\sum_{k=1}^{2n} m_{jk} \dot{u}^k = \frac{\partial G}{\partial u^j}, \tag{42}$$

where

$$m_{jk} = \sum_{A,B=1}^{N} \frac{\partial \xi^A}{\partial u^j} M_{AB} \frac{\partial \xi^B}{\partial u^k} \equiv \frac{\partial}{\partial u^j}\left(\sum_{B=1}^{N} F_B \frac{\partial \xi^B}{\partial u^k}\right) - \frac{\partial}{\partial u^k}\left(\sum_{B=1}^{N} F_B \frac{\partial \xi^B}{\partial u^j}\right). \tag{43}$$

The original equations of motion (18) including the constraints (39) have thus been replaced by a set of equations (42) in a reduced configuration space of exactly the same form as the original ones, but where the quantity (M_{AB}) is replaced by a quantity (m_{jk}), which, considered as a $2n \times 2n$ antisymmetric matrix is expected to be regular, i.e. of rank $2n$. The quantity (m_{jk}) also satisfies the appropriate Bianchi identity,

$$\frac{\partial}{\partial u^i} m_{jk} + \frac{\partial}{\partial u^j} m_{ki} + \frac{\partial}{\partial u^k} m_{ij} = 0. \tag{44}$$

6.2Construction of Canonical Variables

As shown in the previous subsection, one can always in principle eliminate possible constraints from equations of the form (18), and obtain a set of equations of the same form in a reduced configuration space. Such a reduced configuration space is necessarily *even-dimensional*, of dimension $2n$, say. Now we assume that the required reduction (if required) has taken place, and consider again the equations of motion (18), which we write down once more, denoting still the configuration space variables by ξ^A, with the appropriate range of indices $(A, B, \ldots = 1, \ldots, 2n)$,

$$\sum_{B=1}^{2n} M_{AB}(\xi) \dot{\xi}^B = \frac{\partial G(\xi)}{\partial \xi^A},\ A = 1, \ldots, 2n. \tag{45}$$

It is from now on *assumed* that the quantity M_{AB}, considered as an antisymmetric matrix is regular, i.e. of rank 2n, so that the matrix is invertible,

$$det(M_{AB}) \neq 0. \tag{46}$$

The inverse of M_{AB} is denoted by N^{AB},

$$\sum_{B=1}^{2n} M_{AB} N^{BC} = \delta_A^C \tag{47}$$

Then we have

$$\xi^B = \sum_{B=1}^{2n} N^{BA}(\xi) \frac{\partial G(\xi)}{\partial \xi^A},\ B = 1, \ldots, 2n. \tag{48}$$

We now analyse the circumstances under which Eqs. (48) are canonical Hamiltonian equations, with the function $G(\xi)$ acting as a Hamiltonian. We start by assuming the existence of canonical variables p_α, q^α, i.e. a one-to-one correspondence between the variables $\xi^A, A = 1, \ldots, 2n$, and pairs of variables $p_\alpha, q^\alpha, \alpha = 1, \ldots, n$:

$$\{\xi^A, A = 1, \ldots, 2n\} \longleftrightarrow \{p_\alpha, q^\alpha, \alpha = 1, \ldots, n\}. \tag{49}$$

It goes without saying that the correspondence (49) is always assumed to be smooth enough; the functions involved are always assumed to be at least twice continuously differentiable in the domain of validity of the correspondence (49). The canonical equations corresponding to Eqs. (48) are the following:

$$\dot{q}^\alpha = \frac{\partial H(p,q)}{\partial p_\alpha}, \; \dot{p}_\alpha = -\frac{\partial H(p,q)}{\partial q^\alpha}, \tag{50}$$

where

$$H(p,q) \equiv G(\xi(p,q)). \tag{51}$$

It is now a matter of a simple calculation to show that Eq. (45) and (50) are equivalent if and only if the following conditions hold true:

$$\sum_{B=1}^{2n} M_{AB}(\xi)\frac{\partial \xi^B}{\partial q^\alpha} = +\frac{\partial p_\alpha}{\partial \xi^A} \tag{52}$$

and

$$\sum_{B=1}^{2n} M_{AB}(\xi)\frac{\partial \xi^B}{\partial p_\alpha} = -\frac{\partial q^\alpha}{\partial \xi^A} \tag{53}$$

The relations (52) and (53) imply that,

$$M_{AB}(\xi) = \sum_{\alpha=1}^{n}\left(\frac{\partial p_\alpha}{\partial \xi^A}\frac{\partial q^\alpha}{\partial \xi^B} - \frac{\partial q^\alpha}{\partial \xi^A}\frac{\partial p_\alpha}{\partial \xi^B}\right) \tag{54}$$

The quantity ocurring on the right hand side of Eq. (54) is nothing but the *Lagrange bracket* of the quantities ξ^A and ξ^B. The relation (54) can be expressed in a perhaps more familiar form by considering the inverse N^{AB} of the quantity M_{AB} (compare Eq. (47)) i.e. the inverse of the Lagrange bracket (54), which is the familiar *Poisson bracket*,

$$N^{AB}(\xi) = \sum_{\alpha=1}^{n}\left(\frac{\partial \xi^A}{\partial q^\alpha}\frac{\partial \xi^B}{\partial p_\alpha} - \frac{\partial \xi^A}{\partial p_\alpha}\frac{\partial \xi^B}{\partial q^\alpha}\right) \equiv \{\xi^A, \xi^B\}_P. \tag{55}$$

Here and in follows, $\{u,v\}_P$ denotes the Poisson bracket of any two quantities u, v.

We can thus formulate the results so far as follows:

In order that Eqs. (45) may be equivalent to a set of canonical Hamiltonian equations, with the function $G(\xi)$ acting as the Hamiltonian, it is *necessary* that the matrix M_{AB} occuring in these equations be identified with the Lagrange bracket of the configuration space variables ξ in the equations in question. This condition can also be expressed in terms of the inverse matrix N^{AB}; this quantity is to be identified with the corresponding Poisson bracket.

We write the Lagrange bracket equation (54) in the equivalent form,

$$M_{AB}(\xi) = \frac{\partial}{\partial \xi^A}\left(\sum_{\alpha=1}^{N} p_\alpha \frac{\partial q^\alpha}{\partial \xi^B}\right) - \frac{\partial}{\partial \xi^B}\left(\sum_{\alpha=1}^{N} p_\alpha \frac{\partial q^\alpha}{\partial \xi^A}\right). \tag{56}$$

We recall that the quantity M_{AB} is always defined as a curl (e.g. equation (19) or (43)) in terms of given functions F_A. Thus, identifying the expressions (56) and (19) one obtains the following relation,

$$\sum_{\alpha=1}^{N} p_\alpha \frac{\partial q^\alpha}{\partial \xi^A} = F_A(\xi) + \frac{\partial \Phi}{\partial \xi^A} \equiv X_A(\xi), \tag{57}$$

where $\Phi(\xi)$ is an arbitrary function which is at our disposal.

So, the construction of canonical variables has finally boiled down to solving the equations (57) for the variables p_α and q^α for a given set of functions $F_A(\xi)$ and some appropriate choice of the function $\Phi(\xi)$.

It should be noted that the variables p_α and q^α appear in an unsymmetrical fashion in Eq. (57) despite their symmetric appearance in the basic condition (54). This is a result of the choice made in Eq. (56), which was one of two possible choices; one could equally well have switched the roles of p_α and q^α in Eq. (57). The choice described above, as well as the choice of the "arbitrary" function $\Phi(\xi)$ in the equations (57) are related to the possibility of making canonical transformations of the variables p_α and q^α.

The equations (57) define a straightforward mathematical problem, namely a problem of the type known as Pfaffs problem, for which various methods are available in the literature [11].

It then remains to solve the equations (57) for the variables p_α and q^α. We begin by defining a new quantity W^A by means of the following linear equations,

$$\sum_{B=1}^{2n} M_{AB}(\xi)W^B(\xi) = X_A(\xi), \tag{58}$$

where X_A is the known quantity in Eq. (57). Using the relation (54) in Eq. (58) above, we obtain the following,

$$\sum_{\alpha=1}^{n} p_\alpha \frac{\partial q^\alpha}{\partial \xi^A} = \sum_{B=1}^{2n}\sum_{\alpha=1}^{n}\left(\frac{\partial p_\alpha}{\partial \xi^A}\frac{\partial q^\alpha}{\partial \xi^B} - \frac{\partial q^\alpha}{\partial \xi^A}\frac{\partial p_\alpha}{\partial \xi^B}\right)W^B. \tag{59}$$

Defining further the quantities S^α and T_α as follows,

$$S^\alpha = -\sum_{B=1}^{2n} W^B \frac{\partial q^\alpha}{\partial \xi^B}, \; T_\alpha = \sum_{B=1}^{2n} W^B \frac{\partial p_\alpha}{\partial \xi^B} + p_\alpha, \tag{60}$$

one finds that Eqs. (59) are equivalent to the following set of $2n$ linear and homogeneous equations in the $2n$ unknowns S^α and T_α,

$$\sum_{\alpha=1}^{n}\left(\frac{\partial q^\alpha}{\partial \xi^A}T_\alpha + \frac{\partial p_\alpha}{\partial \xi^A}S^\alpha\right) = 0,\ A = 1,\ldots,2n. \tag{61}$$

We denote the determinant of Eq. (61) by D. This determinant is non-zero; it is fairly straightforward to show that

$$D^2 = det(M_{AB}) \neq 0. \tag{62}$$

Thus Eq. (61) has only zero solutions S^α and T_α so that

$$\sum_{B=1}^{2n} W^B \frac{\partial q^\alpha}{\partial \xi^B} = 0 \tag{63}$$

and

$$\sum_{B=1}^{2n} W^B \frac{\partial p_\alpha}{\partial \xi^B} + p_\alpha = 0. \tag{64}$$

Equations (63) and (64) are independent differential equations for the canonical coordinates and momenta, respectively, with known coefficients W^B. However, one should also require that these quantities have the appropriate Poisson brackets, i.e.

$$\{q^\alpha, q^\beta\}_P = 0,\ \alpha,\beta = 1,\ldots,n \tag{65}$$

$$\{p_\alpha, p_\beta\}_P = 0,\ \alpha,\beta = 1,\ldots,n \tag{66}$$

and

$$\{q^\alpha, p_\beta\}_P = \delta^\alpha_\beta,\ \alpha,\beta = 1,\ldots,n \tag{67}$$

Now, from Eq. (55) follows that the Poisson bracket of any two quantities u and v can be expressed as

$$\{u,v\}_P = \sum_{A,B=1}^{2n} N^{AB}(\xi)\frac{\partial u}{\partial \xi^A}\frac{\partial v}{\partial \xi^B}. \tag{68}$$

We can thus finally state the complete set of independent equations for the canonical variables. From Eqs. (58) and (63), it follows that the canonical coordinates q^α satisfy the set of n partial differential equations:

$$\sum_{A,B=1}^{2n} X_A(\xi)N^{AB}(\xi)\frac{\partial q^\alpha}{\partial \xi^B} = 0,\ \alpha = 1,\ldots,n. \tag{69}$$

Furthemore, from the conditions (65) the following set of $\frac{1}{2}n(n-1)$ equations is obtained:

$$\sum_{A,B=1}^{2n} N^{AB}(\xi)\frac{\partial q^\alpha}{\partial \xi^A}\frac{\partial q^\beta}{\partial \xi^B} = 0,\ \alpha,\beta = 1,\ldots,n. \tag{70}$$

The corresponding equations for the canonical momenta p_α are the following, according to Eqs. (58) and (64):

$$\sum_{A,B=1}^{2n} X_A N^{AB}(\xi)\frac{\partial p_\alpha}{\partial \xi^B} = p_\alpha,\ \alpha = 1,\ldots,n. \tag{71}$$

According to the conditions (66), (67) and (68), Eqs. (71) finally have to be completed by the following equations

$$\sum_{A,B=1}^{2n} N^{AB}(\xi)\frac{\partial p_\alpha}{\partial \xi^A}\frac{\partial p_\beta}{\partial \xi^B} = 0, \alpha,\beta = 1,\ldots,n \tag{72}$$

and

$$\sum_{A,B=1}^{2n} N^{AB}(\xi)\frac{\partial q^\alpha}{\partial \xi^A}\frac{\partial p_\beta}{\partial \xi^B} = \delta^\alpha_\beta,\ \alpha,\beta = 1,\ldots,n. \tag{73}$$

The construction of the canonical variables has then finally been reduced to a purely mathematical problem, namely to the solution of the groups of independent partial differential equations (69), (70), and (71)-(73), respectively. The existence of solutions of these equations is a standard question of regularity conditions for the known functions M_{AB} and F_A (as well as the function Φ, which is at our disposal), and need not be elaborated upon further.

I have recently become aware of the circumstance that the construction of canonical variables in a problem which resembles the one considered here, along similar lines to the construction above, has been presented in the literature [12].

In conclusion we may state our results as follows:

The set of equations (45) are canonical Hamiltonian equations, which can be formulated in terms of a set of canonical coordinates and momenta, respectively, provided that the inverse N^{AB} of the matrix M_{AB} occuring in Eq. (45) can be taken to define a Poisson bracket structure as given by Eq. (55). This question in turn is equivalent to that of the existence of solutions q^α and p_α to the partial differential equations (69), (70) and (71)-(73), respectively.

In any given problem, it may be as complicated to bother about the canonical structure in the manner discussed above, as to solve the problem directly, using whatever means seem appropriate. However, in cases where the details of the canonical structure are of importance, the above method can yield fruitful insights.

7. Application to Self-Dual Fields in Two Dimensions

We now consider a self-dual field χ in a two-dimensional space-time, with dynamics specified by the Lagrangian given by Floreanini and Jackiw [1], which was referred to in the Introduction. The action functional is the following,

$$S = \int dt \left\{\frac{1}{4}\int_{-L}^{+L}\int_{-L}^{+L} dxdy\dot{\chi}(x)\epsilon(x-y)\chi(y) - \frac{1}{2}\int_{-L}^{+L} dx\chi^2(x)\right\}, \tag{74}$$

where we follow the notation of Floreanini and Jackiw, with the exception that we initially restrict the (one-dimensional) space to a finite interval $(-L, L)$. The symbol ϵ above denotes the sign-function,

$$\epsilon(x) = \Theta(x) - \Theta(-x) \tag{75}$$

The time variable is suppressed in Eq. (74) and in what follows whenever expedient.

The variational equation following from extremizing the action (74) is the following,

$$\frac{1}{2}\int_{-L}^{L} dy\epsilon(x-y)\dot{\chi}(y) = \chi(x). \tag{76}$$

The boundary conditions for the field χ, which follow from Eq. (76), are as follows:

$$\chi(L) + \chi(-L) = 0. \tag{77}$$

The integrand in Eq. (74) defines a first-order Lagrangian $L^{(1)}$ analogous to the one considered previously. Using the field equation (76) one can immediately identify the analogue of the matrix M_{AB} considered in Sections 3 and 4,

$$M_{AB} \longleftrightarrow M_{xy} = \frac{1}{2}\epsilon(x-y),\ M_{xy}^{-1} = \delta'(x-y). \tag{78}$$

The field theory model considered here thus correspond to the case of a *constant* (i.e. χ- independent) antisymmetric quantity M, but still requires a slight generalization of the formalism developed previously; the discrete indices $(A, B, \ldots)$ simply have to be replaced by continuous ones $(x, y, \ldots)$ and the summation by an integration whenever appropriate. Otherwise most of the considerations in Section 5 can be taken over almost verbatim.

In order to construct the canonical variables corresponding to the system defined by the action (17) we have to solve the following pair of eigenvalue equations, which are analogous to Eqs. (23) and (24) (we change the notation slightly: $x_{\alpha A} \to f_\alpha(x), y_{\alpha A} \to g_\alpha(x)$),

$$\frac{1}{2}\int_{-L}^{L} dy\epsilon(x-y)f_\alpha(y) = -\lambda g_\alpha(x) \tag{79}$$

$$\frac{1}{2}\int_{-L}^{L} dy\epsilon(x-y)g_\alpha(y) = +\lambda_\alpha f_\alpha(x). \tag{80}$$

The orthonormalized solutions of Eqs. (79) and (80), which are consistent with boundary conditions (77), are the following,

$$f_\alpha(x) = \frac{1}{\sqrt{2L}}(\cos(k_\alpha x) + \sin(k_\alpha x)) \tag{81}$$

and

$$g_\alpha(x) = \frac{1}{\sqrt{2L}}(\cos(k_\alpha x) - \sin(k_\alpha x)) \tag{82}$$

with

$$\lambda_\alpha = k_\alpha^{-1},\ k_\alpha = (2\alpha - 1)\frac{\pi}{\sqrt{2L}}, \qquad \alpha = 1, 2, \ldots \tag{83}$$

The canonical momenta p_α and coordinates q_α are simply the following, according to Eqs. (30), (31) and the above results,

$$p_\alpha = \int_{-L}^{L} dx\sqrt{\lambda_\alpha}f_\alpha(x)\chi(x),\ q_\alpha = \int_{-L}^{L} dx\sqrt{\lambda_\alpha}g_\alpha(x)\chi(x). \tag{84}$$

The inverse formula expressing the field χ in terms of the canonical variables is,

$$\chi(x) = \sum_{\alpha=1}^{\infty}\sqrt{k_\alpha}(p_\alpha f_\alpha(x) + q_\alpha g_\alpha(x)). \tag{85}$$

The Hamiltonian G, which immediately can be read off from Eq. (74), is then expressible in terms of the canonical variables,

$$G = \frac{1}{2}\int_{-L}^{L} dx\chi^2(x) = \sum_{\alpha=1}^{\infty}\frac{k_\alpha}{2}(p_\alpha^2 + q_\alpha^2). \tag{86}$$

The canonical variables obey the Poisson algebra,

$$\{q_\alpha, p_\beta\}_P = \delta_{\alpha\beta},\ \{q_\alpha, q_\beta\}_P = 0,\ \{p_\alpha, p_\beta\}_P = 0. \tag{87}$$

It is a simple matter to obtain, e.g. the Poisson bracket of the field χ at different points in space,

$$\{\chi(x), \chi(y)\}_P = -\sum_{\alpha=1}^{\infty}\lambda_\alpha^{-1}(f_\alpha(x)g_\alpha(y) - f_\alpha(y)g_\alpha(x)) \equiv \delta'(x-y). \tag{88}$$

After all this machinery, it is easy to make a transition to the quantized version of the self-dual field making the usual replacement of Poisson brackets by $-i$ times commutators (with $h = 1$, of course). Denoting quantum operators by a caret, the quantum version of the Poison algebra is then the following,

$$[\hat{q}_\alpha, \hat{p}_\beta] = i\delta_{\alpha\beta},\ [\hat{q}_\alpha, \hat{q}_\beta] = 0,\ \ [\hat{p}_\alpha, \hat{p}_\beta] = 0. \tag{89}$$

Introducing new operators, $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$, by means of the following equations,

$$\hat{p}_\alpha = -\frac{1}{2}\left[(1+i)\hat{a}_\alpha + (1-i)\hat{a}_\alpha^\dagger\right],\ \hat{q}_\alpha = -\frac{i}{2}\left[(1+i)\hat{a}_\alpha - (1-i)\hat{a}_\alpha^\dagger\right], \tag{90}$$

one can summarize the commutator algebra (89) as follows,

$$[\hat{a}_\alpha, \hat{a}^\dagger_\alpha] = \delta_{\alpha\beta}. \quad (91)$$

The expansion of the field operator $\hat{\chi}$ at a fixed time ($t = 0$) can then be read off from Eq. (85),

$$\hat{\chi}(x) = \frac{-i}{\sqrt{2L}} \sum_{\alpha=1}^{\infty} \sqrt{k_\alpha} \left(\hat{a}_\alpha \exp(-ik_\alpha x) - \hat{a}^\dagger_\alpha \exp(+ik_\alpha x)\right). \quad (92)$$

Using the commutators (91) and the expansion (92) above, one immediately obtains the equaltime commutator,

$$[\hat{\chi}(x), \hat{\chi}(y)] = i\delta'(x-y), \quad (93)$$

which is the basic commutator given by Floreanini and Jackiw [1] and which is indeed the most natural commutator imaginable.

From Eq. (86) one finally gets the normal ordered quantum Hamiltonian $\hat{H}$

$$\hat{H} = \sum_{\alpha=1}^{\infty} k_\alpha (\hat{a}^\dagger_\alpha \hat{a}_\alpha). \quad (94)$$

We shall not further discuss the physical interpretation of the operator formalism of the self-dual field; this has been done (in the limit $L \to \infty$ with admirable clarity in the paper by Floreanini and Jackiw quoted previously.

8. Summary and Conclusions

In this paper I have given a systematic procedure for constructing unconstrained canonical variables for any system that is described by a first-order Lagrangian, which is linear in the velocities. The circumstances under which there are constraints in the system are analysed, and it is shown that the constraints can always be eliminated (in principle) in order to obtain an unconstrained system. The conditions under which the unconstrained system is canonical are derived explicitly; using the language of forms, one may state the conditions as follows: the Lagrange one-form leads to an exact two-form, which occurs in the equations of motion, and which is invariantly related to the Lagrangian one-form. The system is canonical if and only if the components of the two-form in question can be identified with a Lagrange bracket of the system.

The Poisson structure of the system is thus defined directly by the form of the Lagrangian; there is no freedom in choosing Poisson brackets (the inverse of the Lagrange brackets) for the phase-space variables of the system.

The main part of the analysis has for simplicity been done only for a system with a finite number of degrees of freedom in this paper. In this case, the Lagrange bracket condition reffered to above has been shown to lead to a set of independent partial differential equations for the canonical coordinates and momenta, respectively. These equations have coefficients that are determined partly by the Lagrangian and partly by

what is essentially a gauge choice. This freedom is in turn related to the possibility of making contact transformations among the canonical variables.

The fomalism developed here may be compared with the analysis of first-order Lagrangians which employes the Dirac theory of constraints [13]; an account of this kind of an analysis has recently been given by Govaerts [14].

The formalism of this paper has been applied here also to the case of self-dual fields in a two-dimensional space-time; this has required a slight generalization of the formalism to a system with an infinite number of degrees of freedom. This generalization is straightforward for the case of the self-dual field in 1 + 1 dimensions. The canonical structure of the self-dual field has been exhibited explicitly, and then used as a stepping stone for the quantization of the system in question. The results obtained are in agreement with those obtained previously by Floreanini and Jackiw [1].

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