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NONLINEAR DYNAMICS IN ONE AND TWO DIMENSIONAL ARRAYS OF DISCRETE JOSEPHSON ELEMENTS[1]

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Discrete arrays of Josephson junction elements differ from their continuum counterparts in two essential ways: i) localized dynamic states in discrete arrays, which are not present in the corresponding continuum system, can interact with other excitations that are present; ii) fluxoid quantization for the non-superconducting 'holes' provides a constraint for discrete arrays that is not present in the corresponding continuum system. The consequences of these effects in one-dimensional systems are now beginning to be understood; in two-dimesional systems, on the other hand, the picture is not yet altogether clear. Progress in fabrication technology and potential applications in practical electronic devices – as well as intrinsic interest in nonlinear dynamics – have contributed significantly to the growing interest in these systems.

I. INTRODUCTION

Progress in thin film and photolithographic technology has permitted the construction of large one- and two-dimensional planar arrays of Josephson tunnel junctions having rather precisely designed characteristics; simultaneously, the increasing availability of computing power has permitted large-scale simulations of such arrays, even using small desk-top machines. Planar Josephson junction arrays have attracted research interest both because they display a rich variety of complicated nonlinear behaviors and hence can serve as convenient model systems for studying, *e.g.*, the magnetic behavior of granular superconducting materials, properties of phase transitions in low-dimensional systems, the interplay between coherence, chaos, pattern formation *etc.*, in complex systems, and also because of current or potential applications in practical electronic devices, *e.g.*, voltage standards, logic circuits, and millimeter-wave oscillators and amplifiers for radio astronomy and space-borne receivers. Although much can be intuited about the behavior of discrete planar arrays from the behavior of the corresponding

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continuum systems, discreteness introduces a number of aspects into the dynamics of such arrays that have no counterpart in the continuum systems, a prime example being localized dynamic states.

II. ONE-DIMENSIONAL ARRAYS

A first step in the direction of understanding the dynamics of a one-dimensional *discrete* array may be obtained by considering the dynamics of the corresponding one-dimensional *continuum* structure. For a one-dimensional, overlap-geometry [1] continuum junction, the partial differential equation (PDE) that describes the dynamics of the system, in normalized form, is

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \beta\phi_{xxt} - \gamma\,, \tag{1}$$

with the boundary conditions

$$\phi_x(0,t) + \beta\phi_{xt}(0,t) = \phi_x(L,t) + \beta\phi_{xt}(L,t) = \eta\,. \tag{2}$$

In Eq. (1), ϕ is the quantum phase difference between the two superconducting electrodes of the junction, ϕ_t is the voltage, α is a dissipative term due to quasi-particle tunneling (normally assumed ohmic), β is a dissipative term due to surface resistance of the superconductors, γ is a normalized bias current, and x and t are normalized space and time, respectively. In Eq. (2), η is a normalized magnetic field applied in the plane of the junction, perpendicular to its long dimension, and L is the normalized junction length.

Perhaps the most significant feature of the system described by Eqs. (1-2) is the propagation of solitons – called "fluxons" in this context because they carry one quantum of magnetic flux – in a number of different dynamic configurations [2]; of particular interest are those configurations that give rise to the structures in the current-voltage characteristic of the junction known as zero-field steps, Fiske steps, and flux-flow steps [2].

A fairly standard approach for the numerical integration of Eqs. (1-2) involves spatial discretization, *i.e.*, replacing the spatial derivatives with finite-difference approximations. For example, replacing the first spatial derivative with a central-difference approximation and the second spatial derivative with a three-point approximation [3], and assuming an array of N points having a lattice spacing of a, yields directly (we note in passing that different approximations for the derivatives may change the form of these equations – especially Eqs. (3) and (5) – slightly)

at point 1:

$$\frac{dV_1}{dt} = \frac{2}{a^2}(\phi_2 - \phi_1) + \frac{2\beta}{a^2}(V_2 - V_1) - \sin\phi_1 - \alpha V_1 + \gamma - \frac{2\eta}{a}\,; \tag{3}$$

at point n, $2 \le n \le N-1$:

$$\frac{dV_n}{dt} = \frac{1}{a^2}(\phi_{n-1} - 2\phi_n + \phi_{n+1}) + \frac{\beta}{a^2}(V_{n-1} - 2V_n + V_{n+1}) - \sin\phi_n - \alpha V_n + \gamma\,; \tag{4}$$

at point N:

$$\frac{dV_N}{dt} = \frac{2}{a^2}(\phi_{N-1} - \phi_N) + \frac{2\beta}{a^2}(V_{N-1} - V_N) - \sin\phi_N - \alpha V_N + \gamma + \frac{2\eta}{a}\,; \tag{5}$$

at all points, $1 \le n \le N$:

$$\frac{d\phi_n}{dt} = V_n\,. \tag{6}$$

In this way we have obtained $2N$ first-order ordinary differential equations (ODEs) in $2N$ time-dependent variables (N phases and N voltages), which are just the Kirchhoff circuit-law equations for an array of N discrete Josephson junction elements interconnected via a parallel resistance/inductance combination.

Since Eqs. (3-6) have been obtained as an approximation to Eqs. (1-2), it is reasonable to ask to what extent the solutions of the ODE system will be a reasonable approximation to the solutions of the PDE system. This question was explored numerically some years ago by Currie *et al.* [4], who showed that discreteness effects are small as long as $a \ll 1$, but that as $a \to 1$, they become appreciable. Following through the normalizations employed in Eqs. (1-6), it turns out that for $a \sim 1$, the 'holes' in the discrete array are large enough to contain ~ 1 flux quantum, $h/2e$ (h is Planck's constant and e the electron charge). Thus, a fluxon propagating through a discrete array having $a \ll 1$ feels the discreteness only as a slight 'bumpiness in the road', whereas with $a \sim 1$ it can become trapped in the potential well that is formed between adjacent junctions, in the sense that it must acquire a certain minimum energy, the so-called Peierls-Nabarro barrier energy, in order to proceed.

The numerical work of Currie *et al.* [4] showed that the shape of a fluxon propagating through a discrete array is modulated by the discreteness, thus giving rise to the generation of small-amplitude oscillations. This mechanism was elucidated by Peyrard and Kruskal [5], who proposed an analysis based on linearizing the equations for these small oscillations and seeking stationary solutions of the linearized equations. Their analysis succeeded remarkably in accounting for many of the salient features of the observed numerical solutions of the full equations, in particular, the fact that a fluxon propagating at certain well-defined velocities generates very little radiation and thus propagates in a quasi-stationary manner, whereas at other velocities the radiation of small-amplitude oscillations is large, causing a rapid deceleration of the fluxon. The existence of special velocities for quasi-stationary propagation was given further numerical underpinning by the work of Duncan *et al.* [6], who also gave further confirmation to the suggestion of Peyrard and Kruskal [5] that this phenomenon is not peculiar to Josephson junction arrays, but is present in many nonlinear lattice systems.

The idea of linearizing the equation for the small oscillations radiated by a modulated fluxon was developed further by Ustinov *et al.* [7]. Their point of departure was the observation that the dispersion relation for small-amplitude linear waves in a discrete array is qualitatively different from that in the corresponding continuum system [8]. For example, for Eq. (1) (with dissipative and energy-input terms set to zero and assuming an infinite-length system), the dispersion relation is

$$\omega^2 = 1 + k^2\,, \tag{7}$$

where ω is the angular frequency and k is the wave number. For Eq. (4), instead, the dispersion relation is

$$\omega^2 = 1 + \frac{4}{a^2}\sin^2\left(\frac{ka}{2}\right), \tag{8}$$

which coincides with Eq. (7) only in the limit $a \to 0$. From Eq. (8), Ustinov *et al.* [7] calculated the phase velocity, ω/k, for small-amplitude oscillations and found conditions for resonant interactions with a propagating fluxon. Such superradiant, *i.e.*, phase-locked, interactions give rise to a series of sub-steps in the zero-field step that would be present in the case of simple fluxon propagation. Although the analysis of Ustinov *et al.* [7] was performed for an annular-geometry array, *i.e.*, an array with periodic boundary conditions in place of the finite boundary conditions of Eqs. (3) and (5), studies by Costabile and Sabatino and by Rotoli [9] showed that similar effects are present in arrays with finite boundary conditions, even though the phenomenon is rendered somewhat more complicated by the effects of reflections from the ends of the array.

As mentioned in the Introduction, much of the impetus for the study of Josephson junction arrays has come from various practical electronic applications. Although I cannot here delve deeply into this topic, perhaps it would be appropriate to mention briefly a few a these.

One of the most firmly established applications is that of the Josephson voltage standard, which is based on the fact that the relation between the frequency of a microwave signal applied to a Josephson junction and the resulting voltage that appears across its electrodes depends only on the fundamental quantity, $2e/h$; this provides the possibility of generating voltages that are known to an extremely high degree of precision. However, the voltages that can be obtained using a single junction are on the order of several millivolts, rather lower than the values of 1–10 V which would be most convenient for laboratory use. For this reason, people have for a number of years moved in the direction of employing series-biased arrays of thousands of junctions, an idea which is practicable if and only if all of the individual junctions in the array can be coherently phase locked to a single microwave source. The state of the art in this area has recently been reviewed by Niemeyer [10].

The Rapid Single Flux Quantum (RSFQ) family of logic devices has begun to attract a growing level of research attention in recent years because it offers substantial promise for the realization of computing and other digital signal processing circuits operating at hundreds of GHz, at extremely low power-dissipation levels, and with comfortably wide parameter-margin tolerances. One of the basic elements of RSFQ circuits is the one-dimensional array of small Josephson junctions used essentially as a transmission line [11]. Such arrays provide a convenient means for transfering SFQ pulses between active elements, for amplifying the magnetic field energy connected with a flux quantum, for providing calibrated time delay, and, with appropriate bias currents, for generating and injecting a train of SFQ into a circuit. Likharev [12] has recently reviewed the most recent achievements in this area.

High-frequency amplifiers based on Josephson arrays have also been studied outside of the RSFQ context. Particular attention has been dedicated to arrays of bridge-type Josephson elements constructed by appropriately patterning thin films of high-T_c

superconductors. An indicative example of the performance features obtainable using this technology is the SFFT (superconducting flux-flow transistor) amplifier described by Martens *et al.* [13] at the 1992 Applied Superconductivity Conference: this device showed a gain of 7 dB over a bandwidth of 50 GHz.

Millimeter-wave oscillators using Josephson arrays – both one- and two-dimensional – continue to attract active research interest. A significant stimulus for the study of Josephson millimeter-wave amplifiers and oscillators is undoubtedly the fact that another Josephson element, the SIS (superconductor-insulator-superconductor) mixer [14], is already firmly established as the best choice as a low-noise front-end detector in the range from $\sim$ 100 GHz to $\sim$ 1 THz, since its intrinsic noise temperature seems to limited only by fundamental quantum-uncertainty effects. Consequently, the idea of a fully integrated superconducting receiver assumes considerable importance, especially for space-borne communications and radio-astronomical systems in which high sensitivity and low weight and volume are crucial. The state of the art in the area of Josephson-array millimeter-wave oscillators was reviewed a few years ago by Lukens [15] and up-dated recently by Bi *et al.* [16].

III. TWO-DIMENSIONAL ARRAYS

The passage from a two-dimensional continuum system to the corresponding discrete array is somewhat less transparent than the passage from Eqs. (1-2) to Eqs. (3-6): in addition to the obvious choice of a square (or rectangular) array, one can also well imagine two-dimensional arrays having, *e.g.*, a triangular or hexagonal unit cell [17]. Moreover, two-dimensional continuum systems have only recently (see, *e.g.*, [18]) begun to receive the detailed attention that has for years been dedicated one-dimensional systems, so that acquired intuition provides us less help in this case.

Whatever the form of the unit cell, the condition of fluxoid quantization [19] imposes that the relation between the sum of the phase differences across the junctions in a closed loop and the fluxoid Φ traversing the loop is given by

$$\sum_{loop} \phi = -2\pi\frac{\Phi}{\Phi_0} + 2\pi n, \tag{9}$$

where n is an integer and $\Phi_0 = h/2e$ is the flux quantum. The ratio Φ/Φ_0 is known as the frustration, f. Once the structure of the unit cell is defined, the Kirchhoff circuit laws can be written in a fairly straightforward generalization of Eqs. (3-6). These, together with Eq. (9), completely define the dynamics of the model.

The fluxoid Φ may be divided into two components, one due to an externally-applied magnetic field perpendicular to the array and the other due to self-induced fields stemming from both the self inductance of a given loop and the mutual inductance between the given loop and its neighbors, *i.e.*

$$\Phi = \Phi_{ext} + \Phi_{ind}. \tag{10}$$

There is not yet a universal agreement in the literature regarding the appropriate form to use for the mutual-inductance contribution to Φ_{ind}, even in the simplest case of a square array. There is some indication [20] that when $f \neq 0$ it may be a reasonable approximation simply to neglect this contribution; on the other hand, at least when $\Phi_{ext} = 0$, it appears important to take into account at least nearest-neighbor mutual-inductance terms [21]. The question certainly requires and deserves further study.

Although interest in two-dimensional arrays in fact dates back a number of years [22] in connection with the analogy to the frustrated XY-model in spin glass theory, many of the current experimental and numerical studies of such arrays have been dedicated to understanding the dynamics associated with the fractional giant Shapiro steps first reported by Benz *et al.* [23]: constant-voltage steps in the dc current-voltage characteristic of dc + ac current-driven arrays, in the presence of frustration $f = p/q$, with p and q relative primes, at average-voltage values given by

$$V_q = \frac{m}{q} M \Phi_0 \nu \,, \tag{11}$$

where ν is the frequency of the ac-current drive, M is the number of junctions in the array along the current direction (an $M \times N$ rectangular array is assumed), and m and q are integers. Detailed numerical simulations together with mechanical analog studies [24] suggest that different dynamic mechanisms might be present in different regions of parameter space. The question may possibly be further elucidated by using the powerful technique of low temperature scanning electron microscopy (LTSEM), which has already shown promising results for dc-driven arrays [25]. Undoubtedly as these studies progress, more complicated dynamical states, analogous, *e.g.*, to the chaotic states already observed in one-dimensional arrays [26], will be uncovered.

One of the 'applications' of two dimensional arrays that has stimulated research activity in recent times is the use of such an array as a model for a sample of high-T_c superconductor. Samples of these materials frequently have a granular structure; one can consider such a sample to be a network of superconducting grains mutually coupled to their nearest neighbors via Josephson junctions at their points of contact and containing non-superconducting intergranular regions. A two-dimensional Josephson array can be considered a reasonable model for a thin film of high-T_c granular material in a perpendicular magnetic field or for a cylindrically symmetric sample in a uniform axial field. Study of such arrays has provided useful insight into the mechanism of low-field magnetic penetration into high-T_c samples [27]. For these studies, a correct treatment of the mutual-inductance terms in the array assumes an important rôle [28].

IV. CONCLUSIONS

Research on one- and two-dimensional arrays of discrete Josephson elements is proceeding apace, with new results emerging at a truly surprising rate. Fundamental studies of nonlinear dynamics are proceeding hand in hand with practical electronic applications. So far, one-dimensional arrays are undoubtedly more studied and better understood than their two-dimensional cousins. Numerical simulation studies of two-dimensional arrays have outnumbered experimental measurements, and these experimental studies have been largely limited to systems based on overdamped, SNS-type

junctions, but the picture has already begun to change. I predict that this development will continue for some time as researchers gradually acquire more powerful technological and computational tools as well as a more solid physical intuition regarding the complicated behavior of these systems.

Finally, I would like to mention in passing that – once again stimulated, or at least facilitated, by progress in fabrication technology – in addition to *planar* arrays, the topic of ***vertically-stacked*** junction arrays has recently begun to attract an increasing amount of research attention [29]. Also in this case the use of such structures as model systems, in this case for high-T_c layered cuprate materials [30], together with potential practical electronic applications [31], have provided much of the essential motivation.

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References

[1] N. F. Pedersen: in *Solitons*, S. E. Trullinger, V. E. Zakharov, V. L. Pokrovsky, eds. (Elsevier, Amsterdam, 1986), p. 469;

[2] R. D. Parmentier: in *The New Superconducting Electronics*, H. Weinstock, R. W. Ralston, eds. (Kluwer, Dordrecht, 1993), p. 221;

[3] P. J. Davis, I. Polonsky: in *Handbook of Mathematical Functions*, 9th printing, M. Abramowitz, I. A. Stegun, eds. (Dover, New York, 1965), chap. 25;

[4] J. F. Currie, S. E. Trullinger, A. R. Bishop, J. A. Krumhansl: *Phys. Rev. B* **15** (1977), 5567;

[5] M. Peyrard, M. D. Kruskal: *Physica D* **14** (1984), 88;

[6] D. B. Duncan, J. C. Eilbeck, H. Feddersen, J. A. D. Wattis: *Physica D* **68** (1993), 1;

[7] A. V. Ustinov, M. Cirillo, B. A. Malomed: *Phys. Rev. B* **47** (1993), 8357;

[8] L. Brillouin: *Wave Propagation in Periodic Structures* (Dover, New York, 1953);

[9] G. Costabile, P. Sabatino: in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, P. L. Christiansen, J. C. Eilbeck, R. D. Parmentier, eds. (Plenum, New York, 1993), p. 339; G. Rotoli: *ibid.*, p. 363;

[10] J. Niemeyer: in *Superconducting Devices and their Applications*, H. Koch, H. Lübbig, eds. (Springer-Verlag, Berlin, 1992), p. 430;

[11] V. Kaplunenko: "Nonlinear phenomena in RSFQ logic devices", to be presented at the International Conference on *Nonlinear Superconducting Devices and High-T_c Materials*, Capri, Italy, October 1994;

[12] K. K. Likharev: in *The New Superconducting Electronics*, H. Weinstock, R. W. Ralston, eds. (Kluwer, Dordrecht, 1993), p. 423;

[13] J. S. Martens, V. M. Hietala, T. A. Plut, D. S. Ginley, G. A. Vawter, C. P. Tigges, M. P. Siegal, J. M. Phillips, S. Y. Hou: *IEEE Trans. Appl. Supercond.* **3** (1993), 2295;

[14] R. Blundell, D. Winkler: in *Nonlinear Superconductive Electronics and Josephson Devices*, G. Costabile, S. Pagano, N. F. Pedersen, M. Russo, eds. (Plenum, New York, 1991), p. 55;

[15] J. E. Lukens: in *Modern Superconducting Devices*, S. T. Ruggiero, D. A. Rudman, eds. (Academic Press, New York, 1990), p. 135;

[16] B. Bi, S. Han, J. E. Lukens, K. Wan: *IEEE Trans. Appl. Supercond.* **3** (1993), 2303;

[17] J. A. Zagrodziński: in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, P. L. Christiansen, J. C. Eilbeck, R. D. Parmentier, eds. (Plenum, New York, 1993), p. 367;

[18] S. G. Lachenmann, G. Filatrella, T. Doderer, J. C. Fernandez, R. P. Huebener: *Phys. Rev. B* **48** (1993), 16623;

[19] A. Barone, G. Paternò: *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982), chap. 12;

[20] P. S. Lomdahl, N. Grønbech-Jensen: *Physica D* **68** (1993), 18;

[21] D. Domínguez, J. V. José: *Phys. Rev. Lett.* **69** (1992), 514;

[22] C. J. Lobb, D. W. Abraham, M. Tinkham: *Phys. Rev. B* **27** (1983), 150;

[23] S. P. Benz, M. Rzchouwski, M. Tinkham, C. J. Lobb: *Phys. Rev. Lett.* **64** (1990), 693;

[24] M. Octavio: "Dynamics of classical two-dimensional Josephson arrays", to be presented at the International Conference on *Nonlinear Superconducting Devices and High-T_c Materials*, Capri, Italy, October 1994;

[25] S. G. Lachenmann, T. Doderer, D. Hoffmann, R. P. Huebener, P. A. A. Booi, S. P. Benz: "Observation of vortex dynamics in two-dimensional Josephson junction arrays", *preprint, 1994*;

[26] D. Domínguez, H. A. Cerdeira: *Phys. Rev. Lett.* **71** (1993), 3359;

[27] C. Auletta, R. De Luca, S. Pace, G. Raiconi: *Phys. Rev. B* **47** (1993), 14326;

[28] R. De Luca, S. Pace, B. Savo: *Phys. Lett. A* **154** (1991), 185;

[29] H. Kohlstedt: "Stacked Nb-Al/AlO_x-Nb Josephson tunnel junctions", to be presented at the International Conference on *Nonlinear Superconducting Devices and High-T_c Materials*, Capri, Italy, October 1994;

[30] P. Müller: "Intrinsic Josephson effects in layered superconductors", to be presented at the International Conference on *Nonlinear Superconducting Devices and High-T_c Materials*, Capri, Italy, October 1994;

[31] K. H. Gundlach, D. Billon-Pierron, M. Rosengarten, A. Karpov, T. Lehnert, H. Rothermel: "Double-barrier tunnel junctions for quasiparticle mixers and other applications", *preprint, 1993*.