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SCALING AND UNIVERSALITY IN THE SELF-ORGANIZED CRITICAL FOREST-FIRE MODEL[1]

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We review the properties of the self-organized critical forest-fire model. First, we define critical exponents and scaling laws. In one dimension, we give the exact values of the critical exponents including logarithmic corrections. In higher dimensions, we present simulation results which confirm the scaling theory and seem to agree with mean-field theory above 6 dimensions. We investigate the universality of the critical exponents by changing the lattice symmetry in two dimensions. The critical exponents remain unchanged. We also include immunity against fire as a new parameter in the model. The asymptotic critical behavior is still the same as long as the immunity is below a critical value. Close to this critical value, the system performs a crossover from percolation to self-organized criticality.

1. Introduction

Some years ago, Bak, Tang, and Wiesenfeld introduced the *sandpile model* which evolves into a critical state irrespective of initial conditions and without fine tuning of parameters [1]. Such systems are called *self-organized critical* (SOC) and exhibit power-law correlations in space and time. The concept of SOC has attracted much interest since it might explain the origin of *fractal structures* and $1/f$*-noise*. Other SOC models e.g. for earthquakes [2, 3] or the evolution of populations [4, 5] have been introduced since then, improving our understanding of the mechanisms leading to SOC.

In this paper, we review the properties of a forest-fire model which is SOC under the condition that time scales are separated [6]. In one dimension, where the model is still nontrivial, the exact values of the critical exponents have been calculated [7], thus proving the criticality of the model. Critical exponents have been defined and determined by computer simulations [6, 8, 9, 10, 11] The universality of the values of the critical exponents is investigated by changing the lattice symmetry and by considering the case of nonvanishing immunity [8, 12].

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The outline of the paper is as follows: In Sec. , the rules of the model are introduced and the origin of the SOC behavior is explained. In Sec. , the scaling theory of the model is presented. In Sec. , analytic results in 1 dimension and simulation results in 2 to 6 dimensions are given. Sec. investigates the universality of the critical exponents. In the final section, the results are summarized and discussed.

2. The model

The forest–fire model is a stochastic cellular automaton which is defined on a d–dimensional hypercubic lattice with L^d sites. Each site is occupied by a tree, a burning tree, or it is empty. During one time step, the system is parallely updated according to the following rules

- burning tree ⟶ empty site
- tree ⟶ burning tree, if at least one nearest neighbor is burning
- tree ⟶ burning tree with probability f, if no neighbor is burning
- empty site ⟶ tree with probability p.

An even more general forest–fire model also contains an immunity [13, 14]. In its original version, introduced by P. Bak, K. Chen, and C. Tang, the forest–fire model contained only the tree growth parameter p [15]. This version of the model shows regular spiral–shaped fire fronts in the limit of slow tree growth [16, 17]. Throughout this paper, we will assume that the system size L is large enough such that no finite–size effects occur. In the simulations, we have always chosen periodic boundary conditions.

Starting with arbitrary initial conditions, the system approaches after a transition period a steady state the properties of which depend only on the parameter values. Large–scale structures and therefore criticality can only occur when the ration f/p is very small, since otherwise trees are destroyed by lightning before they become part of large forest clusters. This condition is not yet sufficient to bring about critical behavior in the forest–fire model. When lightning strikes a small forest cluster, it burns down very fast, before any tree can grow at its edge. But when lightning strikes a large forest cluster, it needs some time to burn down, and new trees might grow at the edge of this cluster while it is still burning so that the fire is never extinguished. In order to observe critical, i.e. self–similar behavior, small and large forest clusters must burn down in the same way. We therefore choose the tree growth rate p so small, that even the largest forest cluster burns down, before new trees grow at its edge. In this case, the dynamics of the system depend only on the ratio f/p, but not on f and p separately. When f and p are both decreased by the same factor, the overall time scale of the system is also changed by this factor, but not the number of trees that grow between two lightnings and therefore not the size distribution of forest clusters and of fires. The condition that forest clusters burn down rapidly can be written in the form

$$p \ll T^{-1}(s\text{max}), \qquad (1)$$

where $T(s\text{max})$ is the time the fire needs to burn down a large forest cluster and will be determined below (see Eq. (19)). The two above conditions represent a *double*

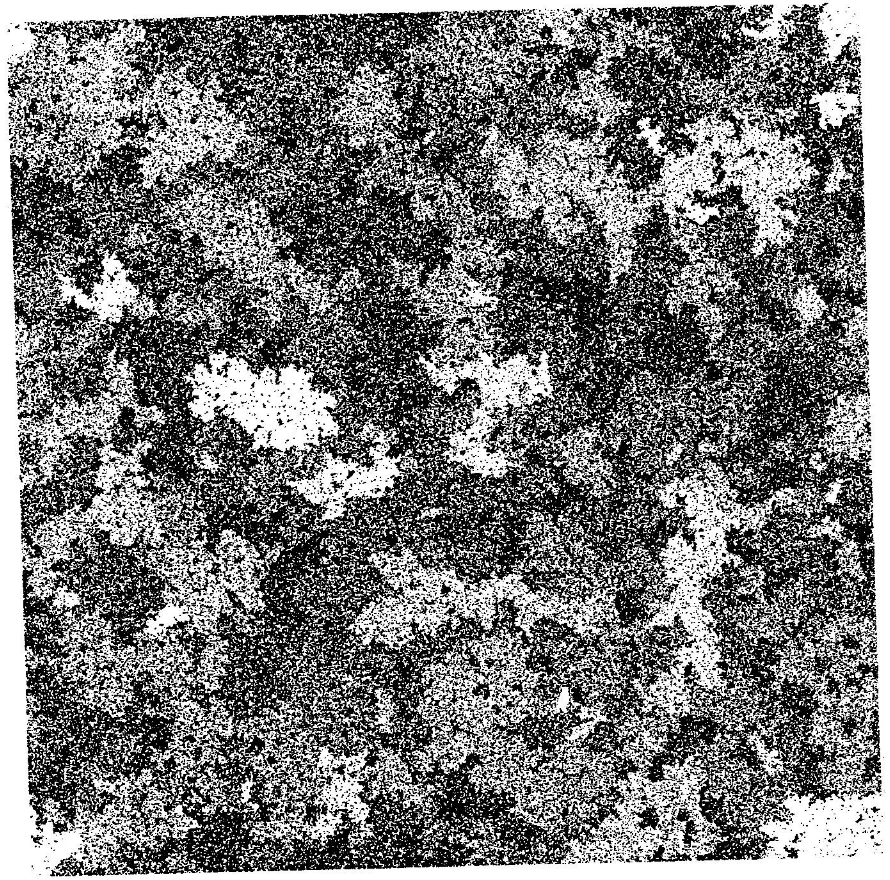

Fig. 1. Snapshot of the SOC state in 2 dimensions. Trees are black, empty sites are white. The parameters are $L = 1000$ and $f/p = 1/500$.

separation of time scales

$$T(s\text{max}) \ll p^{-1} \ll f^{-1}, \qquad (2)$$

which causes SOC behavior in the forest–fire model. The time in which a forest cluster burns down is much shorter than the time in which a tree grows, which again is much shorter than the time between two lightning occurrences. Separation of time scales is quite frequent in nature, while the tuning of parameters to a certain finite value only takes place accidentally. Thus, the forest–fire model is critical over a wide range of parameter values. A snapshot of the critical state is shown in Fig. 1.

3. Scaling laws and critical exponents

In this section, we will derive scaling laws and relations between the critical exponents for the SOC forest–fire model.

First, we calculate the mean number $\bar{s}$ of trees that are destroyed by a lightning stroke. Let ρ be the mean forest density in the steady state. During one time step, there are

$$f\rho L^d$$

lightning strokes in the system and

$$p(1-\rho)L^d$$

growing trees. In the steady state, the number of growing trees equals the number of burning trees, and therefore the mean number of trees destroyed by a lightning stroke is

$$\bar{s} = \frac{p}{f}\frac{1-\rho}{\rho}. \tag{3}$$

For small values of f/p, the forest density ρ assumes a constant value. If this constant value is less than 1, the second factor on the right–hand side of Eq. (3) is also constant for small f/p, and Eq. (3) then represents a power law

$$\bar{s} \propto (f/p)^{-1}. \tag{4}$$

In $d \geq 2$ dimensions, the critical forest density

$$\rho^c = \lim_{f/p\to 0} \rho, \tag{5}$$

in fact, must be less than 1, as the following consideration indicates: If the critical forest density were $\rho^c = 1$ in $d \geq 2$ dimensions, ρ would be very close to 1 for small values of f/p. Then the largest forest cluster would contain a nonvanishing percentage of all trees in the system, and the average number $\bar{s}$ of trees burned by a lightning stroke would diverge in the limit $L \to \infty$ with fixed f/p, in contradiction to Eq. (3). In one dimension, there is no infinitely large forest cluster in the system as long as $\rho < 1$, and therefore the critical forest density is $\rho^c = 1$. Nevertheless Eq. (4) holds also in 1 dimension since the forest density approaches its critical value only logarithmically slowly, as will be shown below. Eq. (4) indicates a critical point in the limit $f/p \to 0$. Close to this critical point, i.e. if $f \ll p$, there is scaling over many orders of magnitude.

Let $n(s)$ be the mean number of forest clusters per unit volume consisting of s trees. Then the mean forest density is

$$\rho = \sum_1^\infty s n(s), \tag{6}$$

and the mean number of trees destroyed by a lightning stroke is

$$\bar{s} = \sum_1^\infty s^2 n(s)/\rho. \tag{7}$$

Since $\lim_{f/p\to 0}\rho$ is finite and $\bar{s}$ diverges $\propto (f/p)^{-1}$, these equations imply that $n(s)$ decreases at least like s^{-2} but not faster than s^{-3}. As long as the system is not exactly at the critical point $f/p = 0$, i.e. for nonvanishing f/p, there must be a cutoff in the cluster size distribution for very large forest clusters. We conclude that [6]

$$n(s) \propto s^{-\tau} \mathcal{C}(s/s_{\max}) \tag{8}$$

with $2 \leq \tau \leq 3$ and

$$s_{\max}(f/p) \propto (f/p)^{-\lambda} \propto \bar{s}^{\lambda}. \tag{9}$$

The cutoff function $\mathcal{C}(x)$ is essentially constant for $x \leq 1$ and decreases to zero for large x. Eqs. (7) - (9) yield $\bar{s} \propto s_{\max}^{3-\tau}$, which leads to the scaling relation

$$\lambda = 1/(3-\tau). \tag{10}$$

In the case $\tau = 2$, the right–hand side of Eq. (8) acquires a factor $1/\ln(s_{\max})$ and reads now

$$n(s) \propto s^{-\tau}\mathcal{C}(s/s_{\max})/\ln(s_{\max}), \tag{11}$$

since the forest density given by Eq. (6) must not diverge in the limit $f/p \to 0$. The mean number of forest clusters per unit volume $\sum_1^\infty n(s)$, therefore, decreases to zero for $f/p \to 0$, and consequently the forest density approaches the value 1.

We also introduce the cluster radius $R(s)$ which is the mean distance of the trees in a cluster from their center of mass. It is related to the cluster size s by

$$s \propto R(s)^\mu \tag{12}$$

with the fractal dimension μ.

The correlation length ξ is defined by

$$\xi^2 = \frac{\sum_1^\infty sn(s)\cdot sR^2(s)}{\sum_1^\infty sn(s)\cdot s} \propto (f/p)^{-2\lambda/\mu}. \tag{13}$$

We conclude

$$\xi \propto (f/p)^{-\nu} \text{ with } \nu = \lambda/\mu. \tag{14}$$

In percolation theory, the ***hyperscaling relation***

$$d = \mu(\tau - 1) \tag{15}$$

is satisfied, but it is not satisfied in the SOC forest–fire model in $d = 2$, as first stated in [11], where also an interpretation of this relation is given: If Eq. (15) is satisfied, every box of $l^d \gg 1$ sites contains a spanning piece of a large cluster when the system is at the critical point. In the forest–fire model, there are at least in $d = 2$ many regions which contain no large forest cluster (see Fig. 1.), and consequently $d < \mu(\tau - 1)$.

The mean forest density ρ approaches its critical value $\rho^c = \lim_{f/p\to 0}\rho$ via a power law

$$\rho^c - \rho \propto (f/p)^{1/\delta}. \tag{16}$$

Finally, we introduce dynamical exponents characterizing the temporal behavior of the fire. Let $T(s)$ be the average time a cluster of size s needs to burn down when ignited, and $N(T)$ the portion of fires that live exactly for T time steps. Then the exponents b and μ' are defined by

$$s \propto T(s)^{\mu'} \text{ and } N(T) \propto T^{-b}. \tag{17}$$

From

$$N(T)dT \propto sn(s)ds$$

follows the scaling relation

$$b = \mu'(\tau - 2) + 1. \tag{18}$$

The time scale of the system is set by

$$T_{\max} = T(s_{\max}) \propto (f/p)^{-\nu'} \text{ with } \nu' = \lambda/\mu'. \tag{19}$$

The dynamical critical exponent z is defined by

$$T_{\max} \propto \xi^z,$$

which leads with (14) and (19) to

$$z = \nu'/\nu = \mu/\mu'. \tag{20}$$

The condition of time scale separation now can be expressed in terms of the critical exponents and reads

$$(f/p)^{-\nu'} \ll p^{-1} \ll f^{-1}, \tag{21}$$

or equivalently

$$f \ll p \ll f^{\nu'/(1+\nu')}. \tag{22}$$

The average number $N_s(t)$ of trees that burn t time steps after a cluster of size s is struck by lightning enters the definition of the temporal fire–fire correlation function $G(\tau)$

$$G(\tau) \propto \sum_{s=1}^{\infty} n(s)s \sum_{t=0}^{\infty} N_s(t)N_s(t+\tau). \tag{23}$$

The power spectrum is the Fourier transform of the fire–fire correlation function

$$G(\omega) = 2\int_0^{\infty} d\tau\, G(\tau)\cos(\omega\tau) \propto \omega^{-\alpha} \text{ for small } \omega. \tag{24}$$

4. Values of the critical exponents in 1 to 6 dimensions

In this section, we determine the values of the critical exponents in 1 to 6 dimensions. In one dimension, the critical exponents can be determined analytically, as was done in [7]. In higher dimensions one has to resort to computer simulations. Here, we shortcut the exact evaluation of the critical exponents in $d = 1$ by using simple arguments.

In one dimension, the critical forest density ρ^c equals 1, since otherwise there were no infinitely large forest cluster in the system. The consideration after Eq. (10) shows that consequently $\tau = 2$ and (via scaling relation Eq. (10)) $\lambda = 1$. In the steady state, the density of forest clusters $\sum n(s)$ is constant, and therefore

$$\sum_{s=1}^{\infty} n(s) = (1 - \rho - (f/p)\rho)/2,$$

which leads together with Eq. (11) to

$$(1-\rho) \propto 1/\ln(s_{\max})$$

and $1/\delta = 0$. One–dimensional forest clusters are compact, therefore $\mu = 1$ and (with Eq. (14)) $\nu = 1$. From $T(s) \propto R(s)$ it follows $\mu' = \nu' = z = 1$. The exact calculation in [7] yields additional logarithmic corrections:

$$s_{\max} \propto \xi \propto T_{\max} \propto (p/f)/\ln(p/f).$$

The fourier transform of the temporal correlation function is [7]

$$G(\omega) \propto \omega^{-2}(1 + \text{const.} \cdot \ln(\omega s_{\max})), \tag{25}$$

indicating a deviation from the trivial ω^{-2}-dependence towards $1/\omega$–noise. Tab. 1. summarizes the values of the critical exponents. They are confirmed by our simulations.

We obtained the values of the critical exponents in $d \geq 2$ dimensions by computer simulations using the same method as in [9]. The values of the critical exponents are given in Tab. 1. They indicate that the SOC forest-fire model is likely to have an upper critical dimension $d_c = 6$, above which the critical exponents are identical with those of mean–field–theory, which again is identical to the mean–field–theory of percolation.

5. Universality of the critical exponents

The critical behavior of a system usually depends only on properties as dimension and conservation laws, but not on microscopic details. We therefore expect that the critical exponents of the SOC forest–fire model are universal under certain changes of the model rules. In [8], we repeated the 2D simulations on a triangular lattice and on a square lattice with next–nearest–neighbor interaction. The simulations of both variations of the model were done on a 4096 × 4096–lattice with f/p ranging from 1/1000 to 1/8000. We compared the exponents τ, μ, μ', ν, ν' and α with the results given in Tab. 1. and found them to be exactly the same (see e.g. Fig. 2.).

Universal behavior is also observed when trees are allowed to be immune against fire. We introduce an immunity g and change rule 2 in the following way [12]:

Table 1. Numerical results for the critical exponents in 1 to 6 dimensions (* = with logarithmic corrections, † = calculated from scaling relations).

d	1	2	3	4	5	6	mean field
L	2^{20}	16384	448	80	32	20	
τ	2	2.14(3)	2.23(3)	2.36(3)	2.45(3)	2.50(3)	2.5
λ	1*	1.15(3)	1.30(6)	1.56(8)†	1.82(10)†	2.01(12)†	2
$1/\delta$	0*	0.48(2)	0.55(12)	-	-	-	1
ρ^c	1	0.4081(7)	0.2190(6)	0.146(1)	0.111(1)	0.090(1)	$1/(2d-1)$
μ	1	1.96(1)	2.51(3)	3.0	3.2(2)	-	4
ν	1*	0.58	0.52(3)†	0.53(3)†	0.57(7)†	-	0.5
μ'	1	1.89(3)	2.04(10)	2.02(10)	1.98(10)	1.94(10)	2
ν'	1*	0.58	0.64(6)†	0.78(8)†	0.92(10)†	1.04(11)†	1
z	1	1.04(2)†	1.24(8)†	1.49(10)†	1.62(18)†	-	2
b	1	1.27(7)†	1.47(9)†	1.73(10)†	1.89(11)†	1.97(11)†	2
α	2*	1.72(5)	2.15(5)	2.00(5)	2.01(5)	1.95(10)	2

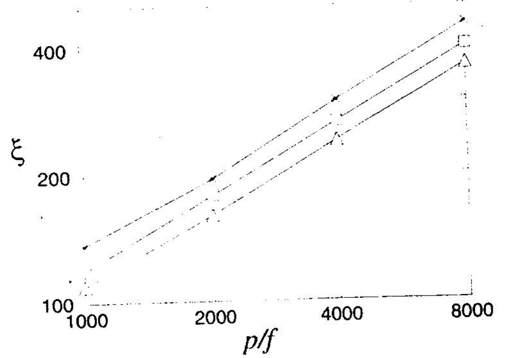

Fig. 2.The correlation length ξ as function of $(f/p)^{-1}$. The slope yields the critical exponent ν. (□ = square lattice, △ = triangular lattice, ∗ = next–nearest–neighbour interaction.)

- tree ⟶ burning tree with probability $1-g^n$, if n nearest neighbors are burning.

When the immunity assumes its critical value $g_c = 1/2$, the model shows percolation-like behavior. As long as the immunity is below its critical value, the asymptotic critical exponents are the same as in the case of vanishing immunity, and the system performs a universal crossover from percolation to self-organized criticality. In the following, we give plausible arguments and simulation results for this crossover behavior.

When the immunity is different from 0, not all trees that are neighbors of a burning tree catch fire, and consequently the fire does no longer burn forest clusters but clusters of trees that are connected by non immune bonds. With increasing immunity, the forest density increases, since fewer trees are burnt. At the critical immunity $g_c = 1/2$, the critical forest density is $\rho^c = 1$. Then we have the following situation: The forest is completely dense in the limit $f/p \to 0$, and clusters that are destroyed by fire are

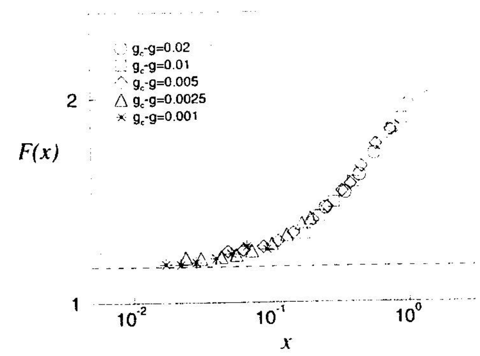

Fig. 3. Crossover scaling function $F(x)$ for the correlation length for different values of the immunity. The dashed line represents $F(0)$ as obtained at $g = g_c$.

percolation clusters of bond percolation. Consequently the exponents τ and μ given by percolation theory: $\tau(g_c) \equiv \tau_c = \tau\text{perc} = 187/91 \simeq 2.05$ and $\mu(g_c) \equiv \mu_c = \mu\text{perc} = 91/48 \simeq 1.90$. When f/p is finite, there is a cutoff in cluster size, since large fires are stopped by empty sites that have been left from earlier fires. The mean forest density is no longer 1. We determined the critical exponents λ, δ, and ν at $g = g_c$ by computer simulations in $d = 2$ dimensions and obtained $\lambda_c = 0.92(3)$, $1/\delta_c = 0.15(1)$, $\nu_c = 0.484(2)$. Since $\rho^c = 1$, Eq. (4) has to be replaced by

$$s \propto (f/p)^{-(1-1/\delta_c)}, \tag{26}$$

and the scaling relation Eq. (10) by

$$\lambda_c = (1-1/\delta_c)/(3-\tau). \tag{27}$$

When the immunity is just below its critical value ($(g_c - g) \ll 1$), the situation becomes more complicated. On length scales smaller than the percolation correlation length $\xi\text{perc} \propto (g_c - g)^{\nu\text{perc}}$, a system close to the percolation threshold cannot be distinguished from a system exactly at the percolation threshold. As long as f/p is so large that the fires do not spread further than ξperc, the exponents are identical to those at $g = g_c$. When f/p becomes very small, there are fires which spread further than the percolation correlation length. These fires are stopped by empty sites that were created by earlier fires. This is the same mechanism as in the limit $g = 0$: fires that would spread indefinitely if there were no empty sites are stopped by empty sites. We conclude that these large fires lead to the critical exponents λ, ν, and δ that have been observed for $g = 0$. We make the following scaling ansatz for the correlation length:

$$\xi = (f/p)^{-\nu_c} F\left(\frac{g_c - g}{(f/p)^{\phi}}\right). \tag{28}$$

It is plausible that the crossover from percolation-like to SOC behavior takes place when f/p becomes so small that the correlation length exceeds the percolation correlation length, which suggests that the crossover exponent ϕ is

$$\phi = \nu_c/\nu_{perc}\,. \qquad (29)$$

The scaling function $F(x)$ is constant for small x and is $\propto x^{(\nu_{soc}-\nu_c)/\phi}$ for large x. Analogous scaling laws hold for $s_{\max}$ and $\rho^c - \rho$. We already mentioned above that the critical forest density is $\rho^c = 1$ at g_c. We therefore expect an additional power law

$$1-\rho^c(g) \propto (g_c - g)^y\,. \qquad (30)$$

The exponent y is obtained from the scaling ansatz

$$1-\rho = (f/p)^{1/\delta_c} G\left(\frac{g_c - g}{(f/p)^\phi}\right). \qquad (31)$$

In the limit $f/p \to 0$, the forest density becomes independent of f/p and assumes a value $\rho^c \neq 1$. Therefore $G(x) \propto x^{1/\phi\delta_c}$ for large x, yielding

$$y = \nu_{\text{perc}}/\nu_c\delta_c\,. \qquad (32)$$

Our simulations confirm all these results. Fig. 3. shows the scaling function for the correlation length $F(x)$ for different values of $g_c - g$. The scaling ansatz Eq. (28) is well confirmed since all curves coincide. The dashed line represents $F(0)$ as obtained from the simulations at g_c.

Thus, we have shown that the forest-fire model performs a crossover from percolation to SOC when the immunity is close to its critical value. This crossover is characterized by scaling functions which are defined in the same way as in crossover phenomena in equilibrium phase transitions.

Although all simulations were performed in $d = 2$ dimensions, we expect that this crossover behavior can also be observed in higher dimensions. In $d = 1$, the critical immunity is $g_c = 0$, and no crossover can take place. For $d \geq 6$, simulations suggest that the critical exponents assume their mean-field values which are identical to those of percolation [8, 10]. Consequently there is no crossover in $d \geq 6$ dimensions.

6. Summary and discussion

In this paper, we have reviewed the properties of the SOC forest–fire model. The appropriate critical exponents were defined, and scaling relations between them were derived.

The critical exponents in one dimension, which are known exactly [7], were rederived by simple arguments. In dimensions ≥ 2, computer simulations then determined the values of the critical exponents and confirmed the scaling relations.

The simulations suggest that the critical exponents of the SOC forest–fire model in dimensions $d \geq d_c = 6$ are given by its mean–field theory, which is identical with the mean–field theory of percolation.

Finally, we investigated the universality of the critical properties by changing the lattice symmetry and by introducing the new parameter immunity. The critical exponents turned out to be universal under these modifications.

As already pointed out in earlier publications [13, 14, 8], there is a close relationship between the forest–fire model and excitable media which comprise phenomena so different as spreading of deseases, oscillating chemical reactions, propagation of electrical activity in neurons or heart muscles, and many more (For a review on excitable systems see e.g. [18, 19]). These systems essentially have three states which are called quiescent (corresponds to tree), excited (corresponds to burning tree), and refractory (corresponds to empty site). Excitation spreads from one place to its neighbors if they are quiescent. After excitation, a refractory site needs some time to recover its quiescent state. In many of these systems, spiral–waves have been observed. We expect that a SOC state can be found in some of these systems, if the appropriate range of parameter values is investigated, i.e. if spontaneous excitation occurs rarely and if excitation spreads much faster than the system recovers from the refractory state.

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