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MODE ENTANGLEMENT IN NONLINEAR QUANTUM-OPTICAL PARAMETRIC PROCESSES

Gabriel Drobný[1]
Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia

Igor Jex[2]
Research Institute for Theoretical Physics University of Helsinki P.O. Box 9, FIN - 00014 Helsinki, Finland

Vladimír Bužek
Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia

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We analyze quantum-statistical properties of field modes in nonlinear optical processes. In particular we study mode entanglement in a process of degenerate and non-degenerate multi-photon down-conversion with quantized pump. We study in detail how the degree of entanglement depends on initial statistics of the pump and the down converted modes. We analyze the efficiency of the energy transfer between quantum fields in the processes under consideration. Other nonclassical effects such as spontaneous disentanglement, and production of squeezed and sub-Poissonian states are discussed.

1. Introduction

In recent years a great deal of interest has been paid to an investigation of nonclassical effects [1] appearing during an interaction of a field mode(s) with a material medium. Among various processes a privileged role was assigned to a degenerate and non-degenerate two-photon down-conversion [2]. These processes refer to a situation when pairs of highly correlated photons are generated out of the pump mode. In the degenerate down conversion two photons produced out of a single pump photon have the same frequency, polarization and a wave vector (i.e. in the degenerate process

[1]E-mail address: drobny@savba.sk
[2]Permanent address: Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia

two identical photons are produced). On the other hand, in a non-degenerate down conversion process two "distinguishable" photons are produced.

It is generally assumed that if the pump mode is initially prepared in a highly excited coherent state, then it can be treated as a classical field. In this case the interaction Hamiltonian describing the degenerate down conversion takes the form (in what follows we will use the interaction picture)

$$\hat{H}_d^{(par)} = \lambda_d[\gamma(\hat{a}^\dagger)^2 + \gamma^* \hat{a}^2], \tag{1}$$

while the non-degenerate process is described by the interaction Hamiltonian

$$\hat{H}_n^{(par)} = \lambda_n[\gamma \hat{a}^\dagger \hat{b}^\dagger + \gamma^* \hat{a}\hat{b}]. \tag{2}$$

The operators $\hat{a}, \hat{a}^\dagger (\hat{b}, \hat{b}^\dagger)$ describe an annihilation and creation of photons of the down-converted modes. The coupling constant $\lambda_{d(n)}$ is proportional to the second order polarizability of the medium (crystal) mediating the process. The pump mode is assumed to be classical with the amplitude γ and frequency ω_c.

It is well known today that two-photon nonlinear processes described by the Hamiltonians (1) and (2) give rise to quadrature squeezing, i.e. in these processes light fields with reduced quantum fluctuations can be produced [3]. In real experimental setups the system is optimalized to obtain maximum squeezing. Production of highly squeezed light is closely connected with an intense transfer of energy from the pump mode to the down-converted mode(s). Generally speaking, in the parametric processes described by Eqs.(1) and (2) it is valid that higher the amount of energy transferred from the pump to the down-converted mode(s), higher the degree of squeezing in the down-converted mode(s). Large amounts of energy (even for an intense pump) can be transferred only when the interaction time is sufficiently long. The analysis of the down-conversion processes on a longer time scale forces us to take into consideration at least two effects. Firstly, we have to take into account the depletion of the pump mode when large amounts of energy are transferred to the down-converted mode(s). Secondly, we have to include the back action of the down-converted mode(s) on the pump mode. The pump depletion and the back-action of the down-converted mode(s) on the pump mode can be taken into consideration by treating the pump mode as a quantized field, i.e. by introducing into the models (1) and (2) quantum variable describing the pump mode. The simplest and most straightforward way is to treat instead of the interaction Hamiltonians (1) and (2) the models [4]

$$\hat{H}_d = \lambda_d[(\hat{a}^\dagger)^2 \hat{c} + \hat{a}^2 \hat{c}^\dagger], \tag{3}$$

and

$$\hat{H}_n = \lambda_n[\hat{a}^\dagger \hat{b}^\dagger \hat{c} + \hat{a}\hat{b}\hat{c}^\dagger], . \tag{4}$$

The Hamiltonians (3) and (4) describe a physical situation when a nonlinear crystal is placed into a resonator which supports the modes entering the interaction. Further, we will assume exact resonance between the pump mode and the down-converted modes, i.e.

$\omega_c = 2\omega_a$ for the degenerate case and $\omega_c = \omega_a + \omega_b$ for the non-degenerate process. This formulation of the problem enables us to take into consideration besides the depletion of the pump mode and the back action of the down-converted mode(s) also the effect of initial quantum-statistical properties of the pump mode on the dynamics of the system and, in particular, on nonclassical properties of down-converted modes.

The relation between the process with the quantized pump [Hamiltonians (3) and (4)] and its parametric-approximation version is not as straightforward as it can be assumed at the first glance. Generally it is believed that the parametric approximation can be easily performed providing the pump mode is initially in an intense coherent state. In this case the substitution $\hat{c} \to \gamma$ is usually performed (working in the interaction picture). Nevertheless we have to stress that such substitution (parametric approximation) is justified only for a short interaction times, i.e. for times smaller than $t_c = (\lambda_{d(n)}|\gamma|)^{-1}$ (for more details see below). Besides this restriction we have to take into account the fact that the parametric approximation cannot be straightforwardly applied for multi-photon processes [5].

The degenerate two-photon down-conversion model in the parametric approximation (1) can be considered as a special case of a more general process of k-photon down-conversion with the interaction Hamiltonian

$$\hat{H}_k^{(par)} = \lambda_k[\gamma(\hat{a}^\dagger)^k + \gamma^* \hat{a}^k]. \tag{5}$$

In analogy with two-photon (squeezed) states for $k = 2$, it can be naively expected that k-photon process should give rise to k-photon states. A detailed analysis of the model (5) (for $k > 2$) revealed that this approach faces serious mathematical difficulties [5]. In particular, it has been shown that the time evolution operator $\exp[-it\hat{H}_k^{(par)}]$ related to the Hamiltonian (5) leads to divergencies in the mean photon number of the down converted mode in *finite* times. One possibility how to overcome these divergences has been proposed by Hillery [5], who has pointed out that the divergences can be removed by quantizing the pump mode, i.e. by using instead of (5) the interaction Hamiltonian of the form

$$\hat{H}_k = \lambda_k[(\hat{a}^\dagger)^k \hat{c} + \hat{a}^k \hat{c}^\dagger]. \tag{6}$$

While for $k \leq 2$ the replacement of the pump mode operators by c numbers does not formally limit the parametric approximation in the case $k > 2$ the parametric approximation will be either impossible or at least limited to finite times [5].

The parametric approximation (i.e. the substitution $\hat{c} \to \gamma$ for a coherent pump with amplitude γ) for any pump state was analyzed recently by Hillery and co-workers [6]. Following their treatment we consider the degenerate two-photon down-conversion process (3) with an initial-state vector

$$|\psi(0)\rangle = |\psi(0)\rangle_a |\psi(0)\rangle_c \tag{7}$$

and the time-evolution operator given in the interaction picture by

$$\hat{U}_2(t) = \exp[-i\lambda_2 t(\hat{a}^{\dagger 2}\hat{c} + \hat{a}^2 \hat{c}^\dagger)]. \tag{8}$$

Now we have to expand the given pump state $|\psi(0)\rangle_c$ in terms of coherent states

$$|\psi(0)\rangle_c = \frac{1}{\pi}\int d^2\beta\ {}_c\langle\beta|\psi(0)\rangle_c|\beta\rangle_c. \qquad (9)$$

and then the "standard" parametric approximation can be applied for each coherent state separately, i.e. [6]

$$|\psi(t)\rangle = \frac{1}{\pi}\int d^2\beta\ {}_c\langle\beta|\psi(0)\rangle_c \exp[-i\lambda_2 t(\beta\hat{a}^{\dagger 2} + \beta^*\hat{a}^2)]|\psi(0)\rangle_a|\beta\rangle_c. \qquad (10)$$

This kind of the parametric approximation clearly shows that an "incoherent" pump mode, i.e. the pump mode prepared in another state than the coherent one, leads to a strong correlation (entanglement) between the pump and the down converted mode(s) [in analogous way this treatment can be adopted also for the non-degenerate case]. More complicated is the question about the range of validity of this treatment. Roughly speaking, the generalized parametric approximation is not valid for times beyond $t_c = (\lambda_2\sqrt{n_c})^{-1}$ where n_c is the initial pump intensity.

The origin of the limited applicability of the parametric approximation is not only dictated by the amount of energy transferred from the pump to the down-converted mode(s), i.e. by the pump depletion, but as we said earlier also by the back action of down-converted mode(s) on the pump mode. The back-action leads to an entanglement (correlation) between the pump and the down-converted mode(s). Which means that if each mode is in a pure state initially, i.e. the initial state vector of the whole system can be written in a factorized form $|\psi(t=0)\rangle = |\psi\rangle_a \otimes |\psi\rangle_c$, then at $t > 0$ the state vector $|\psi(t)\rangle$ *cannot* be expressed in a form

$$|\psi(t)\rangle = |\psi(t)\rangle_a \otimes |\psi(t)\rangle_c.$$

From the fact that the total state vector cannot be factorized into a product of vectors describing the pump and the down-converted mode(s) separately it follows that due to the quantum interaction the pump modes evolves from its pure initial state into a statistical mixture state at $t > 0$. This loss of purity of the pump mode represents an additional restriction on the applicability of the parametric approximation.

In the present paper we will discuss in detail the time evolution of light fields in the quantum nonlinear processes. In particular, we will study how the initial photon statistics of the pump mode affects the time evolution of the system under consideration as well as quantum-statistical properties of the down-converted modes. In our discussion we will concentrate our attention only on those physical processes in which dissipative influence of an environment can be neglected.

2. Dynamics of lossless down-converters

The dynamics of the model governed by the fully-quantized Hamiltonians (4) and (6) cannot be generally described in terms of exact analytical solutions [7]. Nevertheless, a numerical treatment of the problem is very effective [4, 8, 9]. This numerical approach

is based on a numerical diagonalization of the interaction Hamiltonian on dynamically invariant finite subspaces labeled by the eigenvalues of the integrals of motion:

$$\left.\begin{aligned}\hat{L} &= \hat{a}^\dagger\hat{a} + \hat{c}^\dagger\hat{c}\\ \hat{D} &= \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\end{aligned}\right\} \quad \text{for non-degenerate down conversion,}$$
$$\hat{M} = \hat{a}^\dagger\hat{a} + k\hat{c}^\dagger\hat{c}, \qquad \text{for } k\text{-photon down conversion.} \qquad (11)$$

From the above definitions it follows that $[\hat{H}_n, \hat{L}] = [\hat{H}_n, \hat{D}] = 0$, $[\hat{H}_k, \hat{M}] = 0$ which enables us to label the Hilbert space $\mathcal{H}$ of the quantum system in the case of the non-degenerate process (4) as

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c = \oplus_{D,L}\mathcal{H}_{D,L}, \qquad (12)$$

and in the case of the degenerate k-photon down-conversion we label the corresponding Hilbert space as

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_c = \oplus_M \mathcal{H}_M, \qquad (13)$$

where $\mathcal{H}_x$ denotes the Hilbert space of the particular mode x. The subspace $\mathcal{H}_{D,L}$ with $D \geq 0$ is formed out of the Fock basis vectors $\{|D+m\rangle_a|m\rangle_b|L-m\rangle_c, m = 0,\ldots,L\}$ (with the dimension $L+1$). The subspace $\mathcal{H}_M$ is formed out of the basis of Fock state vectors $\{|M-km\rangle_a|m\rangle_c\}$. The dimension of this subspace is $[M/k]+1$ where $[x]$ is the integer part of x. The values D, L (characterizing the non-degenerate process) and M (corresponding to the degenerate process) are integers being the eigenvalues of the corresponding operators $\hat{D}, \hat{L}$ and $\hat{M}$. On the subspaces $\mathcal{H}_{D,L}$ and $\mathcal{H}_M$ the interaction Hamiltonians in the Fock basis take the tridiagonal form

$$\begin{pmatrix} 0 & h_0 & 0 & 0 & \ldots \\ h_0 & 0 & h_1 & 0 & \ldots \\ 0 & h_1 & 0 & h_2 & \ldots \\ 0 & 0 & h_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (14)$$

where $h_m = [(L-m)(D+m+1)(m+1)]^{1/2}$ with $m = 0,\ldots,L-1$ for the non-degenerate process and $h_m = [(m+1)(M-km)!/(M-km-k)!]$ with $m = 0,\ldots,[M/k])$ for the degenerate case. The numerical treatment of the problem consists now in the numerical solution of the eigenvalue problem. In the case of the degenerate k-photon down conversion we have to solve the eigenvalue problem for the interaction Hamiltonian (6) (we remind that we work in the interaction representation)

$$\hat{H}_k|E_j(M)\rangle = E_j(M)|E_j(M)\rangle, \qquad (15)$$

with eigenvectors $|E_j(M)\rangle$ given in the Fock basis by the relation

$$|E_j(M)\rangle = \sum_{m=0}^{[M/k]} c_{jm}(M)|km+z\rangle_a|M-m\rangle_c, \qquad z = M - k[M/k]. \qquad (16)$$

In the case of the non-degenerate process (4) we solve the eigenvalue problem

$$\hat{H}_n|E_j(D,L)\rangle = E_j(D,L)|E_j(D,L)\rangle, \quad (17)$$

with the eigenvectors

$$|E_j(D,L)\rangle = \sum_{m=0}^{L} c_{jm}(D,L)|D+m\rangle_a|m\rangle_b|L-m\rangle_c. \quad (18)$$

Finding the actual values of the eigenvalues $E_j(D,L)$ $[E_j(M)]$ and the coefficients $c_{jm}(D,L)$ $[c_{jm}(M)]$ we can construct the evolution operator describing the time evolution of the quantum system under consideration (we assume the exact resonance between modes). For example, in the case of the degenerate k-photon down conversion the evolution operator in the interaction picture reads

$$\hat{U}_k(t) = \exp(-i\hat{H}_k t) = \sum_{M=0}^{\infty}\sum_{j=0}^{[M/k]} e^{-itE_j(M)}|E_j(M)\rangle\langle E_j(M)|. \quad (19)$$

Obviously, applying this operator on any initial state $|\psi(0)\rangle$ we obtain the corresponding time evolution, i.e. $|\psi(t)\rangle = \hat{U}_k(t)|\psi(0)\rangle$. The given numerical treatment limits us (from the computational point of view) to the region of smaller intensities of the input states (up to a few hundreds input photons).

Due to the complex dynamics induced by (4), (6) the state vector $|\psi(t)\rangle$ cannot be, in general, for time $t>0$ factorized, which means that the modes under consideration become entangled during the time evolution. This entanglement leads to an increase of marginal entropies of the particular modes. These entropies can be used as a measure of the degree of entanglement between the modes [9]. The von Neumann entropy S_x [10] of the particular mode x is a measure of the purity of the mode and it enables us to quantify the degree of entanglement or correlation between the modes. The von Neumann entropy S_x is defined as (we use notation with the Boltzmann constant $k_B = 1$)

$$S_x = -\mathrm{Tr}_x\{\hat{\rho}_x \ln \rho_x\}, \quad (20)$$

where $\hat{\rho}_x$ is the reduced density matrix of the particular mode

$$\hat{\rho}_x = \mathrm{Tr}_{\{y\neq x\}}\hat{\rho}, \quad (21)$$

and $\hat{\rho}$ is the density matrix of the whole system. The trace is performed through all modes except x.

In the present paper we limit ourselves to the case when the initial state vector of the whole system [pump+down converted mode(s)] can be factorized, i.e. to those states that can be written as a product of pure states of the particular modes. Consequently, initially the marginal entropies of modes are equal to zero

$$S_x|_{t=0} = 0 \qquad \text{for all } x. \quad (22)$$

In addition, if we suppose that the quantum mechanical system is isolated from an environmental influence, then the entropy of the whole system at $t \geq 0$ equals to zero

$$S \equiv -\mathrm{Tr}\{\hat{\rho}\ln\rho\} = 0. \quad (23)$$

Using the Araki-Lieb theorem [10] we can find the following inequalities for marginal entropies of corresponding field modes in the processes under consideration:

$$|S_{ab} - S_c| \leq S \leq S_{ab} + S_c, \qquad \text{non-degenerate down conversion}$$
$$|S_a - S_c| \leq S \leq S_a + S_c, \qquad k\text{-photon down conversion.} \quad (24)$$

Moreover, with respect to Eq.(23), the following relations between marginal entropies are valid [9]

$$S_{bc} = S_a \;,\; S_{ac} = S_b \;,\; S_{ab} = S_c, \qquad \text{for non-degenerate down conversion}$$
$$S_a = S_c, \qquad \text{for } k\text{-photon down conversion.} \quad (25)$$

The relations (25) can be used to define the index of correlation [11] between modes involved in a nonlinear quantum-optical process. In particular, for processes (4), (6) the index of correlation can be expressed in a form:

$$I_{ab-c} = S_{ab} + S_c - S = 2S_c, \qquad \text{for non-degenerate down conversion}$$
$$I_{a-c} = S_a + S_c - S = 2S_c, \qquad \text{for } k\text{-photon down conversion.} \quad (26)$$

From relations (26) it directly follows that in the parametric approximation there is no entanglement between the pump and the down-converted modes (because $S_c = 0$). Nevertheless, in the non-degenerate case the signal and idler modes can still be highly correlated.

In some situations it is not straightforward to calculate the marginal entropy of the field mode under consideration even though the reduced density matrix $\hat{\rho}_x$ is known. Therefore it turns out that it is more convenient to use the linearized entropy [10] defined as

$$S_x^{corr} = 1 - \mathrm{Tr}_x\{\hat{\rho}_x^2\}, \quad (27)$$

instead of the von Neumann entropy S_x in the definition (26) of the index of correlation. The linear entropy S_x^{corr} equals to zero for any pure state and for any statistical mixture state $S_x^{corr} > 0$. Moreover S_x^{corr} represents a lower bound of the corresponding von Neumann entropy S_x [i.e. $S_x^{corr}(t) \leq S_x(t)$].

3. Degenerate k-photon down conversion

The k-photon down conversion process with the quantized pump mode is in the case of the exact resonance (i.e. $\omega_c = k\omega_a$) governed by the Hamiltonian (6). It is instructive to start our analysis of the dynamics described by the Hamiltonian (6) with the most simple case of $k=1$, corresponding to a resonant linear coupling between modes a and c.

3.1. The quantum linear coupler

The linear coupler [12] represents a special type of the degenerate down conversion described by the Hamiltonian (6) with $k=1$. The dynamics of the linear coupler can be solved in a closed form. This can be shown easily by noticing that the interaction Hamiltonian can be written in terms of the generators SU(2) Lie algebra [13]

$$\hat{H}_1 = \lambda_1(\hat{J}_+ + \hat{J}_-) \tag{28}$$

where

$$\hat{J}_+ = \hat{a}^\dagger\hat{c} \quad , \quad \hat{J}_- = \hat{a}\hat{c}^\dagger \quad , \quad \hat{J}_3 = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{c}^\dagger\hat{c}). \tag{29}$$

Using the disentanglement theorem for SU(2) algebra [13] we can express the evolution operator $\hat{U}_1(t)$ as

$$\begin{aligned}\hat{U}_1(t) &= \exp[-i\lambda_1 t(\hat{J}_+ + \hat{J}_-)] \\ &= \exp[-i\tan(\lambda_1 t)\hat{J}_+]\exp[-2\ln\cos(\lambda_1 t)\hat{J}_3]\exp[-i\tan(\lambda_1 t)\hat{J}_-].\end{aligned} \tag{30}$$

Let us assume the pump mode is initially (i.e. at time $t=0$) prepared in a coherent state $|\gamma\rangle_c$ and the down-converted mode is at $t=0$ in the vacuum state $|0\rangle_a$. Using (30) we obtain for the state vector of the total system $|\psi(t)\rangle$ at $t>0$ the expression

$$|\psi(t)\rangle = |\alpha(t)\rangle_a|\gamma(t)\rangle_c, \tag{31}$$

describing the signal and the pump as a product of two coherent states with amplitudes

$$\alpha(t) = -i\gamma\sin(\lambda_1 t) \quad , \quad \gamma(t) = \gamma\cos(\lambda_1 t). \tag{32}$$

The obtained results (31), (32) reveal several interesting features. Firstly, the two-mode linear coupler does not lead for the chosen initial state to entanglement between the pump and the signal [$S_a(t) = S_a^{corr}(t) = 0$]. Secondly, the coupled modes exchange their energy in a coherent manner, i.e., they remain in coherent states for any $t>0$. The pump mode can transfer completely its energy to the down-converted mode, i.e. at certain moments of the time evolution the pump becomes completely depleted and the parametric approximation cannot be adopted . Nevertheless at the early stages of the time evolution ($\lambda_1 t \ll 1$), when just a small fraction of the pump energy is transferred to the signal mode, the parametric approximation can be safely performed. This parametric approximation then changes the evolution operator (30) into the Glauber-Sudarshan displacement operator [14].

We have to stress here that even in the case of the linear coupler two modes under consideration can become entangled. The degree of the entanglement depends on the initial state of the pump–signal system. To illustrate this sensitivity of the entanglement on initial conditions we consider three cases. In the first case we will study consequences when the pump is initially prepared in some other state than a coherent state, namely in a squeezed vacuum state (at the input the signal is considered to be in the vacuum state). In two other cases we will assume again a strong coherent pump mode, but we will assume that the a mode (i.e. the signal) is also excited. In this case we will consider firstly the signal mode to be initially in a coherent state and secondly, we will analyze the situation when the signal is initially in a Fock (number) state.

Now, let us consider the case when the signal mode is prepared initially in the vacuum state and the pump mode is initially in a squeezed vacuum state [1], i.e.

$$|\psi(t=0)\rangle = |0\rangle_a|\eta\rangle_c, \qquad |\eta\rangle_c = \hat{S}_c(\eta)|0\rangle_c, \tag{33}$$

where $\hat{S}_c(\eta) = \exp[\eta(\hat{c}^\dagger)^2 - \eta^*\hat{c}^2]$ is the squeeze operator. Using the unitarity of the evolution operator (30) and the fact that $\hat{U}_1(t)|0\rangle_a|0\rangle_c = |0\rangle_a|0\rangle_c$, we can write the state vector $|\psi(t)\rangle$ which at $t=0$ takes the form (33) as:

$$|\psi(t)\rangle = \hat{U}_1(t)\hat{S}_c(\eta)|0\rangle_a|0\rangle_c = \hat{S}(\eta,t)|0\rangle_a|0\rangle_c, \tag{34}$$

where

$$\hat{S}(\eta,t) = \exp[\eta(\hat{c}^\dagger(t))^2 - \eta^*(\hat{c}(t))^2], \qquad \hat{c}(t) = \hat{U}_1(t)\hat{c}\hat{U}_1^\dagger(t). \tag{35}$$

Simplifying the expression (34) we can write $|\psi(t)\rangle$ as

$$\begin{aligned}|\psi(t)\rangle = \exp[&-\sin^2\lambda_1 t(\eta\hat{a}^{\dagger 2} - \eta^*\hat{a}^2) + \cos^2\lambda_1 t(\eta\hat{c}^{\dagger 2} - \eta^*\hat{c}^2) \\ &-i\sin 2\lambda_1 t(\eta\hat{a}^\dagger\hat{c}^\dagger + \eta^*\hat{a}\hat{c})]|0\rangle_a|0\rangle_c,\end{aligned} \tag{36}$$

from which it follows that $|\psi(t)\rangle$ cannot be factorized at $t>0$ and that the pump and the signal mode become entangled at $t>0$. This entanglement has its origin in the presence of the cross term $\exp(\xi\hat{a}^\dagger\hat{c}^\dagger - \xi^*\hat{a}\hat{c})$ in Eq.(36). This term is responsible not only for the entanglement between modes but it also gives rise to the two-mode squeezing [1, 11, 15] which can be obtained in the state (36). Moreover as seen from Eq.(36), the signal mode becomes also squeezed when it is initially linearly coupled to the squeezed pump. In other words, the fluctuations from the pump are transferred in a phase-sensitive manner to the signal mode. For details on statistical properties of two-mode states (36) see Ref.[16]. To evaluate an explicit expression for the degree of the entanglement between modes we rewrite the initial squeezed vacuum state (33) as a one-dimensional superposition of coherent states [17]:

$$|\eta\rangle_c = B_\xi \int_{-\infty}^{\infty} d\gamma\ \exp\left(-\frac{1-\xi}{2\xi}\gamma^2\right)|\gamma\rangle_c, \qquad \xi = \tanh\eta \tag{37}$$

where $B_\xi = (1-\xi^2)^{1/4}(2\pi\xi)^{-1/2}$ is the normalization constant. Due to Eq.(32) and superposition principle (the initial squeezed vacuum is represented as a continuous superposition of coherent states on the line) we obtain the time evolution in the form:

$$|\psi(t)\rangle = B_\xi \int_{-\infty}^{\infty} d\gamma\ \exp\left(-\frac{1-\xi}{2\xi}\gamma^2\right)|-i\gamma\sin\lambda_1 t\rangle_a|\gamma\cos\lambda_1 t\rangle_c. \tag{38}$$

If we analyze the time evolution at the initial stages, i.e. for times $\lambda_1 t \ll 1$ where we can use the approximation $|-i\gamma\sin\lambda_1 t\rangle_a|\gamma\cos\lambda_1 t\rangle_c \simeq |-i\gamma\lambda_1 t\rangle_a|\gamma\rangle_c$, it is then clear that the modes becomes strongly entangled. This is a consequence of the fact that the pump mode was initially prepared in a highly nonclassical state. In a straightforward way we can derive the expression for the entanglement parameter

$$S_c^{corr} = 1 - \sqrt{\frac{2-2\xi^2}{2-\xi^2(1+\cos 4\lambda_1 t)}} \tag{39}$$

which reaches its first maximum at $\lambda_1 t = \pi/4$ and is equal to $1-\sqrt{1-\xi^2}$. In other words, higher the degree of squeezing of the initial pump mode is higher the entanglement between the modes at $t > 0$. In general, it is true that the maximum degree of the entanglement between the modes can be observed at the time moment $\lambda_1 t = \pi/4$ for any initial pure states of the modes.

Now we will change the input statistics of the signal mode. If the signal mode is initially prepared in a coherent state $|\alpha\rangle_a$ and the pump mode is in the coherent state $|\gamma\rangle_c$, i.e.

$$|\psi(0)\rangle = |\alpha\rangle_a|\gamma\rangle_c, \tag{40}$$

then, using the explicit expression (30) for the time evolution operator $\hat{U}_1(t)$, we find that at $t > 0$ the state vector of the pump–signal system can be expressed as a product of two coherent states

$$|\psi(t)\rangle = |\alpha(t)\rangle_a|\gamma(t)\rangle_c, \tag{41}$$

with the amplitudes

$$\begin{aligned} \alpha(t) &= \alpha\cos(\lambda_1 t) - i\gamma\sin(\lambda_1 t), \\ \gamma(t) &= \gamma\cos(\lambda_1 t) - i\alpha\sin(\lambda_1 t). \end{aligned} \tag{42}$$

From Eq.(41) it follows that the two modes (the pump and the signal) are disentangled for any $t > 0$ providing they are prepared initially in coherent states $|\alpha\rangle_a$ and $|\gamma\rangle_c$, respectively. Nevertheless, as it is seen from Eq.(42), the parametric approximation can consistently be applied only in the case when the amplitude of the initial pump mode is much larger than the amplitude of the signal mode ($|\gamma| \gg |\alpha|$). Simultaneously, this approximation is valid only for times for which $\lambda_1 t \ll 1$ (i.e. the pump depletion can be neglected). Up till now we have considered the c mode as the pump because its initial intensity is assumed to be much higher then that of the a mode. In fact a better choice is to identify the pump with that mode which in the initial stages of the time evolution losses photons, i.e. $d\langle\hat{n}_{pump}\rangle/dt < 0$. From this point of view the role of the pump for coherent inputs in both modes depends on the relation between their phases. From Eq.(42) it is seen that if $\phi_a - \phi_c = -\pi/2$, then in the initial moments the energy is transferred from the c-mode (as the pump) to the a-mode, but for the phase relation

$\phi_a - \phi_c = \pi/2$ the situation is opposite and the a-mode functions as the pump (for any nonzero amplitude even smaller than an amplitude of the c mode).

As a third example, in which we illustrate the sensitiveness of the degree of entanglement between two modes of the linear coupler on initial conditions, we assume the pump mode to be initially prepared in the coherent state $|\gamma\rangle_c$ and the signal mode in the Fock state $|N\rangle_a$ with N photons, i.e.

$$|\psi(0)\rangle = |N\rangle_a|\gamma\rangle_c. \tag{43}$$

To find a compact expression for the time evolution of the initial state (43) we expand the number state $|N\rangle_a$ in terms of coherent states [18], i.e.

$$|\psi(t=0)\rangle = |N\rangle_a|\gamma\rangle_c = B_N(\alpha)\int_0^{2\pi} d\varphi \exp(-iN\varphi)|\alpha e^{i\varphi}\rangle_a \otimes |\gamma\rangle_c, \tag{44}$$

where α can be chosen arbitrary and $B_N(\alpha) = \sqrt{N!}\,\alpha^{-N}e^{\alpha^2/2}/(2\pi)$ is the corresponding normalization constant (for concreteness one can choose $\alpha = \sqrt{N}$). According to Eqs.(41), (42) and using the superposition principle we obtain:

$$|\psi(t)\rangle = B_N(\alpha)\int_0^{2\pi} d\varphi \exp(-iN\varphi) \times$$

$$|\alpha e^{i\varphi}\cos(\lambda_1 t) - i\gamma\sin(\lambda_1 t)\rangle_a|\gamma\cos(\lambda_1 t) - i\alpha e^{i\varphi}\sin(\lambda_1 t)\rangle_c. \tag{45}$$

From Eq.(45) it is clear that the state vector $|\psi(t)\rangle$ cannot be factorized and that the two modes under consideration become entangled, i.e. $S_a^{corr}(t) > 0$ for $t > 0$. It can be shown that the entanglement parameter S_c^{corr} reads

$$S_c^{corr} = 1 - \sum_{m=0}^{N}\left[\binom{N}{m}(\cos^2\lambda_1 t)^{N-m}(\sin^2\lambda_1 t)^m\right]^2 \tag{46}$$

and is independent on the initial amplitude γ of the coherent pump. As we have noted earlier maximum entanglement [in this case equal to $1-4^{-N}\sum_m \binom{N}{m}^2$] can be observed at time $\lambda_1 t = \pi/4$. On the other hand, during the first instants of the time evolution, i.e. for $\lambda_1 t \ll 1$, the modes do not entangle, providing $\gamma \gg N$. In this case the state vector (45) can be approximately written in the form

$$|\psi(t)\rangle \approx \exp[-i\lambda_1\gamma t(\hat{a}^\dagger + \hat{a})]|N\rangle_a|\gamma\rangle_c, \tag{47}$$

i.e. under certain conditions the parametric approximation can still be adopted. From Eq.(47) it follows that the down-converted mode evolves into a displaced number state [19].

We can conclude that as soon as one of the two modes of the linear coupler is initially prepared in a nonclassical state, the modes become entangled at $t > 0$ and the parametric approximation cannot be adopted except the region $\lambda_1 t \ll 1$ for a coherent pump.

We have already demonstrated that the fields in the linear coupler can entangle during the time evolution. However it can be shown that there are moments (independent on the initial state of the system) at which the field modes become disentangled. To show this we write the initial state of the system in the form

$$|\psi(0)\rangle = |\psi_1\rangle_a|\psi_2\rangle_c = \hat{O}_a(\hat{a},\hat{a}^\dagger)|0\rangle_a\hat{O}_c(\hat{c},\hat{c}^\dagger)|0\rangle_c, \tag{48}$$

where $\hat{O}_x$ are operators relating the actual initial state $|\psi_j\rangle_x$ to the vacuum $|0\rangle_x$. We know that the relations for the transformation of the operators $\hat{a}$ and $\hat{c}$ under the action of the evolution operator (30) are:

$$\begin{aligned} \hat{U}_1\hat{a}\hat{U}_1^\dagger &= \hat{a}\cos(\lambda_1 t) + i\hat{c}\sin(\lambda_1 t) \\ \hat{U}_1\hat{c}\hat{U}_1^\dagger &= i\hat{a}\sin(\lambda_1 t) + \hat{c}\cos(\lambda_1 t). \end{aligned} \tag{49}$$

From these equations we can find three important time moments. First, at $\lambda_1 t = 2\pi$, the state of the system is restored, i.e. the dynamics of the linear coupler is periodic in time. The other two time moments are $\lambda_1 t = \pi$ and $\lambda_1 t = \pi/2$. At the time $\lambda_1 t = \pi$ the modes are disentangled and they differ from the initial state only by phase shifts. At the moment $\lambda_1 t = \pi/2$ the modes are disentangled as well, but they interchange their statistical properties completely. Such a behavior is typical only for the linear coupler and is absent in the case of the down-conversion processes with $k > 1$.

The linear coupler, which represents a lossless beam splitter, can be used also to disentangle states [20]. For example, if a two-mode squeezed vacuum state is sent into a lossless symmetrical beam splitter then two single-mode squeezed vacuum states emerge (the output of the 50-50 beam splitter is equivalent to the state in the linear coupler after $\lambda_1 t = \pi/4$). The initial correlated state is transformed by this device into two identical uncorrelated single-mode states which can be used to measure an optical phase shift with accuracy inversely proportional to the intensity of the measured single-mode squeezed vacuum state [20]. In the improved scheme one can use instead of the two-mode vacuum a displaced two-mode vacuum as an input correlated state which is transformed by the linear coupler (after $\lambda_1 t = \pi/4$) into two displaced single-mode vacuum states. This kind of disentanglement is a consequence of the periodicity of the lossless linear coupler, which is fully absent in the case of multi-photon down conversion processes.

3.2. Signal-pump entanglement in two-photon down conversion

In the previous section we have shown that for coherent input states of the linear coupler the pump-signal state vector can be factorized for any $t > 0$, i.e. the pump and the signal remain in the pure state during the time evolution. This preservation of the purity of the pump and the signal mode is a very exceptional property of the linear coupler. In this section we show that in the two-photon down-conversion process described in the interaction picture by the Hamiltonian

$$\hat{H}_2 = \lambda_2(\hat{a}^{\dagger 2}\hat{c} + \hat{a}^2\hat{c}^\dagger) \tag{50}$$

the situation is different. First of all, we should stress that dynamics of the quantum mechanical system corresponding to the Hamiltonian (50) cannot be described in an analytically closed form [7], but has to be studied numerically. For details of our numerical approach we refer the reader to the papers [8].

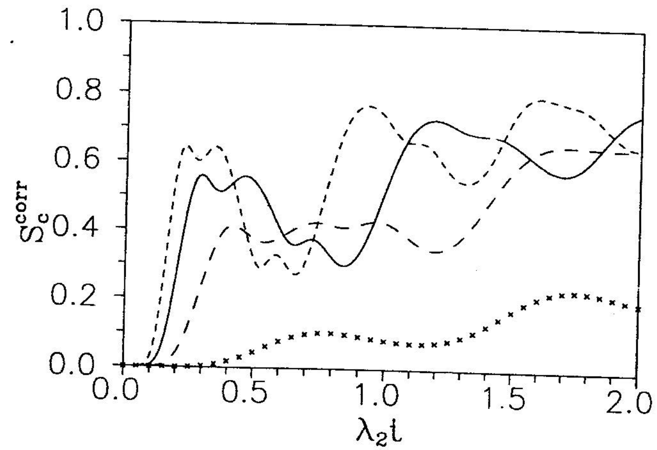

Fig. 1. The short time evolution of the entanglement parameter $S_c^{corr}(t)$ for the two-photon down-conversion with the initial state $|0\rangle_a|\gamma\rangle_c$. The intensities of the initial coherent field were set $n_c = 1$ (× points), $n_c = 9$ (long-dashed curve), $n_c = 25$ (solid curve) and $n_c = 49$ (short-dashed curve). The disentangled period becomes shorter for higher intensities.

Let us assume the signal mode initially in the vacuum state $|0\rangle_a$ and the pump mode in the coherent state $|\gamma\rangle_c$, i.e. $|\psi(0)\rangle = |0\rangle_a|\gamma\rangle_c$. With this initial state the signal and the pump in the linear coupler remain in pure states for $t > 0$. In the two-photon down converter the modes become entangled. Their entanglement parameter S_c^{corr} [see Eq.(27)] starts to be positive at $t > 0$ and none of the modes will evolve into a pure state again. Nevertheless, for a short range of time [i.e. for times $t < (\lambda_2|\gamma|)^{-1}$] the entanglement between the modes is still very small and approximately equal to zero. During this time interval the parametric approximation can be adopted. To see this we perform a formal expansion of the evolution operator $\hat{U}_2(t)$ given in the interaction picture by expression

$$\hat{U}_2(t) = \exp(-it\hat{H}_2) = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\left(\hat{H}_2\right)^n \tag{51}$$

and we keep only two first terms in this expansion, i.e.

$$\hat{U}_2(t) \approx \hat{1} - it\hat{H}_2. \tag{52}$$

Using (52) we find the approximate solution for the state vector of the pump-signal system in the form:

$$|\psi(t)\rangle = \hat{U}_2(t)|\psi(t=0)\rangle \approx [|0\rangle_a - i\sqrt{2}\lambda_2\gamma t|2\rangle_a]|\gamma\rangle_c = |\psi(t)\rangle_a|\psi(t)\rangle_c. \tag{53}$$

This approximation can be adopted for times $t < t_c$, where

$$t_c = \frac{1}{\sqrt{2}\lambda_2|\gamma|}. \tag{54}$$

It is obvious now that for these times the entanglement parameter S_c^{corr} is equal to zero (as the state vector (53) can be written in a factorized form) and the parametric approximation can be safely used.

In Fig. 1 we plot the time evolution of the purity parameter for several values of the initial intensity of the pump mode. From this figure it follows that higher the initial intensity of the pump more rapidly the entanglement parameter increases and shorter is the time during which S_c^{corr} can be approximated by zero [in accordance with Eq.(54)].

The increase of a mutual correlation (entanglement) between the two modes is accompanied with a significant changes in statistical properties of both the pump and the signal modes. To study these changes we evaluate the Q-function, the photon number distribution and a degree of squeezing corresponding to the signal and the pump mode.

The Q-function (quasidistribution in the phase space) is defined as [21]

$$Q_x(\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}_x|\beta\rangle, \tag{55}$$

the marginal photon number distribution (PND) of each mode is given by the relation

$$P_x(n) = \langle n|\hat{\rho}_x|n\rangle, \tag{56}$$

where the reduced density operator $\hat{\rho}_x$ (with $x = a, c$) is defined by Eq.(21). To measure the quadrature squeezing we utilize two quadrature operators [22] for each mode (i.e. $x = a$ or $x = c$)

$$\hat{X}_x = \frac{\hat{x}e^{-i\phi_x} + \hat{x}^\dagger e^{i\phi_x}}{2}, \qquad \hat{Y}_x = \frac{\hat{x}e^{-i\phi_x} - \hat{x}^\dagger e^{i\phi_x}}{2i}. \tag{57}$$

The degree of squeezing can be defined as

$$S_x^X = 4\langle(\Delta\hat{X}_x)^2\rangle - 1, \qquad S_x^Y = 4\langle(\Delta\hat{Y}_x)^2\rangle - 1, \tag{58}$$

where $\langle(\Delta\hat{X}_x)^2\rangle = \langle\hat{X}_x^2\rangle - \langle\hat{X}_x\rangle^2$, $\langle(\Delta\hat{Y}_x)^2\rangle = \langle\hat{Y}_x^2\rangle - \langle\hat{Y}_x\rangle^2$. When S_x^X (S_x^Y) becomes negative quadrature squeezing appears and $S_x^X = -1$ ($S_x^Y = -1$) corresponds to 100% squeezing.

To describe statistical properties of the modes in more detail we evaluate also the deviation of the given state from the minimum uncertainty state (MUS), i.e. we evaluate the parameter

$$u_x = \langle(\Delta\hat{X}_x)^2\rangle\langle(\Delta\hat{Y}_x)^2\rangle - \frac{1}{16}. \tag{59}$$

The u_x parameter equals to zero only for MUS, i.e. states minimizing the uncertainty relations.

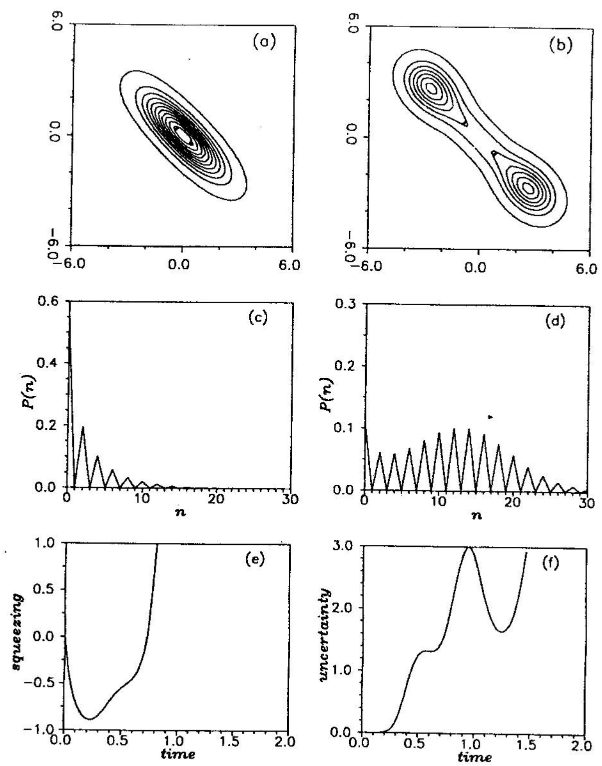

Fig. 2. Statistical parameters of the down-converted mode for $n_c = 9$: (a) the Q_a function for $\lambda_2 t = 0.2$; (b) the Q_a function for $\lambda_2 t = 0.5$; (c) the photon number distribution at $\lambda_2 t = 0.2$; (d) the photon number distribution at $\lambda_2 t = 0.5$; (e) the time evolution of the squeezing parameter and (f) the time evolution of the MUS parameter.

In Fig. 2 we plot various parameters describing statistical properties of the signal mode obtained via two-photon down-conversion with the pump intensity γ^2 equal to 9 and for two values of the interaction time, $\lambda_2 t = 0.2$ and $\lambda_2 t = 0.5$ As seen from Fig. 1 the entanglement parameter at $\lambda_2 t = 0.2$ is approximately equal to zero (i.e. the signal mode is in a pure state), while S_c^{corr} at $\lambda_2 t = 0.5$ is significantly greater than zero. In Fig. 2a contour plots of the Q-function of the signal mode at $\lambda_2 t = 0.2$ are shown. We see that the initial circle contours corresponding to the vacuum state are transformed into elliptical (squeezed) contours. The photon number distribution (see Fig.2c) exhibits significant oscillations. Taking into account that at $\lambda_2 t = 0.2$ the degree of squeezing in the signal mode is very large (see Fig. 2e) and the fact that the signal mode is in a pure state which simultaneously is a MUS (see Fig. 2f) we can conclude that under the given conditions the squeezed vacuum is produced in the signal mode,

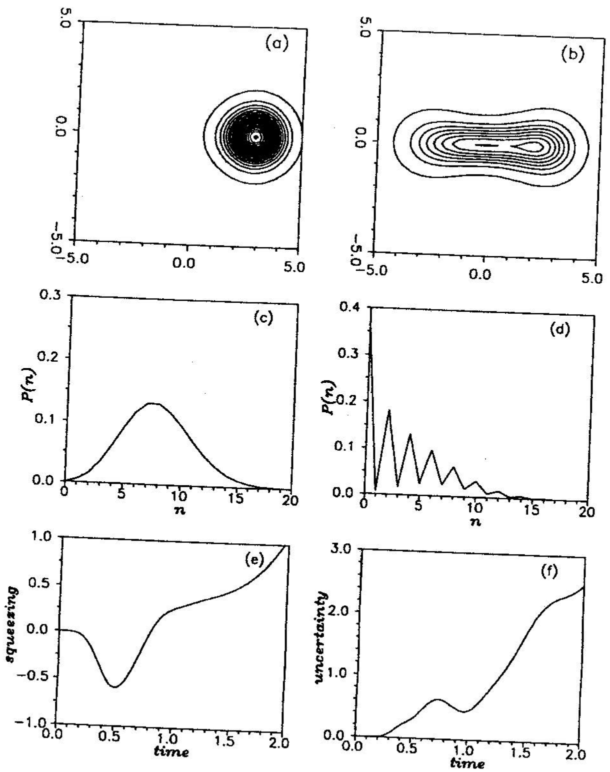

Fig. 3. The same as *Fig.2* but for the pump mode.

which means that during time period for which $t < t_c$ [see Eq.(54)] the state vector $|\psi(t)\rangle$ of the pump-signal system can be written approximately in a factorized form

$$|\psi(t)\rangle \approx \hat{S}_a(-i\gamma t)|0\rangle_a|\gamma\rangle_c, \tag{60}$$

where $\hat{S}_a(\eta)$ is the squeezing operator. From here it is clear, why the two-photon down converter can serve as device in which squeezed states can be produced.

If we analyze statistical properties of the pump mode in the process under consideration at $\lambda_2 t = 0.2$ we find that the pump mode at this time is approximately in a coherent state, i.e. the Q-function is represented by circle contours (see Fig. 3a), the photon number distribution is Poissonian (Fig. 3c), there is no squeezing exhibited by the pump (Fig. 3e) and finally the pump mode is in the MUS (Fig. 3f). On the other hand at $\lambda_2 t = 0.5$ the quantum statistical properties of the pump mode are significantly changed by the back action of the signal mode. First of all, the pump is not in a pure state at this moment (Fig. 1), the Q-function is "deformed" (Fig. 3b), and the photon number distribution exhibits oscillations (Fig. 3d). Moreover the pump mode exhibits large degree of squeezing (Fig. 3c). Needless to say the pump mode is not in the MUS at this moment (Fig. 3f). Obviously under this circumstances one cannot adopt the parametric approximation.

Except the above discussed non-classical properties of the modes we should also mention an interesting effect related to the energy exchange between modes, namely the fact that the pump mode cannot be fully depleted in the case of the initial state $|0\rangle_a|\gamma\rangle_c$. Moreover, for $\gamma \gg 1$ the pump mode cannot lose more than 2/3 of its initial intensity. The energy transfer between the modes in the process of multi-photon down conversion was studied in [24]. Let us briefly discussed the origin of the "imperfection" of the energy transfer from the pump to the signal a mode which is initially empty. The time evolution of the mean photon number for the initial state $|\psi(t=0)\rangle = |0\rangle_a|\gamma\rangle_c$ (due to the mutual orthogonality of subspaces $\mathcal{H}_M$) can be written in the form

$$\bar{n}_c(t) = \langle\psi(t)|\hat{n}_c|\psi(t)\rangle = \mathrm{e}^{-|\gamma|^2}\sum_{N=0}^{\infty}\frac{|\gamma|^{2N}}{N!}\bar{n}_c(N;t), \tag{61}$$

where

$$\bar{n}_c(N;t) = {}_c\langle N|_a\langle 0|\hat{U}_2^\dagger(t)\hat{n}_c\hat{U}_2(t)|0\rangle_a|N\rangle_c \tag{62}$$

represents the time evolution of the mean photon number for the input state $|0\rangle_a|N\rangle_c$ (the pump prepared in the number state). In the Fig. 4 we plot overlaps of the state $|0\rangle_a|N\rangle_c$ with eigenvectors on the corresponding dynamically independent Hilbert subspace $\mathcal{H}_M$ $(M = 2N)$, namely the probabilities $P(j) = |\langle E_j(N)|0\rangle_a|N\rangle_c|^2$ where the eigenstates $|E_j(N)\rangle$ are given by Eq.(16) [the eigenvalues are labeled by N instead of $M = 2N$]. We plot probabilities P(j) for N even and for N odd separately, because they differ by the number of eigenstates which "form" the given input state. It is seen that for N even the state $|0\rangle_a|N\rangle_c$ has significant overlap with three eigenstates (nevertheless a dominant overlap is only with one of them) while for N odd the state $|0\rangle_a|N\rangle_c$ has the overlap with two eigenstates, i.e. the time evolution $|\psi(N;t)\rangle$ of the initial state $|0\rangle_a|N\rangle_c$ can be approximately written as

$$|\psi(N;t)\rangle \sim \begin{cases} c_{0,0}(N)|E_0(N)\rangle + \\ c_{1,0}(N)\mathrm{e}^{-itE_1(N)}|E_1(N)\rangle + c_{1,0}(N)\mathrm{e}^{itE_1(N)}|-E_1(N)\rangle, & N \text{ even} \\ \\ c_{0,0}(N)\mathrm{e}^{-itE_0(N)}|E_0(N)\rangle + c_{0,0}(N)\mathrm{e}^{itE_0(N)}|-E_0(N)\rangle, & N \text{ odd} \end{cases} \tag{63}$$

Here lies the origin of the "inhibition" of the full energy transfer from the pump mode into the empty a mode. For N even the dominant overlap with the eigenstate $|E_0(N)\rangle$ directly suggests a significant "trapping" of the energy in this eigenstate with zero eigenenergy. For N odd the state $|0\rangle_a|N\rangle_c$ consists of two eigenstates $|E_0(N)\rangle$, $|E_1(N)\rangle$ which have up to the sign the same decomposition into a Fock state basis, i.e. $|E_0(N)\rangle = \sum_m c_{0m}(N)|2m, N-m\rangle$, $|E_1(N)\rangle = \sum_m(-1)^m c_{0m}(N)|2m, N-m\rangle$ and their eigenenergies are realted as $E_1(N) = -E_0(N)$. The eigenstates $|E_0(N)\rangle$ and $|E_1(N)\rangle$ are orthogonal which means that $\sum_m(-1)^m[c_{0m}(N)]^2 = 0$. This relation also suggests that the matrix element $\langle E_1(N)|\hat{n}_c|E_0(N)\rangle = \sum_m m(-1)^2 c_{0m}^2$ is a small number [at

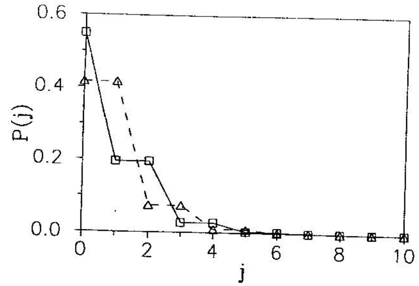

Fig. 4. The overlap probabilities $P(j) = |c_{j0}(N)|^2$ of the Fock state $|0\rangle_a|N\rangle_c$ with eigenvectors $|E_j(N)\rangle$ for $N = 25$ (line with triangles) and $N = 26$ (line with squares). Eigenstates are labeled in order of increasing absolute values of eigenenergies E_j.

least in comparison with the element $\langle E_0(N)|\hat{n}_c|E_0(N)\rangle$, for details see [23]]. From here it follows that in the basis of the eigenstates $|E_j(N)\rangle$ there is a reduced exchange of energy between two dominant eigenstates. As a result the pump mode for $N \gg 1$ cannot transfer into signal mode more than 2/3 of its initial photons. Such "imperfection" of down conversion leads to important consequences. In particular, it was shown by Hillery and co-workers [6] that to obtain the signal mode more sub-Poissonian than the pump mode (i.e. $q_a < q_c$), the pump mode has to be considerably depleted, namely it has to loose more than 2/3 of its initial intensity. Due to the "imperfect" down conversion the Mandel's q_a-parameter of the signal mode will be always higher than that of the pump mode. Generalization of the obtained results to the case of the initial coherent state of the pump mode is straightforward: the resulting mean photon number $\hat{n}_c(t)$ is given as a sum of weighted mean photon numbers $n_c(N;t)$ from particular subspaces [see Eq.(61)]. Moreover, the degree of the squeezing of the signal mode depends on the intensity of this mode. Because the intensity of the signal is bounded by the amount of the energy transferred from the pump we can expect the maximum degree of squeezing to be bounded as well [see [25]]. In more details these question were analyzed in [24] where it was shown that for $\gamma \gg 1$ the efficiency of the down conversion is less than 2/3 (the pump cannot lose more than 2/3 of its initial intensity) and it was predicted that for the pump mode prepared in the highly excited squeezed vacuum state this efficiency is even smaller. We will confirm this effect using numerical calculations. We will also show that the "efficiency" of the energy transfer in down conversion process can be improved if initially the signal mode is excited. The presence of the signal photons will "stimulate" a more intensive transfer of the energy from the pump mode to the signal mode.

Till now we have focused our attention mainly on the short time scale which is relevant from the point of view of possible production of non-classical states and effects. In this limit we can neglect energy losses due to dissipative coupling of the signal-pump system to an environment because we can expect that phenomenological damping constants Γ_x ($x = a, c$) are much smaller than $\lambda_2\gamma$, i.e. a photon can be lost into environment only after many oscillations of the mean photon number which characterize

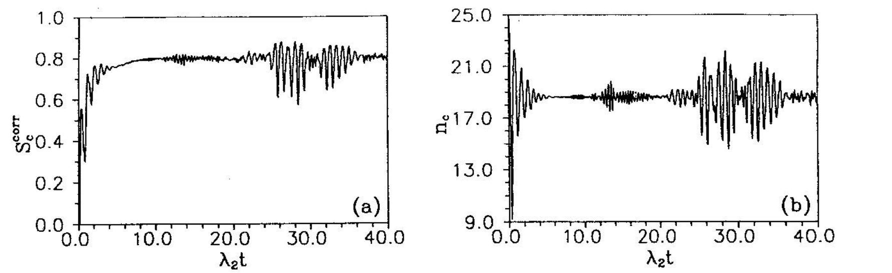

Fig. 5. (a) The long time evolution of the entanglement parameter $S_c^{corr}(t)$ and (b) the evolution of the mean photon number for the initial state $|0\rangle_a|\gamma\rangle_c$. The initial intensity was set $n_c = 25$.

the energy exchange between the modes.

Let us now turn our attention to a long time evolution of the modes in the two-photon down converter, when the pump is initially prepared in a coherent state and the signal mode is initially empty, i.e. $|\psi(t=0)\rangle = |0\rangle_a|\gamma\rangle_c$. On the long-time scale the degree of the entanglement S_c^{corr} between the pump and the signal modes exhibits a purely quantum effect of "spontaneous disentanglement" (which for the first time was reported by Phoenix and Knight in the framework of the Jaynes-Cummins model [26]). From Fig. 5a it is clearly seen that at certain moments the entanglement parameter becomes significantly reduced. Comparing the time evolution of $S_c^{corr}(t)$ and the time evolution of the mean photon number of signal photons (Fig. 5b) we find that the decrease of the entanglement parameter is accompanied with the "revival" in the mean photon number.

In what follows we will estimate the revival-time of the mean photon number $\langle \hat{n}_c \rangle$. We start our estimation rewriting the time evolution of the mean photon number for an initial Fock state $|0\rangle_a|N\rangle_c$ using the matrix elements of $\hat{n}_b$ in the basis of the eigenstates $|E_j(N)\rangle$:

$$\bar{n}_c(N;t) = \sum_i [c_{j0}(N)]^2 \langle E_j(N)|\hat{n}_b|E_j(N)\rangle + \sum_{i<j} c_{i0}(N)c_{j0}(N)\langle E_i(N)|\hat{n}_b|E_j(N)\rangle \cos\left([E_i(N) - E_j(N)]t\right) \quad (64)$$

where $c_{j0}(N) = \langle E_j(N)|0\rangle_a|N\rangle_c$ is an overlap of the initial state with the given eigenvector [Eq.(16)]. The mean photon number is given as a superposition of the finite number $[N(N+1)/2]$ of periodic contributions (note that the time dependence enters the mean photon number only via cosine term). Nevertheless, as was shown above [see Eq.(jj14)], for even $N = 2l$ there are only three eigenstates, namely $|E_0(2l) = 0\rangle$, $|E_1(2l)\rangle$ and $|E_2(2l)\rangle \equiv |-E_1(2l)\rangle$, which have a significant overlap with the input state and therefore the main contribution to the time dependence of $n_c(N = 2l;t)$ comes from terms with $\cos[E_1(2l)t]$ and $\cos[2E_1(2l)t]$. For N odd (i.e. $N = 2l+1$) there is only one important time-dependent contribution which is proportional to $\cos[2E_1(2l+1)t]$.

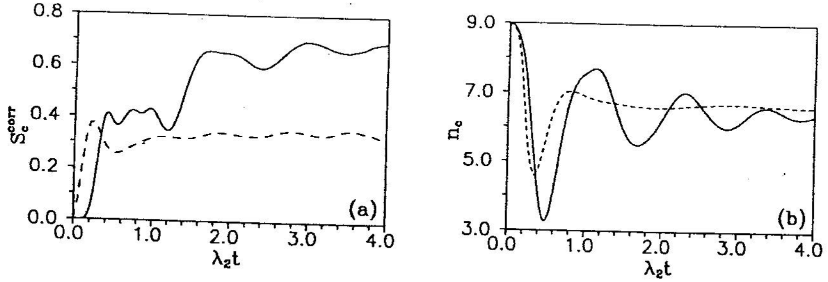

Fig. 6. (a) The short time evolution of the entanglement parameter $S_c^{corr}(t)$ for the pump mode initially prepared in the squeezed vacuum state (dashed line) and the coherent state (solid line). In both cases initially $n_c = 9$. (b) The time records of corresponding mean photon numbers.

According to Eq.(61) in order to obtain the mean photon number for an initially coherent pump (i.e. the input state is considered to be $|0\rangle_a|\gamma\rangle_c$) we have to superpose these three types of cosine terms with Poissonian weight factors (probability to find an input within a given subspace). The revival appears when two neighboring cosine terms get a mutual phase shift of 2π. If γ is large enough then we can limit ourselves to those terms with $N \approx |\gamma|^2$. Consequently we get three revival time estimations

$$\begin{aligned} 2\left[E_1(2\bar{l}+2) - E_1(2\bar{l})\right] t_{R_1} &= 2\pi, \\ \left[E_1(2\bar{l}+2) - E_1(2\bar{l})\right] t_{R_2} &= 2\pi, \\ 2\left[E_0(2\bar{l}+1) - E_0(2\bar{l}-1)\right] t_{R_3} &= 2\pi. \end{aligned} \qquad (65)$$

A numerical check reveals that $t_{R_2} \approx t_{R_3}$ and so we are left with main revival time estimation

$$t_{R_2} = \frac{2\pi}{E_1(2\bar{l}+2) - E_1(2\bar{l})} = 2t_{R_1} \qquad (66)$$

which is in very good agreement with numerical calculations (see Fig. 5).

In the preceding part we assumed the pump mode to be prepared in a coherent state and the empty signal. We found that the modes become entangled and that the parametric approximation cannot be applied beyond $t_c \approx (\lambda_2\gamma)^{-1}$. This behaviour seems to be quite different from that of the linear coupler for the same input (when no entanglement between two linearly coupled modes is established). Nevertheless, the applicability of the parametric approximation in both cases is proportional to the time scale of the maximum energy exchange between modes. In the linear coupler the modes exchange fully their initial intensities at $t = \pi/\lambda_1$ [compare with Eq.(49)] while for the two-photon down conversion the time of the maximum pump depletion is proportional to $(\lambda_2\gamma)^{-1}$.

Analogously to the previous section we will now analyze the influence of the initial statistics of the modes on the of entanglement between the pump and the signal mode. We will study the same cases as those for the linear coupler. Firstly, it is of interest

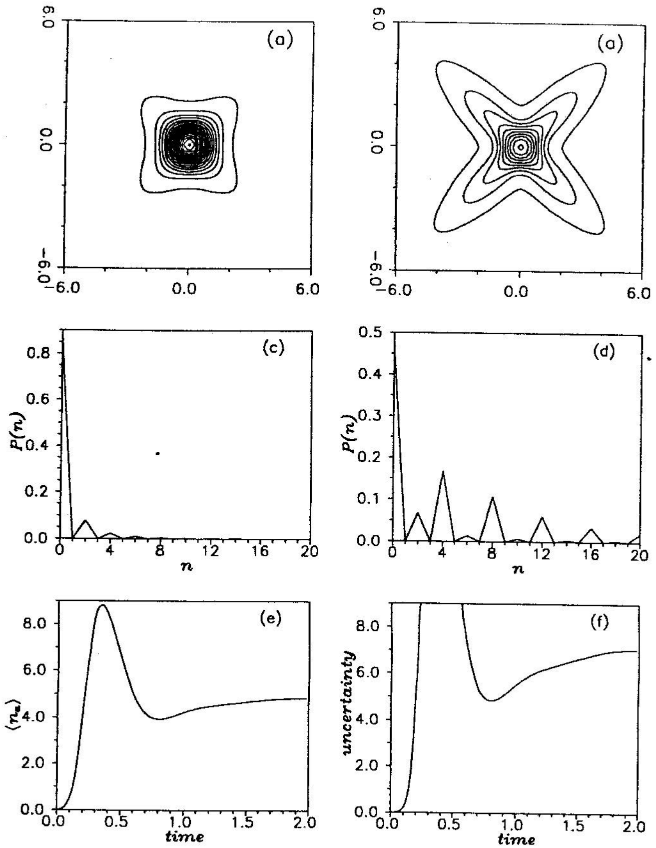

Fig. 7. Statistical properties of the down-converted mode for the initially squeezed pump with $n_c = 9$: (a) the Q_a function at $\lambda_2 t = 0.1$; (b) the Q_a function at $\lambda_2 t = 0.6$; (c) the photon number distribution at $\lambda_2 t = 0.1$; (d) the photon number distribution at $\lambda_2 t = 0.6$; (e) the time evolution of the mean photon number and (f) the evolution of the MUS parameter. Time is scaled as $1/\lambda_2$.

to analyze how the mutual correlations and statistics of the modes under consideration are changed when the pump mode is initially prepared in a highly nonclassical state. In particular, if we assume the pump mode to be initially prepared in a squeezed vacuum state $|\xi\rangle_c$ and the signal mode in the vacuum state $|0\rangle_a$ [see (33)], then we can find that in this case the strong entanglement between the modes is established much faster than in the case when the pump is initially prepared in a coherent state (see Fig. 6a). Simultaneously we should stress that the signal mode in the process under consideration possesses very interesting nonclassical behaviour which is reflected by the shape of Q-function (Figs. 7a,b). Hillery and co-workers showed in [6] that rotational symmetry in phase space of the signal mode is twice that of the pump-mode state. Therefore the Q-

function has a symmetrical four-fold structure and the corresponding PND (Figs. 7c,d) exhibits oscillations which reflects this four-fold symmetry. From Figs.7 it follows that in the given case the signal mode evolves into a multicomponent superposition state [27]. We present in Fig. 8 pictures illustrating statistical properties of the pump mode. It is seen that the pump prepared in a highly squeezed vacuum losses its nonclassical properties slowly in comparison with the time scale of the highest pump depletion (i.e. the scale of oscillations of the energy exchange). Fig. 6b confirms the prediction (see above) that the energy transfer from the squeezed-vacuum pump to the signal mode is less intensive than in the case of a coherent pump (with the same initial intensity). Moreover, in the case of the squeezed pump the "stationary" value of the entanglement (i.e. the value of S_c^{corr} for large times) is approximately equal to one half of the value of the entanglement in the case when the pump is initially in a coherent pump.

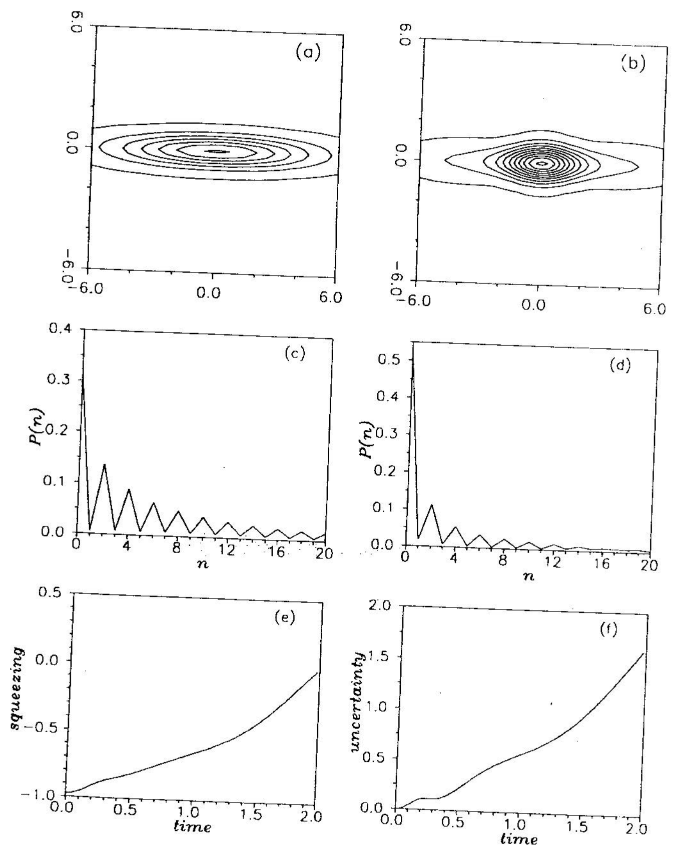

Fig. 8. (a)-(d) Statistical properties of the initially squeezed pump mode plotted analogously as in *Fig.7*; (e) the time evolution of the squeezing parameter. Time is scaled as $1/\lambda_2$.

From the above we can conclude that photon statistics of the pump and the signal mode during the first instants of the time evolution is very sensitive with respect to statistics of input states of the two-photon down converter.

In the preceding part it was shown that a nonclassical pump input leads to a more rapid establishment of the entanglement between modes in comparison with a coherent pump input. In what follows we will investigate how the variation of statistics of the signal mode at $t = 0$ changes the character of the output states and the entanglement between modes. We will study the case when both modes are prepared in coherent states $|\alpha\rangle_a$ and $|\gamma\rangle_c$, respectively, and the intensity of the c mode is stronger than the intensity of the a mode, i.e. $|\gamma| \gg |\alpha|$. Such input state can approximate (for high enough intensities) the classical initial conditions with two input laser beams. It can be expected that such input leads to a production of squeezed coherent states in the a mode at least at initial stages of time evolution. We should stress that although we call the c mode as the pump, for non-empty a mode such identification can be misleading even when the c mode is much more excited than the a mode. As in the case of the linear coupler, we define the pump as that mode from which the energy is transferred to the other mode at the first instants of the time evolution. To illustrate this definition we assume the initial state vector to be

$$|\psi(0)\rangle = |\alpha\rangle \otimes |\gamma\rangle = ||\alpha|e^{i\varphi_a}\rangle \otimes ||\gamma|e^{i\varphi_c}\rangle. \tag{67}$$

From the expression for the first derivative of the mean photon number in the c mode, when the initial state vector is given by (67) :

$$\left.\frac{d\bar{n}_c}{dt}\right|_0 = 2\lambda_2|\alpha|^2|\gamma|\cos(2\varphi_a - \varphi_c - \pi/2) \tag{68}$$

it is evident that for $2\varphi_a - \varphi_c = \pi/2$ there is a gain in the c mode during the first stages of the time evolution even though the intensity of the a mode is negligible in comparison with the excitation of the c mode. For $2\varphi_a - \varphi_c = -\pi/2$ we obtain a gain in the a mode, i.e. a typical down conversion regime. In the intermediate regime when $2\varphi_a - \varphi_c = 0$, we have to calculate the second derivative of the mean photon number (which does not depend on the phase relation between the modes) to find which mode is amplified during the initial moments of the time evolution:

$$\left.\frac{d^2\bar{n}_c}{dt^2}\right|_0 = 2\lambda_2^2(|\alpha|^4 - 4|\alpha|^2|\gamma|^2 - 2|\gamma|^2). \tag{69}$$

From this expression it is clear that a gain in the c mode for the phase relation $2\varphi_a - \varphi_c = 0$ is obtained only when the intensity of the a mode is at least four-times higher then that of the c mode ($|\alpha|^2 > 4|\gamma|^2$). So in this case the definition of the c mode as the pump mode is perfectly fine.

It is of interest to study changes in dynamics when the a mode is excited at $t = 0$ and how this dynamics and the entanglement between the modes can be manipulated through the phase relation between the modes. For simplicity we fix the zero phase of the intensive coherent c mode ($\varphi_c = 0$). Dynamics at the early stages of the time

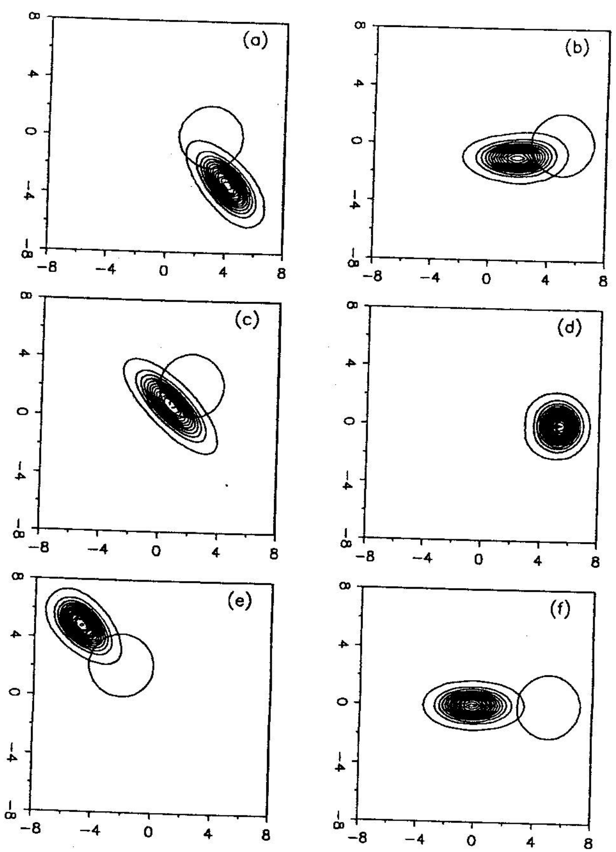

Fig. 9. Q-functions of the modes in the degenerate two-photon down conversion with coherent inputs. The amplitudes are $\gamma = 5$ and $|\alpha| = 3$. (a) The Q_a function for $\alpha = 3$ at $\lambda_2 t = 0.1$; (b) the Q_c function for $\alpha = 3$ at $\lambda_2 t = 0.15$; (c) the Q_a function for $\alpha = 3i$ at $\lambda_2 t = 0.1$; (d) the Q_c function for $\alpha = 3i$ at $\lambda_2 t = 0.15$; (e) the Q_a function for $\alpha = -3i$ at $\lambda_2 t = 0.1$; (f) the Q_c function for $\alpha = -3i$ at $\lambda_2 t = 0.15$. The position of the initial Q-function (at $t = 0$) is sketched by circle.

evolution can be visualized via Q-functions of the modes for $\gamma = 5$ and $|\alpha| = 3$ (see the Fig. 9). The Q_a-function of the a mode coincide for $\lambda_2 t\gamma \ll 1$ with that one which can be obtained using the parametric approximation, i.e. which corresponds to the squeezed coherent state

$$|\psi\rangle_a^{(par)} = \exp[\xi(\hat{a}^\dagger)^2 - \xi^* \hat{a}^2]|\alpha\rangle_a, \qquad \xi = -i\lambda_2 t\gamma \ll 1. \tag{70}$$

This squeezed coherent state with the reduction of fluctuations in one quadrature remain the MUS. Such state has a better defined phase than a coherent state if it is "stretched" in the direction of the displacement in the phase space. In our case such situation

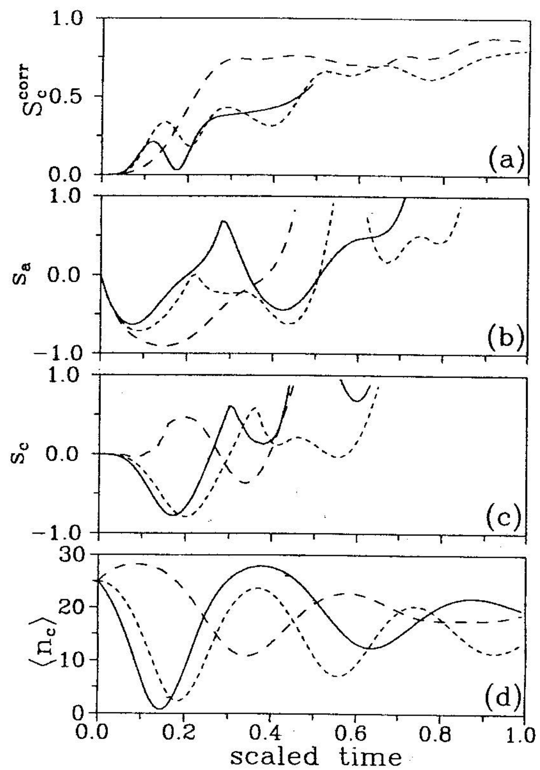

Fig. 10. Statistical properties of the modes for the degenerate two-photon down conversion with coherent inputs. For the fixed value of the amplitude $\gamma = 5$ we set $\alpha = 3$ (short-dashed lines), $\alpha = 3i$ (long-dashed lines) and $\alpha = -3i$ (solid lines). (a) The time evolution of the entanglement parameter S_c^{corr}, (b) the squeezing s_a of the a mode, (c) the squeezing s_c of the c mode and (d) the mean photon number $\bar{n}_c$ of the pump c mode. The scaled time is $\lambda_2 t$.

happens for $k\varphi_a - \varphi_c = -\pi/2$. In the Fig. 10a we plot the degree of the entanglement between the modes for the initial state (67) with $\gamma = 5$ and $|\alpha| = 3$ and for different phases of the a mode. In general, with the increase of the number of photons in the a mode (at the given intensity of the pump) the modes become more rapidly entangled. Nevertheless, change of the phase of the a mode influences significantly the dynamics. For the phase matching condition $2\varphi_a - \varphi_c = \pi/2$ and small intensities of the a mode we obtain compared with other phase relation the smallest initial rate of the increase of the entanglement between the modes. In this case the c mode is amplified (during the first instants of the time evolution). For other phase relation when $2\varphi_a - \varphi_c = -\pi/2$ (which corresponds to the amplification of the a mode) the modes become more correlated during the first instants of the time evolution. Nevertheless, with the increase of the intensity of the a mode (which stimulates the transfer of the energy from the c mode to the a mode) there appears the effect of the spontaneous disentanglement, i.e. the entanglement parameter S_c^{corr} is significantly reduced after its first local maximum. Moreover, with the increase of the intensity of the a mode the modes can be treated as fully disentangled, i.e. $S_c^{corr} \simeq 0$. Comparing the evolution of the parameter S_c^{corr} with the evolution of the mean photon number (Fig. 10d) we can conclude, that for intensities $|\alpha| \sim \gamma$ the c mode transfers all its energy in a coherent way into the a mode. During this transfer there is no entanglement between modes and in the moment when the whole energy is transferred into the a mode the process of the harmonic generation with the coherent a mode (but now with intensity equal to $|\alpha|^2 + 2|\gamma|^2$) and the empty c mode is "tviggered". For completeness we should mention that for any phase relation the c mode never exhibits sub-Poissonian statistics, i.e. the Mandel q_x-parameter ($x = a, c$) defined as [28]

$$q_x = \frac{\langle(\Delta\hat{n}_x)^2\rangle}{\langle\hat{n}_x\rangle} - 1 \tag{71}$$

is positive for any $t > 0$ and $\alpha \leq \gamma$. On the other hand, the initial presence of photons in the a mode leads to an appearance of the sub-Poissonian statistics ($q_a < 0$) in the case when the phase relation $2\varphi_a - \varphi_c = \pi/2$ is fulfilled. The sub-Poissonian character of the a mode is restricted to the initial region when photons are transferred from a mode to the c mode and the minimum value of the Mandel parameter is reached in the "middle" of this process of the c mode amplification. For the case $\gamma = 5$ and $\alpha = 3i$ we find the minimum $(q_a)_{min} = -0.48$ at $\lambda_2 t = 0.04$. This sub-Poissonian character becomes more pronounced with the increase of the input intensity of the a mode. In Figs. 10b,c we plot minima s_x ($x = a, c$) of the squeezing parameters S_x^X, S_x^Y given by Eq.(58). For both phase matching condition $2\varphi_a - \varphi_c = \pm\pi/2$ the maximum squeezing reached by the a mode is reduced in comparison with the case $2\varphi_a - \varphi_c = 0$ (as well as the case $\alpha = 0$). On the other hand, the maximum squeezing of the pump mode (even compared with the case $\alpha = 0$) is enhanced.

From Eq.(70) it follows that the parametric approximation in the non-degenerate down conversion process with the initial state (67) can be performed only if the pump mode is initially in a highly excited coherent state ($|\gamma| \gg |\alpha|$). Simultaneously, we have to stress that this approximation is restricted to times such that $\lambda_2 t\gamma \ll 1$. On the other hand if $\lambda_2 t\gamma \sim 1$ (i.e. when one of the modes transfers a significant portion of its initial energy into other modes) then the particular value of the entanglement parameter S_c^{corr} depends on the phase of the initial coherent state of the a mode. From here we see a strong dependence of the entanglement parameter on statistical properties of the initial state of the a mode. Therefore, we will now investigate changes in correlation between the modes in the case when the a mode (signal) is prepared in a highly nonclassical state.

In particular, if we assume the signal mode to be prepared in the Fock state $|n\rangle_a$, then the entanglement between the pump and the signal mode becomes stronger. Generally, the larger the initial number of photons n in the signal mode the stronger is the entanglement for given γ and $\lambda_2 t$ and consequently the larger is the value of the entanglement parameter S_c^{corr} (see Fig. 11). From here we can conclude that the variation

in the initial statistics of the signal mode can significantly constrain the applicability of the parametric approximation.

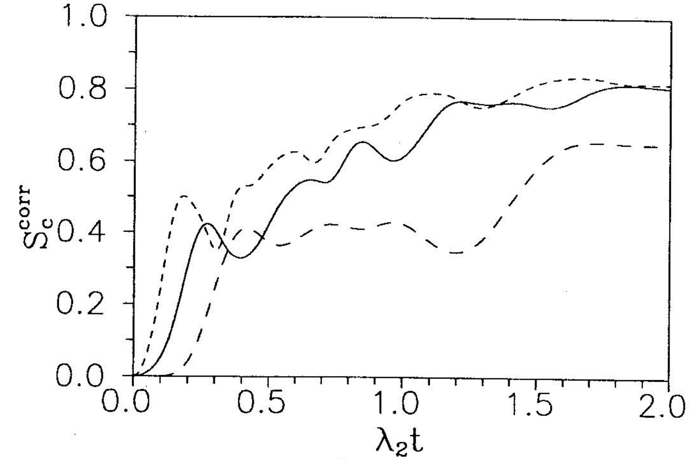

Fig. 11. The short time evolution of the entanglement parameter $S_c^{corr}(t)$ for the pump mode initially in the coherent state with $n_c = 9$. The signal mode is prepared initially in the Fock state $|n\rangle_a$ with photon number: $n = 0$ (long-dashed curve), $n = 2$ (solid curve) and $n = 4$ (short-dashed curve).

One of the consequences of the above observation is the fact that the state vector $|\psi(t)\rangle$ of the pump-signal system, which is initially in the state

$$|\psi(t=0)\rangle = |n\rangle_a|\gamma\rangle_c, \tag{72}$$

cannot be for $t > 0$ and $n > 1$, even approximately expressed as

$$|\psi(t)\rangle = \hat{U}_2(t)|\psi(t=0)\rangle \neq \hat{S}_a(-i\gamma t)|n\rangle_a|\gamma\rangle_c. \tag{73}$$

We have to note here that approximation (73) can be used for $n = 1$ but only on a time scale at which $t \ll (\sqrt{6}\lambda_2|\gamma|)^{-1}$.

We can conclude that the parametric approximation which is represented by the replacement of the c-mode operators $\hat{c}$, $\hat{c}^\dagger$ by complex numbers can be done only in the case of coherent inputs in both modes and its applicability is restricted to the region $\lambda_2 t\gamma \ll 1$ for $\gamma > \alpha$. If a nonclassical state (such as squeezed vacuum or number state) is used as an input in one of the modes then the modes become very rapidly entangled.

3.3. Multiphoton ($k > 2$) down-conversion

In the k-photon down-conversion process one photon of the pump mode with the frequency $k\omega$ is transformed into k photons of the signal mode with the frequency ω. If initially the down-converted mode (signal) is empty then the state vector of the whole system takes at $t > 0$ the form

$$|\psi(t)\rangle = \sum_{L=0}^{\infty}\sum_{m=0}^{L} d_m(L;t)|km\rangle_a|L-m\rangle_c, \tag{74}$$

$$d_m(L;t) = \sum_j c_{j0}(L)c_{jm}(L)e^{-itE_j(L)}, \tag{75}$$

where coefficients c_{jm} are given by Eq.(16) [we label the coefficients c_{jm} and eigenvalues E_j by L instead of $M = kL$]. This multiphoton character of the process under consideration results in the fact that if the signal mode is initially prepared in the vacuum state, then at $t \geq 0$ we find

$$\langle\hat{a}\rangle_a = \langle\hat{a}^2\rangle_a = \ldots = \langle\hat{a}^{k-1}\rangle_a \equiv 0, \tag{76}$$

which means that in the k-photon down conversion with $k > 2$ the signal mode does not exhibit quadrature squeezing. Nevertheless, it is not excluded that the higher-order squeezing [5], the amplitude-squared [29] or the amplitude k-the power squeezing [30] can be observed.

In the k-photon down conversion the signal and the pump mode become entangled in the same way as in the two-photon down conversion. Generally speaking, the higher the order of the process (i.e. the higher the k) the stronger is the entanglement (during the first instants of the time evolution). This is seen in Fig. 12 for initially empty signal mode and a coherent pump. For initial states of the signal mode other then the vacuum state the entanglement is even stronger. Nevertheless, the maximum entanglement should be expected to decrease with the increase of the order of the nonlinear process under consideration. This effect is due to a decreasing energy-transfer efficiency in the k-photon conversion [24].

The pump-signal state vector in the k-photon down conversion process for an initial state $|\psi(t=0)\rangle = |0\rangle_a|\gamma\rangle_c$ can be written in the factorized form (see below) only during the first instants of the time evolution when

$$t < \frac{1}{\sqrt{k!}\gamma\lambda_k}. \tag{77}$$

For these times the pump-signal state vector can be factorized as $|\psi(t)\rangle = |\psi(t)\rangle_a|\psi(t)\rangle_c$ where

$$|\psi(t)\rangle_a = |0\rangle_a - i\lambda_k t\sqrt{k!}\gamma|k\rangle_a, \qquad |\psi(t)\rangle_c = |\gamma\rangle_c. \tag{78}$$

The Q_a-function corresponding to the state $|\psi(t)\rangle_a$ exhibits k-fold rotational symmetry and can be written in the form

$$Q_a(\alpha) \sim e^{-|\alpha|^2}\left(1 - 2\gamma t|\alpha|^k \sin k\vartheta + \gamma^2t^2|\alpha|^{2k}\right), \tag{79}$$

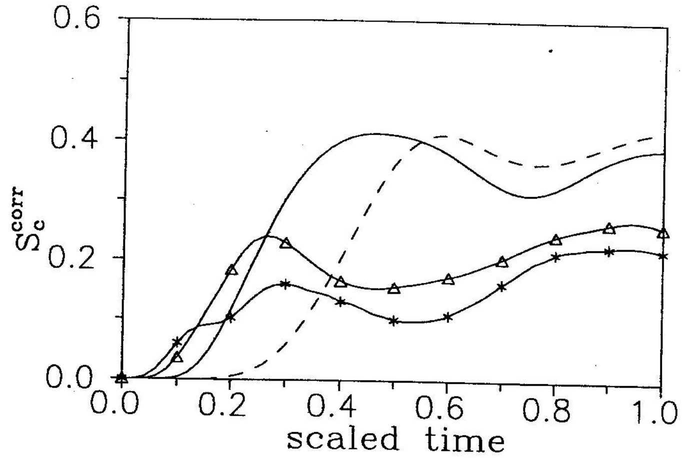

Fig. 12. The short time evolution of the entanglement parameter $S_c^{corr}(t)$ for the k-photon down-conversion with the initial state $|0\rangle_a|\gamma = 3\rangle_c$ with $k = 2$ (dashed line), $k = 3$ (solid line), $k = 4$ (line with triangles) and $k = 5$ (line with stars). The scaled time equals $\sqrt{k!}\lambda_k t$.

where $\alpha = |\alpha|\exp(i\vartheta)$. This Q_a-function as well as phase properties of the signal mode in the process under consideration have been analyzed in details in [31].

In the Introduction we have shown that the application of the parametric application in the k-photon down conversion ($k > 2$) is questionable because of the divergence of the vacuum-to-vacuum matrix element of the evolution operator $\hat{U}_k^{(par)}(t) = \exp[-it\hat{H}_k^{(par)}]$ [see Eq.(5)]. In other words, it is questionable to describe the signal mode in the k-photon down conversion at $t > 0$ by the state vector

$$|\psi(t)\rangle_a^{(par)} = \hat{U}_k^{(par)}(t)|\psi(0)\rangle_a = \exp\left[\xi(\hat{a}^\dagger)^k - \xi^*\hat{a}^k\right]|0\rangle_a, \qquad \xi = -i\lambda_k t\gamma \tag{80}$$

even though that some numerical calculations can be performed by using the Pade' approximants [for instance, Braunstein and McLachlan [5] have shown that for small values of ξ, the Q_a-function of the state (80) is equal to the expression (79)]. Due to problems associated with the parametric approximation in the Hamiltonian (6) it was proposed to use instead of the evolution operator (we use the interaction picture) $\hat{U}_k^{(par)}(t) = \exp\left[\xi(\hat{a}^\dagger)^k - \xi^*\hat{a}^k\right]|0\rangle_a$ the approximation [9, 29]

$$\hat{U}_k^{(reg)}(t) = \exp(\sqrt{k!}\xi\hat{A}_k^\dagger - \sqrt{k!}\xi^*\hat{A}_k), \qquad \xi = -i\lambda_k t\gamma \tag{81}$$

where the multiphoton operators [29]

$$\hat{A}_k^\dagger = \sqrt{\left[\frac{\hat{n}}{k}\right]\frac{(\hat{n}-k)!}{\hat{n}!}}\hat{a}^{\dagger k}, \qquad \hat{A}_k = (\hat{A}_k^\dagger)^\dagger, \tag{82}$$

which obey the Weyl-Heisenberg commutation relations (i.e. $[\hat{A}_k, \hat{A}_k^\dagger] = 1$) are used. Within the times $t < t_c$ differences between the time evolution induced by (6) and (81) are not significant.

The long time behavior of the entanglement parameter is similar to the two photon case (*Fig.* 5). Only the maximum value of the entanglement parameter S_c^{corr} is reduced with the increase of the order k of the nonlinear process. This fact is due a smaller exchange of energy between modes for higher k. The origin of this effect was discussed in the previous section (case $k = 2$). For $k > 2$ analogous arguments are valid [see Eqs.(61), (63) and (64)]. Moreover, for $k > 2$ the initial state $|0\rangle_a|N\rangle_c$ for even $N = 2l$ has dominant overlap with only one eigenstate $|E_0(2l)\rangle$ with zero eigenenergy (we can say that the energy is "trapped" in this eigenstate). For the odd $N = 2l + 1$ the state $|0\rangle_a|N\rangle_c$ can be approximated as a superposition of two eigenstates [as in the case $k = 2$ (63)] but the matrix element of the operator $\hat{n}_c$ between these two eigenvectors is smaller and smaller with the increase the nonlinearity-order k which results in a drastic suppression of the energy-transfer efficiency in the k-photon down conversion for $k > 2$. For example, for $k = 3$ this efficiency (a ratio of the maximum possible depletion of the pump c mode to its initial intensity) is less than 0.15 for $\gamma^2 \approx 100$ and for $k = 4$ is smaller than 0.04 (i.e. the pump can lose only 4% from its initial number of photons). One of the consequences is that the Mandel q_a parameter of the signal a mode is always higher than that of the pump c mode, i.e. the pump mode is always more sub-Poissonian than the signal mode (for details see ref.[24]). The revival time of the mean photon number can be derived along the same lines as in the case $k = 2$.

Analogously to the two-photon down conversion we can study the influence of the initial mode statistics on the entanglement and on non-classical properties of the modes. One can expect, in principle, the similar behaviour as in the case of the two-photon down conversion. Therefore we will only briefly discuss results for other initial states such as a squeezed pump or an excited signal mode.

In the case when the pump is prepared in the squeezed vacuum state and the signal mode is in the vacuum state the modes become very rapidly entangled during the first stages of the time evolution. Nevertheless, because the initial state is a superposition of the states $|0\rangle_a|N\rangle_c$ with even $N = 2l$ there is a significantly smaller transfer of the energy from the pump to the signal mode (compared with the case of a coherent pump) which determines a reduction of the maximum of correlations (measured by S_c^{corr}) between the modes. Following the arguments given by Hillery and co-workers [6] the symmetry of the signal Q_a-function is twice of that of the pump-mode Q_c-function. Therefore for the squeezed pump we find the signal-mode Q_a-function which possesses $(2k)$-fold symmetry.

For the other initial state with coherent inputs in both modes, i.e. $\psi(0)\rangle = |\alpha\rangle_a|\gamma\rangle_c$, the situation is analogous to the case $k = 2$. The relation $k\varphi_a - \varphi_c$ between the phases of the coherent amplitudes characterizes the initial regime. If the phase condition $k\varphi_a - \varphi_c = \pi/2$ is fulfilled then the c mode is during initial stage of the time evolution amplified; the condition $k\varphi_a - \varphi_c = -\pi/2$ corresponds to the amplification of the a mode. In the case $k\varphi_a - \varphi_c = 0$ the c mode is amplified if $|\alpha| \geq k|\gamma| \gg 1$.

In the summary: the degenerate k-photon down conversion leads (except the case $k = 1$) to a strong entanglement between the pump and the down-converted mode. Disentanglement between the modes is typically limited to very short initial times $\approx (\lambda_k|\gamma|)^{-1}$ providing the pump mode is initially in the coherent state $|\gamma\rangle_c$. The parametric approximation is limited to this time period. For other initial states (different from coherent states) the modes become entangled faster, so the parametric approximation cannot be adopted.

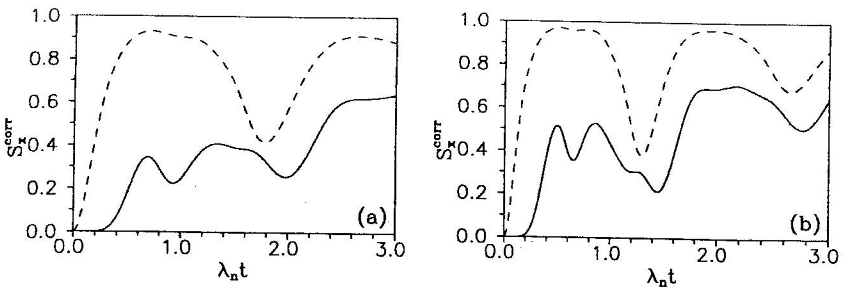

Fig. 13. The short time evolution of the entanglement parameters of the signal mode S_a^{corr} (dashed lines) and the pump mode S_c^{corr} (solid lines) for the non-degenerate two-photon down-conversion process. The initial state vector $|0\rangle_a|0\rangle_b|\gamma\rangle_c$ was taken with intensities of the coherent pump mode as follows: (a) $n_c = 9$ and (b) $n_c = 25$.

4. Non-degenerate two–photon down conversion

The process of the non-degenerate two-photon down conversion is governed by the Hamiltonian (4). In this process the pump photon (mode c) gives rise to a pair of highly correlated photons (of the signal and the idler modes). In what follows we will analyze the entanglement of the field modes in the case when the pump mode is initially prepared in a coherent state and the signal and the idler mode are in the vacuum state:

$$|\psi(t=0)\rangle = |0\rangle_a|0\rangle_b|\gamma\rangle_c, \qquad (83)$$

where $|\gamma\rangle_c$ describes the coherent state with an amplitude γ.

Using the numerical approach [8] we can study the time evolution of the initial state-vector (83) governed by the Hamiltonian (4). The entanglement parameter S_c^{corr} of the pump mode describing the degree of entanglement between the pump and the signal-idler subsystem at the early stages of the time evolution is shown in Fig.13 for various initial intensities $|\gamma|^2$. From this picture we learn that higher the intensity of the initial pump mode the larger is the entanglement parameter S_c^{corr} of the pump, i.e. the stronger is the entanglement between the pump and the signal-idler system. Through a sequence of local minima and maxima the entanglement parameter reaches an almost steady state with small oscillations. From the time evolution of the entanglement parameter it is clearly seen that the pump mode remains in a pure state only at very early stages of the time evolution. In addition, the larger is the intensity of the initial pump mode the shorter is the interval at which $S_c^{corr} \simeq 0$ [see Eq.(84)].

At the early stages of the time evolution the state vector of the whole system can be approximated as (working in the interaction picture we drop the free evolution term):

$$|\psi(t)\rangle = \hat{U}_n(t)|0\rangle_a|0\rangle_b|\gamma\rangle_c \approx (|0\rangle_a|0\rangle_b - it\lambda\gamma|1\rangle_a|1\rangle_b)|\gamma\rangle_c = |\xi\rangle_{ab}|\gamma\rangle_c \qquad (84)$$

where $|\xi\rangle_{ab}$ can be considered as the first order approximation of the two-mode squeezed vacuum state

$$|\xi\rangle_{ab} = \hat{S}_{ab}|0\rangle_a|0\rangle_b = \exp[\xi\hat{a}^\dagger\hat{b}^\dagger - \xi^*\hat{a}\hat{b}]|0\rangle_a|0\rangle_b, \qquad \xi = -i\lambda_n t\gamma \ll 1. \tag{85}$$

Factorization (84) of the state vector describing the pump mode and the down-converted subsystem is possible only for times t for which

$$t < t_c = \frac{1}{\lambda_n \gamma}. \tag{86}$$

From Eq.(84) it follows that for $t < t_c$ the pump mode remains almost completely disentangled from the down-converted modes. In other words during the time interval $t < t_c$ the statistical properties of the pump are not affected by the interaction with the signal-idler subsystem. To see this we plot in Fig. 14 the Q_c-function and the photon number distribution of the pump mode with $|\gamma| = 5$ for two different time moments. In Figs. 14a and 14b we plot Q_c and PND at time $\lambda_n t = 0.2$. The shape of these function is almost identical to that of the initial coherent state, i.e. the Q_c-function has a Gaussian profile and the PND is the Poissonian distribution. As an additional check of the behaviour of the state of the pump mode during the early stages of time the evolution we show in Fig. 14e a deviation of the pump mode from the minimum uncertainty state, i.e. the parameter u_c [for definition see Eq.(59)]. We see that for short times the parameter u_c equals to zero and then it starts to grow rapidly, which means that under the influence of the down-converted modes the pump mode starts to deviate from MUS. Comparing Figs. 13 and 14e we can conclude that the pump mode starts to deviate from the MUS at the same time when it starts to be strongly entangled with the signal-idler subsystem. It means that the back action of the down-converted modes significantly affects statistical properties of the pump mode. In Fig. 14c (14d) we plot the Q_c-function (and the PND) of the pump mode at time $\lambda t = 0.5$ (for $\gamma = 5$). At this time the pump mode is no longer in a pure state but in a statistical mixture, i.e. the back action of the down-converted modes causes a deterioration of the purity of the state. Although being in a mixture, the c mode exhibits some nonclassical properties. We see that the Q_c function is "squeezed" in one of the directions and stretched in the other. The photon number distribution exhibits oscillations which are typical for squeezed vacuum states. Moreover, the PND becomes much broader when compared with the Poissonian distribution. From the above we can conclude that the back action of the down-converted modes leads to squeezing of the pump mode. In Fig. 14f we plot the time evolution of squeezing parameters defined by Eq.(58) which confirm the conclusions based on the analysis of the Q_c-function. The maximum degree of squeezing is obtained at times around $\lambda t \approx 0.6$. After this moment quadrature fluctuations become rapidly superfluctuant and do not become squeezed again.

Now we turn our attention to the down-converted modes. The entanglement parameter S_{ab}^{corr} of the signal-idler subsystem is equal to that of the pump mode S_c^{corr} [in analogy with (25)]. This naturally means that at the very early stages of the time evolution the signal-idler subsystem is in a pure state. While the pump is still in a pure state a two-mode squeezed vacuum state (85) is generated in the signal-idler modes.

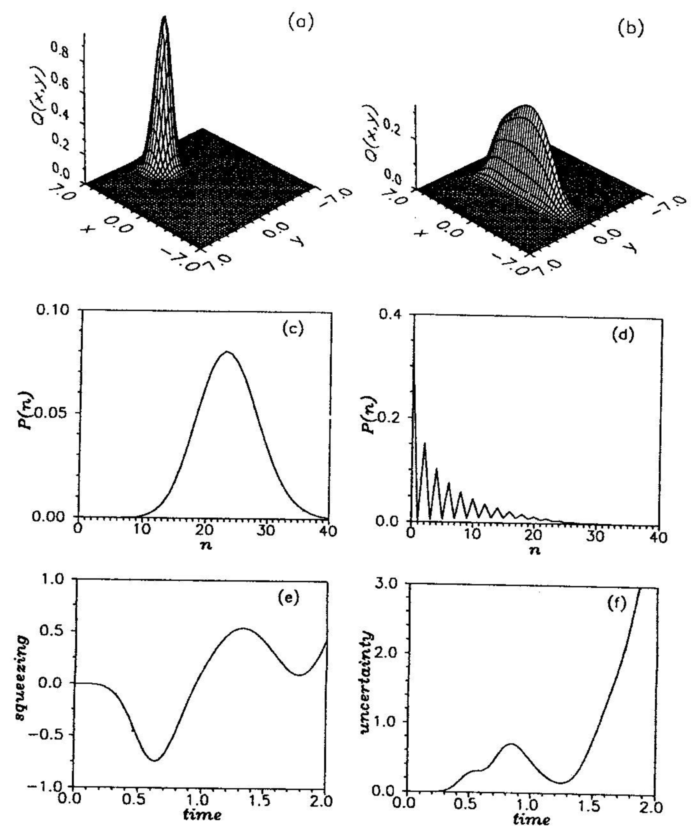

Fig. 14. Statistical parameters of the pump mode for the non-degenerate process initially with $n_c = 25$. (a) The Q_c function at $\lambda_n t = 0.2$; (b) the Q_c function at $\lambda_n t = 0.6$; (c) the photon number distribution at $\lambda_n t = 0.2$; (d) the photon number distribution at $\lambda_n t = 0.6$; (e) the time evolution of the one-mode squeezing; (f) the time evolution of the MUS parameter. Time is scaled as $1/\lambda_n$.

The two-mode squeezing can be described introducing two quadrature operators $\hat{Z}_i$:

$$\hat{Z}_1 = \frac{\hat{d}(t)+\hat{d}^\dagger(t)}{\sqrt{2}}, \qquad \hat{Z}_2 = \frac{\hat{d}(t)-\hat{d}^\dagger(t)}{i\sqrt{2}},$$
$$\hat{d}(t) = \exp(-i\phi)\frac{\hat{a}\exp(i\omega_a t)+\hat{b}\exp(i\omega_b t)}{2}. \tag{87}$$

The degree of two-mode squeezing can be quantified using two parameters $z_i(t)$ ($i = 1,2$):

$$z_i(t) = 4\langle(\Delta\hat{Z}_i)^2\rangle - 1, \tag{88}$$

where $\langle(\Delta\hat{Z}_i)^2\rangle = \langle\hat{Z}_i^2\rangle - \langle\hat{Z}_i\rangle^2$ and 100% squeezing is obtained for $z_i(t) = -1$. The time evolution of $z_i(t)$ is shown in Fig. 15a. During the first instants of the time evolution a high degree of two-mode squeezing is obtained and the energy is transferred from the pump mode to the signal-idler modes. With the increase of the transferred energy the degree of squeezing becomes larger. This scenario is valid until the moment when the back-action of the signal-idler modes on the pump mode becomes significant. Before that a pure two-mode squeezed state is generated in the signal-idler modes. This pure two-mode state ($S_{ab}^{corr} \simeq 0$) is a minimum uncertainty state. One can check this by inspecting (see Fig. 15b) the time evolution of the parameter $u_{ab}(t)$ which is defined as

$$u_{ab} = \langle(\Delta\hat{Z}_1)^2\rangle\langle(\Delta\hat{Z}_2)^2\rangle - \frac{1}{16}. \tag{89}$$

With the initial condition (83) the reduced density operators of the signal and the idler modes $\hat{\rho}_a, \hat{\rho}_b$ have the same form, which means that the signal and the idler modes have identical statistical properties, i.e.:

$$S_a^{corr} = S_b^{corr}; \quad P_a(n) = P_b(n); \quad \text{and} \quad Q_a(\alpha) = Q_b(\alpha). \tag{90}$$

The time evolution of the entanglement parameter S_a^{corr} of the signal mode is shown in Fig. 13. It can be proved that for the initial conditions given by Eq.(83) an inequality $S_a^{corr}(t) \geq S_c^{corr}(t)$ is valid for any $t \geq 0$. Fig. 13 confirms that for the given intensity of the initial pump mode the purity parameter S_a^{corr} increases during the initial period of the time evolution much more rapidly than the parameter $S_{ab}^{corr} = S_c^{corr}$. From here it follows that at initial stages of the time evolution the index of correlation $I_{a-b}^{\rm corr}$ takes its maximal values (i.e. $I_{a-b}^{\rm corr} \simeq 2S_a$), which is an additional proof that the two-mode squeezed vacuum [11] is produced in the signal-idler modes. As soon as the pump mode is affected by the action of the down-converted modes the entanglement parameter $S_{ab}^{corr} = S_c^{corr}$ becomes larger than zero, which consequently results in the deterioration of the degree of correlation between the signal and the idler.

In Fig. 15c (15d) we plot the Q-function (the PND) of the signal mode at the moment $\lambda t = 0.2$ (we assume $\gamma = 5$). We clearly see the thermal-like character of the marginal photon number distribution and of the Q_a-function which is the characteristic feature of the two-mode squeezed vacuum state. At later moments, when the pump and the down-converted modes becomes entangled ($S_c^{corr} > 0$), the field statistics of the signal mode is significantly different from the thermal-like field. In particular, the photon number distribution becomes very broad (compare Figs. 15d and 15f). At times much longer than t_c given by Eq.(86) the three modes under consideration become strongly entangled. They are not in pure states anymore, but the quantum nature of the dynamics leads to some new interesting features.

On the long time scale the time evolution of the entanglement parameters S_c^{corr} and S_a^{corr} is similar, i.e. both parameters exhibit small oscillation around some stationary value and a very significant decrease and significant oscillations at some particular moments. We can call this behaviour as the collapse and revival of the purity of the field modes. We should note here that the minima of the parameters S_a^{corr} and S_c^{corr} appear simultaneously and moments when they appear coincide with the "revivals" of

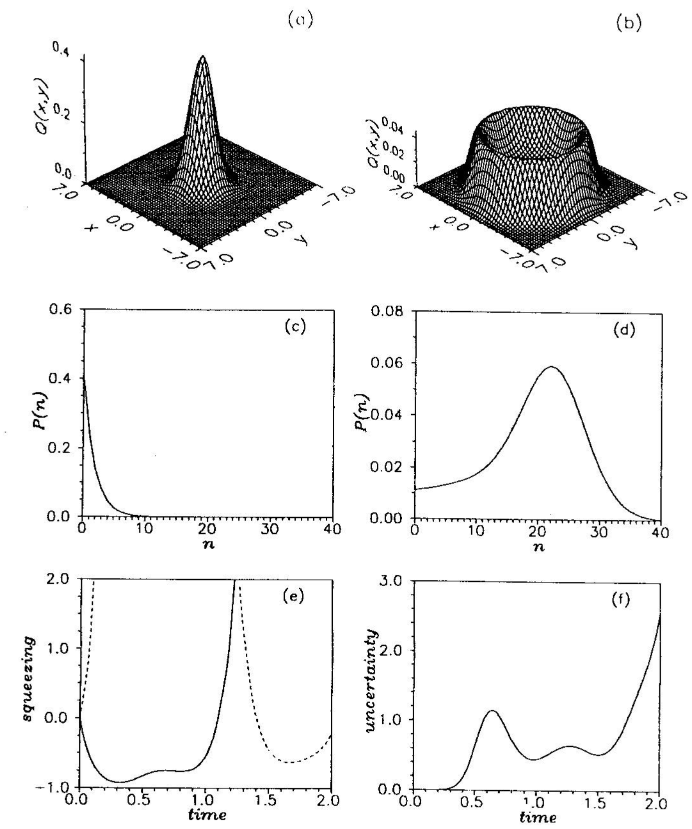

Fig. 15. Statistical parameters of the down-converted modes for the same situations as in Figs. 14a-d; (e) the time evolution of the two-mode squeezing parameters z_1 (dashed line) and z_2 (solid line); (f) the time evolution of the two-mode MUS parameter. Time is scaled as $1/\lambda_n$.

the mean photon number of the signal (idler) mode [9]. At this moments we can observe a *partial* restoration of the initial purity of the modes under consideration. One can estimate the "revival" time of the mean photon number in exactly the same way as we have described in the previous section.

If we compare the above results for the non-degenerate down conversion and the degenerate two-photon down conversion we see that dynamics of the pump mode in both processes is very similar. On the other hand, as one can expect, quantum statistics of down converted modes in these two processes (degenerate and non-degenerate) is different.

Analogously as in the case of the degenerate process we can study the influence of the change of initial conditions on output properties of the modes. If the pump is prepared in a squeezed vacuum state instead of a coherent state then results concerning the entanglement parameter S_c^{corr} as well as properties of the pump mode are analogous to those for the degenerate two-photon down conversion. To obtain results which have no analogy in the degenerate case we turn our attention to the case when either the signal mode or the idler is initially excited.

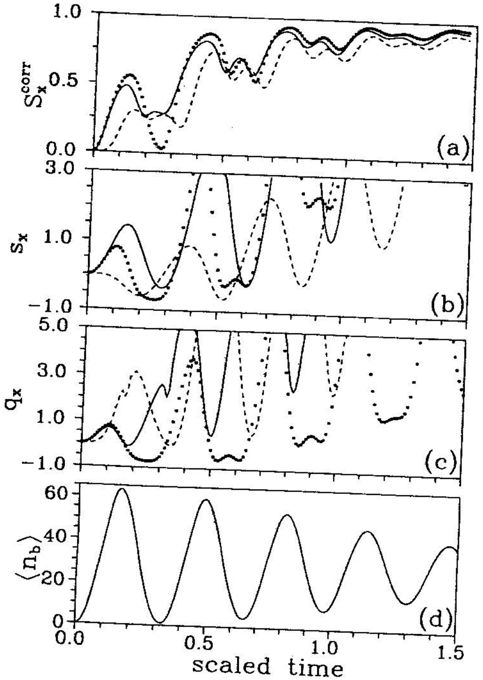

Fig. 16. The time evolution of statistical properties of the modes in the non-degenerate down conversion with the initial state $|\alpha = 8\rangle_a|0\rangle_b|\gamma = 8\rangle_c$. (a) The entanglement parameters S_x^{corr} ($x = a, b, c$), (b) the squeezing parameters s_x, (c) the Mandel's q_x parameters for the signal a mode (□), the idler b mode (solid line) and the pump c mode (dashed line); (d) the mean photon number $\langle \hat{n}_b \rangle$. The scaled time is $\lambda_n t$.

We explore the model (4) for the input state when the pump as well as one of the down-converted modes are assumed to be initially in a coherent state [32]

$$|\psi(t=0)\rangle = |\alpha\rangle_a|0\rangle_b|\gamma\rangle_c. \tag{91}$$

For this kind of the input no longer the entanglement parameters S_a^{corr} and S_b^{corr} of the down-converted modes are the same, i.e. in general $S_a^{corr} \neq S_b^{corr}$ for $t > 0$. It means that the signal and the idler are entangled with the rest of the system (i.e. with the pump) differently. The results are collected in Fig. 16 for $\alpha = \gamma = 8$. It can be said that the presence of the initial signal photon causes a deterioration of the nonclassical properties of the pump mode, namely squeezing is decreasing with the increase of α. Both the idler and the pump modes exhibit only a negligible sub-Poissonian photon statistics. On the other hand, nonclassical properties of the signal a mode are enhanced and it exhibits both a high degree of squeezing as well as a sub-Poissonian character. Moreover, this highly nonclassical behaviour appears at the time of the "spontaneous" disentanglement of the signal a mode from the two other modes, i.e. $S_a^{corr} \to 0$. From here we can conclude that a pure nonclassical state in the signal a mode is produced. The pump c mode and the idler exhibit some degree of squeezing even though they both are in mixture states. The effect of the spontaneous disentanglement appears in the a mode only when its initial intensity is precisely chosen, namely if $\alpha \approx \gamma$ (as was pointed out elsewhere [32], further increase of the input signal intensity leads to a deterioration of nonclassical effects in all modes) and is restricted to the short time scale. From Fig. 16d it is seen that the disentanglement appears during those moments when the initially amplified signal and idler modes partly return their gained energy back to the pump (c mode). The time at which the disentanglement appears in the non-degenerate down-conversion with the input (91) characterized by $\alpha \approx \gamma$ is proportional to $(\lambda_n \gamma)^{-1}$.

Finally, we want to compare the case when an input coherent signal was considered with the case when the signal is prepared in a number state $|D\rangle_a$, i.e.

$$|\psi(t=0)\rangle = |D\rangle_a|0\rangle_b|\gamma\rangle_c. \tag{92}$$

This input was studied in details in [9]. For number states with $D \ll |\gamma|^2$ the dynamics behaves in a rather obvious way, i.e the pump mode becomes faster (and more strongly) entangled to the down-converted modes compared with the case of the initially empty signal ($D = 0$). Nevertheless, quite unexpected behaviour can be found in the regime $D \gg |\gamma|^2$ when the entanglement of the pump to the other modes becomes weaker and the pump can be treated as decoupled from the down-converted modes at the time region proportional to $\sqrt{D}/\lambda_n$. The origin of this effect can be found from the analysis of the matrix element (14) of the interaction Hamiltonian (4). On the subspace $\mathcal{H}_{D,L}$ the matrix element

$$\begin{aligned} h_m &= {}_c\langle L-m-1|\,{}_b\langle m+1|\,{}_a\langle D+m+1|\hat{H}_n|D+m\rangle_a|m\rangle_b|L-m\rangle_c \\ &= \lambda_n\sqrt{(D+m+1)(m+1)(L-m)} \end{aligned} \tag{93}$$

can be in a good approximation (for $D \gg L$) rewritten as

$$h_m \approx (\lambda_n\sqrt{D})\sqrt{(m+1)(L-m)} \qquad (D \gg L). \tag{94}$$

The same matrix elements possesses the linear coupler with the coupling constant $\lambda_1 = \lambda_n\sqrt{D}$. Effectively the pump mode interaction with the system composed of the signal and idler modes is linear as the state $|D+m\rangle_a|m\rangle_b$ represents the state $|m\rangle\rangle_{\tilde{a}}$ of the "mode" $\tilde{a}$ to which the pump c mode is coupled via interaction Hamiltonian (28) with $\lambda_1 = \lambda_n\sqrt{D}$. Therefore the initial coherent pump is disentangled from the $\tilde{a}$ "mode" for times $\lambda_n t \ll \sqrt{D}$ remaining in a coherent state with the amplitude $\gamma(t) = \gamma\cos\lambda_1 t$ [see (31),(32)]. Later the "basic" nonlinear interaction causes a loss of the purity of the pump mode. Almost full restoration of the initial pure pump state (and inevitably also signal and idler states) appears at the revival time of the mean photon number $t_R = 4\pi\sqrt{D}/\lambda_n$.

Now we will shortly discuss the efficiency of the energy transfer from the pump mode into an initially empty mode (idler). The presence of initial signal photons increases efficiency of the energy transfer from the pump mode. For an initially empty signal mode [i.e. for the input (83)] less than 80% of the pump photons can be transferred into idler mode while with the initial states (91) and (92) the efficiency tends to 100% for $\alpha, D \to \infty$. Signal photons stimulate an energy transfer from the pump mode and make the interaction more "linear" [see linearization (94)]. The origin of the limited efficiency of the non-degenerate down conversion (with empty signal and idler) is analogous to that in the degenerate two-photon down conversion.

Conclusions

We have analyzed quantum-statistical properties (such as squeezing and sub-Poissonian character) of the field modes in the process of degenerate k-photon down conversion as well as non-degenerate down conversion with quantized pump. The main attention was paid to the study of the degree of entanglement between modes. It was shown that the parametric approximation (i.e. a classical description of the intensive pump) cannot be applied for arbitrary initial state of the system. Typically, a coherent pump with intensity n_c can be treated classically at times $t \ll (\lambda_{k(n)}\sqrt{n_c})^{-1}$. In contrary, the pump prepared in a squeezed vacuum state has to be treated quantum-mechanically even in this time region. Moreover, the degree of the entanglement between the modes depends also on initial statistics of the down converted mode(s).

Although we have investigated some particular nonlinear optical processes one can expect that our conclusions concerning the applicability of the parametric approximation are valid also for other nonlinear processes, such as the degenerate and non-degenerate hyper-Raman scattering. Even a two-mode coupler with intensity dependent coupling leads to this kind of the results [33]. This work was devoted to quantum-correlations between field modes, but the mutual entanglement of subsystems can be observed also in processes when one quantized field mode interacts with an ensemble of atoms. In fact, dynamics of the non-degenerate down conversion process can be directly mapped on the Dicke model without dissipations [34] (so called the Tavis-Cumming model) due to the same algebraic structure (the b and c modes form the bosonic representation of the atomic subsystem composed of two-level atoms which interact with the a mode).

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References

[1] H. Takahashi, *Adv. Commun. Syst.* 1, 227 (1965); H. P. Yuen, *Phys. Rev. A* **13**, 2226 (1976); R. Loudon,, P. L. Knight, *J. Mod. Optics* **34**, 709 (1987); K. Zaheer, M. S. Zubairy, 1990, in: *Advances in Molecular and Optical Physics*, vol. 28, edited by D.Bates and B.Bederson (Academic Press, New York), p.143

[2] W. H. Louisell, A. Yariv, A. E. Siegman, *Phys. Rev.* **124**, 1646 (1961); for a review of optical parametric amplifiers see R.G.Smith, in: Laser Handbook, vol. 1, edited by F.T.Arecchi and E.O.Schulz-Dubois (North-Holland, Amsterdam, 1972). Peřina, J., 1991 *Quantum Statistics of Linear and Nonlinear Phenomena*, 2nd edition (Kluwer Academic, Dordrecht).

[3] L.-A. Wu, H. J. Kimble, J. L. Hall, H. Wu, *Phys. Rev. Lett.* **57**, 2520 (1986); see also C. Fabre, G. Gammy, *Phys. Rev. Lett.* **59**, 2555 (1987); R. Slusher, P. Grangier, A. LaPorta, B. Yurke, M. Potasek, *Phys. Rev. Lett.* **59**, 2566 (1987); S. Pereira, M. Xiao, H. J. Kimble, J. Hall, *Phys. Rev. A* **38**, 4931 (1988); P. R. Tapster, J. G. Rarity, J. S. Satchell, *Phys. Rev. A* **37**, 2963 (1988); T. Debuischer, S. Reynaud, A. Heidmann, E. Giacobino, C. Fabre, *Quantum Opt.* **1**, 3 (1989).

[4] D. F. Walls, C. T. Tindle, *J. Phys. A*, **8**, 534 (1972); *Lett. Nuovo Cim.* **2**, 915 (1971); D. F. Walls,, R. Barakat, *Phys. Rev. A* **1**, 446 (1970).

[5] R. A. Fisher, M. M. Nieto, V. D. Sandberg, *Phys. Rev. D* **29**, 1107 (1984); S. L. Braunstein, R. I. McLachlan, *Phys. Rev. A* **35**, 1659 (1987). M. Hillery, *Phys. Rev. A* **42**, 498 (1990).

[6] M. Hillery, D. Yu, J. Bergou, *Phys. Rev. A* **49**, 1288 (1994).

[7] S. Carusotto, *Phys. Rev. A* **40**, 1848 (1989); B. Jurčo, *J. Math. Phys.*, **30**, 1739 (1989).

[8] G. Drobný, I. Jex, *Phys. Rev. A* **45**, 1816 (1992); G. Drobný, I. Jex, *Phys. Rev. A* **46**, 499 (1992).

[9] V. Bužek, G. Drobný, *Phys. Rev. A* **47**, 1237 (1993); G. Drobný, I. Jex, V. Bužek, V., *Phys. Rev. A* **48**, 569 (1993).

[10] J. von Neumann, *Gott. Nachr.*, 273 (1927); M. Araki, E. Lieb, *Commun. Math. Phys.* **18**, 160 (1970); A. Wehrl, *Rev. Mod. Phys.* **50**, 221 (1978).

[11] S. M. Barnett, S. J. D. Phoenix, *Phys. Rev. A* **40**, 2404 (1989);

[12] W. K. Lai, V. Bužek, P. L. Knight, *Phys. Rev. A* **43**, 6323 (1991), and references therein; H. Fearn, R. Loudon, *Opt. Commun.* **64**, 485 (1987); *J. Opt. Soc. Am. B* **6**, 917 (1989).

[13] Wodkiewicz, K.,, Eberly, J. H., *J. Opt. Soc. Am. B* **6**, 2447 (1985).

[14] R. J. Glauber, *Phys. Rev. Lett.* **10**, 84 (1963); E. C. G. Sudarshan, *Phys. Rev. Lett.* **10**, 277 (1963); R. J. Glauber, *Phys. Rev.* **131**, 202 (1963).

[15] B. L. Schumaker, *Phys. Rep.* **135**, 317 (1985); J. N. Holenhorst, *Phys. Rev. D* **23**, 1669 (1979); C. M. Caves, *Phys. Rev. D* **23**, 1693 (1979).

[16] M. S. Abdalla, *J. Mod. Opt.* **39**, 1067 (1992).

[17] J. Janszky, An. V. Vinogradov, *Phys. Rev. Lett.* **64**, 2771 (1990); V. Bužek, P. L. Knight, *Opt. Commun.* **81**, 331 (1991).

[18] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983); J.Janszky, P.Domokos, P.Adam, *Phys. Rev. A* **48**, (1993).

[19] F. A. M. deOliveira, M. S. Kim, P. L. Knight, V. Bužek, *Phys. Rev. A* **41**, 2645 (1990); P. Král, *J. Mod. Opt.* **37**, 889 (1990).

[20] V. Bužek, M. Hillery, unpublished.

[21] K. E. Cahill, R. J. Glauber, *Phys. Rev.* **177**, 1857 (1969); ibid. **177**, 1882 (1969).

[22] C. K. Hong, L. Mandel, *Phys. Rev. Lett.* **54**, 323 (1985).

[23] I. Jex, *J. Mod. Optics* **40**, 1753 (1993).

[24] G. Drobný, V. Bužek, unpublished.

[25] M. Hillery, M. S. Zubairy, *Phys. Rev. A* **29**, 1275 (1984); D. D. Crouch, S. L. Braunstein, *Phys. Rev. A* **38**, 4696 (1988).

[26] E. T. Jaynes, F. W. Cummings, *IEEE* **51**, 89 (1963); B. Shore, P. L. Knight, *J. Mod. Opt.* **40** (1993) and references therein.

[27] I. Jex, V. Bužek, *J. Mod. Opt.* **40**, 771 (1993).

[28] L. Mandel *Opt. Lett.* **4** 205 (1979).

[29] G. D'Ariano, M. Rasetti, M. Vadacchio, *Phys. Rev. D* **32**, 1034 (1985); J. Katriel, A. Solomon, G. D'Ariano, M. Rasetti, *Phys. Rev. D* **34**, 332 (1986).

[30] C. K. Hong, L. Mandel, *Phys. Rev. A* **32**, 974 (1985).

[31] R. Tanaś, Ts. Gantsog, *Phys. Rev. A* **45**, 5031 (1992); Ts. Gantsog, R. Tanaś, R. Zawodny, *Opt. Commun.* **82**, 345 (1991); I. Jex, G. Drobný, M. Matsuoka, *Opt. Commun.* **94**, 619 (1992).

[32] G. Drobný, I. Jex, unpublished.

[33] I. Jex, G. Drobný, *Phys. Rev. A* **47**, 3251 (1993) 3251.

[34] R. Dicke, *Phys. Rev. A* **93**, 99 (1954); M. Gross, S. Haroche, *Phys. Rep.* **93**, 301 (1982) and references therein; G. Drobný, I. Jex, *Opt. Commun.* **102**, 141 (1993); G. Drobný, Ts. Gantsog, I. Jex, *Phys. Rev. A* **48**, 622 (1994).