acta physica slovaca vol. 44 No. 2, 93 – 98 April 1994

CANONICAL HAMILTONIAN FORMULATION OF THE LOTKA-VOLTERRA EQUATIONS[1]

C. Cronström[2,3]

Department of Theoretical Physics, University of Helsinki,
Siltavuorenpenger 20C, SF-00170 Helsinki, Finland

Received 5 January 1994, in final form 25 February 1994, accepted 2 March 1994

A canonical Hamiltonian formulation is given for a fairly general system of Lotka-Volterra equations. This requires certain restrictions on the equations in question, which are spelled out in detail. A set of canonical coordinates and momenta are constructed explicitly. The Hamiltonian (which is bounded from below) is also given explicitly.

1. Introduction

The Lotka-Volterra model [1] is defined by the following system of non-linear differential equations,

$$\frac{dN_i}{dt} = k_i N_i + \sum_{j=1}^{n} A_{ij} N_i N_j \tag{1}$$

The equations above were originally proposed as evolution equations in population dynamics, in which case the quantity N_i denotes the (normalised) population of the i:th species in a system with n competing species. The quantities k_i in the equations above are so-called rate-constants, and the matrix (A_{ij}) with constant matrix elements defines the non-linear interaction between the species. The matrix (A_{ij}) ought to fulfill certain conditions which relate to the concept "crowding inhibits growth". This condition will in this paper be implemented by the requirement that the matrix (A_{ij}) be antisymmetric,

$$A_{ij} = -A_{ji} \tag{2}$$

$$i, j = 1, 2, ..., n\ .$$

The system defined by (1) is interesting in its own right, and has found various applications in physics and chemistry [2].

In what follows I will describe the conditions under which the system (1) admits a canonical Hamiltonian formulation. This is a joint work with Milan Noga, which will be described in more detail elsewhere [3].

---

[1] Presented at School and Workshop on Heavy Ion Collisions, Bratislava, 13-18 September 1993
[2] E-mail address: CRONSTROM@PHCU.HELSINKI.FI
[3] On leave from the Dept. Theor. Physics, presently at the Academy of Finland

2. New variables

We first consider the stationary (*i.e.* time-independent) solutions z_i of the equations (1),

$$z_i(k_i + \sum_{j=1}^{n} A_{ij} z_j) = 0 \tag{3}$$

Assuming $z_i \neq 0$ in general, we have

$$\sum_{j=1}^{n} A_{ij} z_j = -k_i \tag{4}$$

$$i = 1, ..., n$$

In the view of the assumed antisymmetry of the matrix (A_{ij}), the Eqns. (4) have solutions for general k_i only if the dimension of the state space (the number n in Eq. (14)) is an even number. It is furthermore desirable that the solutions z_i of Eqns. (4) be *positive*; this requires some further conditions on the rate parameters k_i and matrix elements A_{ij}, which do not restrict these quantities unduly. In what follows we simply assume the requisite positivity conditions,

$$z_i > 0 \quad , \quad i = 1, ..., n \tag{5}$$

It is now convenient to introduce new variables v_i as follows,

$$v_i := \log \frac{N_i}{z_i} \tag{6}$$

The equations (1) take the following form in the new variables v_i,

$$\dot{v}_i = \sum_{j=1}^{n} A_{ij} z_j (e^{v_j} - 1) \tag{7}$$

In view of the antisymmetry condition (2) one finds immediately a conserved quantity which we call $G(v)$,

$$G(v) := \sum_{i=1}^{n} z_i (e^{v_i} - v_i) \tag{8}$$

It should be noted that the quantity $G(v)$ defined by Eq. (8) is bounded form below (for positive z_i). This is an indication that the quantity $G(v)$ defined by Eq. (8) might be a suitable Hamiltonian for the system (7); that this is indeed the case will be shown below.

It should be noted that the variables v introduced in Eq. (6) as well as the conserved quantity $G(v)$ given in Eq. (8) were considered already in the paper by Goel et al. mentioned above in Ref. [2]. However, these authors did not analyse the canonical structure of the Lotha-Volterra system. We consider this question below in great detail.

In order to discuss the canonical formulation of the system (7) it is necessary to introduce canonical coordinates and momenta in stead of the variables v_i. At this stage one does of course not know that such variables exist; it will be demonstrated below by explicit construction.

3. Canonical variables

Let us to begin with ***assume*** that there exist canonical coordinates q_a and momenta p_a, $(a = 1, .., \frac{1}{2}n)$ which are linear combinations of the variables v_i,

$$q_a = \sum_{i=1}^{n} \alpha_{ai} v_i \quad , \quad p_a = \sum_{i=1}^{n} \beta_{ai} v_i \tag{9}$$

$$a = 1, 2, ..., \frac{1}{2} n$$

The relations (9) ought to be invertible, (remember that n is an even number),

$$v_i = \sum_{a=1}^{\frac{1}{2}n} (U_{ia} q_a + V_{ia} p_a) \tag{10}$$

¿From Eqns. (9) and (10) follows immediately that

$$\sum_{i=1}^{n} \alpha_{ai} U_{ib} = \delta_{ab} \tag{11}$$

$$\sum_{i=1}^{n} \alpha_{ai} V_{ib} = 0 \tag{12}$$

$$\sum_{i=1}^{n} \beta_{ai} V_{ib} = \delta_{ab} \tag{13}$$

$$\sum_{i=1}^{n} \beta_{ai} U_{ib} = 0 \tag{14}$$

as well as

$$\sum_{a=1}^{\frac{1}{2}n} (U_{ia} \alpha_{aj} + V_{ia} \beta_{aj}) = \delta_{ij} \tag{15}$$

The equations (11)-(15) are of course nothing but consequences of the assumed linear relations (9)-(10) between the quantities q_a, p_a and v_i. Let us then return to the question of the canonical structure. At this stage we ***assume*** that the system (7) admits a canonical Hamiltonian formulation, with the Hamiltonian H proportional to the conserved quantity $G(v)$ defined in Eq. (8),

$$H = CG(v) \tag{16}$$

where C is some suitable constant. The equations (7) then ought to be equivalent to the following canonical equations

$$\dot{q}_a = \frac{\partial H}{\partial p_a} \quad , \quad \dot{p}_a = -\frac{\partial H}{\partial q_a} \tag{17}$$

Using the relations (9) one finds the following set of differential equations from Eqns. (7),

$$\dot{q}_a = \sum_{i,j} \alpha_{ai} A_{ij} z_j (e^{v_j} - 1) \tag{18}$$

$$\dot{p}_a = \sum_{i,j} \beta_{ai} A_{ij} z_j (e^{v_i} - 1) \tag{19}$$

Using Eqns. (10) and (16) one finds

$$\frac{\partial H}{\partial p_a} = C \sum_{j=1}^{n} z_j (e^{v_j} - 1) V_{ja} \tag{20}$$

$$\frac{\partial H}{\partial q_a} = C \sum_{j=1}^{n} z_j (e^{v_j} - 1) U_{ja} \tag{21}$$

Comparing the pairs of equations (18),(19) and (20),(21), respectively, with the canonical equations (17) one finds indeed that the differential equations (18),(19) are canonical Hamiltonian equations provided the following conditions hold true,

$$\sum_{i=1}^{n} \alpha_{ai} A_{ij} = C V_{ja} \tag{22}$$

$$\sum_{i=1}^{n} \beta_{ai} A_{ij} = -C U_{ja} \tag{23}$$

So far we have shown, that the system (7) admits a canonical Hamiltonian formulation, with canonical coordinates q_a and momenta p_a given by Eq. (9), and Hamiltonian given by Eq. (16) if and only if constants α_{ai}, β_{ai} and U_{ia}, V_{ia} can be found, which satisfy all the conditions (11)-(15) and (20)-(23), respectively. The existence of such constants will be demonstrated below.

4. Auxiliary linear problem

We now consider a linear eigen-value problem, which is related to the existence of coefficients α_{ai}, β_{ai} and U_{ia}, V_{ia} which fulfill the conditions given above in Sec. III. The eigen-value probblem in question consists of the following pair of eigen-value equations

$$\sum_{k=1}^{n} A_{jk} x_{ak} = \lambda_a y_{aj} \tag{24}$$

$$\sum_{k=1}^{n} A_{jk} y_{ak} = -\lambda_a x_{aj} \tag{25}$$

It is a simple matter to demonstrate the existence of real-valued eigenvectors x_a and y_a with eigenvalues λ_a which can be taken to be non-negative without loss of generality. This can most easily be demonstrated by considering first *e.g.* the eigenvalue problem for the matrix $A^T A$, where A^T denotes the transpose of the antisymmetric matrix (A_{ij}). Furthermore, we have assumed the matrix (A_{ij}) to be non-singular (*i.e.* $\det A \neq 0$) which implies that the eigenvalues λ_a are different from zero, *i.e.* can be chosen to be strictly positive,

$$\lambda_a > 0$$
$$a = 1, 2, .., \frac{1}{2}n \tag{26}$$

For simplicity we will in what follows restrict attention to the case in which the eigenvalues λ_a are distinct,

$$a \neq b \Leftrightarrow \lambda_a \neq \lambda_b \tag{27}$$

Under the stated conditions one finds that the eigenvectors x_a and y_a (if properly normalised) form a bi-orthogonal and complete set,

$$(x_a, y_b) := \sum_{j=1}^{n} x_{aj} y_{bj} = 0 \tag{28}$$

$$a, b = 1, .., \frac{1}{2}n$$

$$(x_a, x_b) = \delta_{ab} \quad , \quad (y_a, y_b) = \delta_{ab} \tag{29}$$

and

$$\sum_{a=1}^{\frac{1}{2}n} (x_{ak} x_{a\ell} + y_{ak} y_{a\ell}) = \delta_{k\ell} \tag{30}$$

We now claim that the constants α_{ai}, β_{ai} and U_{ia}, V_{ia} introduced in Sec. III can be taken to be proportional to the eigen-vector quantities x_{ai} and y_{ai}, respectively.

Thus, let

$$\alpha_{ak} := \psi_a y_{ak} \tag{31}$$

$$\beta_{ak} := \varphi_a x_{ak} \tag{32}$$

and

$$U_{ja} := \frac{\varphi_a \lambda_a}{C} y_{aj} \tag{33}$$

$$V_{ja} := \frac{\psi_a \lambda_a}{C} x_{aj} \tag{34}$$

where the quantities φ_a and ψ_a are constants to be determined.

All the necessary (and sufficient) conditions (11)-(15) and (22)-(23) are now satisfied with the choices (31)-(32) and (33)-(34), respectively, provided the constants φ_a and ψ_a satisfy the following condition,

$$\varphi_a \psi_a \lambda_a = C \tag{35}$$

The validity of the assertion above follows straightforwardly from the conditions of bi-orhtogonality (28)-(29) and completeness (30) of the eigenvector quantities x_a and y_a.

Without essential loss of generality we choose to satisfy the condition (35) as follows,

$$\varphi_a = \psi_a = \sqrt{\frac{C}{\lambda_a}} \tag{36}$$

To summarise: we have demonstrated that the following variables q_a and p_a are canonical coordinates and momenta, respectively, for the system of equations (7), with the Hamiltonian given by Eq. (16),

$$p_a = \sqrt{\frac{C}{\lambda_a}} \sum_{i=1}^{n} x_{ai} v_i \tag{37}$$

$$q_a = \sqrt{\frac{C}{\lambda_a}} \sum_{i=1}^{n} y_{ai} v_i \tag{38}$$

The quantities x_{ai}, y_{ai} and λ_a are defined by the eigen-value problem (24)-(25).

No doubt the consuderations above could be replaced by general arguments from symplectic geometry. However, we have in this lecture preferred an elementary discussion based on linear algebra, which is perfectly adequate in the present case.

Needless to say, the discussion above can be modified so as to include the special cases of coincident eigenvalues or of singular matrices (A_{ij}), which implies the occurence of zero eigenvalues λ_a. A particular case of the latter kind has recently been analysed by A.Yu. Volkov [4], who refers to L.D. Faddeev and L.A. Takhtajan [5] for background material. The Hamiltonian obtained by the methods given in this paper differs (in a substantial, non-trivial way) from that obtained by Volkov.

Acknowledgements

The work presented here has been made in collaboration with Professor Milan Noga. It is a pleasure to record my indebtness to my friend and colleague Milan Noga for a most stimulating collaboration. I also wish to record my indebtness to Professor Briitta Koskiaho for many important and valuable discussions on non-linear dynamics as well as various aspects of complex systems in general.

References

[1] A.J. Lotka: *J. Phys. Chem.* **14** (1910), 271;
Proc. Natl. Acad. Sci. U.S. **6** (1920), 410
V. Volterra: *Leçons sur la theorie Mathematique de la Lutte pour la Vie.* Gauthier-Villors, Paris 1931

[2] See *eg.* N.S. Goel, S.C. Maitra, E.W. Montroll: *Rev. Mod. Phys.* **43** (1971), 231;
S. Novikov, S.V. Manakov, L.P. Pitaevski, V.E. Zakharov: *Theory of Solitons, The Inverse Problem Method.* Plenum Press, New York 1984

[3] C. Cronström, M. Noga *(in preparation).*

[4] A.Yu. Volkov: *Phys. Lett.* **A167** (1992), 345

[5] L.D. Faddeev, L.A. Takhtajan: *Springer Lecture Notes in Physics* **246** (1986), 166