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A MODIFICATION OF THE WIENER PROCESS DUE TO A GENERAL MARKOVIAN TRAIN OF DIFFUSION-ENHANCING PULSES

V. Bezák [1]

Faculty of Mathematics and Physics,
Comenius University,
842 15 Bratislava, Slovakia

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A new modification of the Wiener process is considered generalizing previous results of the author. In the present paper, a multiplicative stochastic process is analyzed combining two processes: the Wiener process defined by a diffusion constant D and a general Markovian random train of diffusion-enhancing delta-pulses of strength M. The waiting time τ between the M-pulses is defined by an arbitrary probability $\Phi(\tau)$, whilst in author's former paper, $\Phi(\tau)$ was chosen rather specially (in the Poisson form). The conditional probability density and three forms of the evolution equation for the process are derived. Green's function of the evolution equation is represented as a functional integral generalizing the Feynman-Kac integral. Possible applications are discussed, including the dechannelling kinetics of high-energy particles in crystals.

I. INTRODUCTION

In our former paper [1], we have analyzed a modification of the Wiener process due to a Poisson random train of diffusion-enhancing pulses. This means, if we say it more precisely, we dealt with a multiplicative one-dimensional stochastic process $\xi(u)$, $0 \le u \le t$, which was defined by the stochastic differential equation

$$\xi(u) = [2D(u)]^{1/2} f(u), \tag{1}$$

with some fixed starting point

$$\xi(0) = x_0, \tag{2}$$

with the standard Gaussian white-noise random function $f(u)$ defined by the conditions

$$\langle f(u)\rangle_G = 0, \quad \langle f(u_1)f(u_2)\rangle_G = \delta(u_1 - u_2), \tag{3}$$

[1] E-mail address : BEZAK@FMPH.UNIBA.CS

and with a two-valued positive random function $D(u)$ defined in a Poisson way. Having interpreted $\xi(u)$ as the Wiener process (i.e. Brownian motion) on random sections of the time interval $(0,t)$, we gave the following meaning to the function $D(u)$: it was defined by two randomly alternating diffusion constants, D and D_p (so that $0 < D < D_p$). In other words, we spoke of some random alternation of the diffusive D-state and D_p -state. We might equally speak of D_p -state pulses on the D-state background. Each pulse was taken with the same duration τ_p, but the waiting time τ between the pulses was considered as a random variable with the Poisson probability

$$\Phi_p(\tau) = \exp(-\nu\tau), \quad \nu = \text{const} > 0. \tag{4}$$

Moreover, we simplified the problem by the condition

$$\lim D_p \tau_p = M = \text{const} > 0 \tag{5}$$

for $D_p \to \infty$, $\tau_p \to +0$. In this case, we spoke of M-pulses (i.e. delta-function-shaped pulses with the "strength" M).

In the present paper, we again deal with the process $\xi(u)$ defined by eqs. (1), (2) and (3), with a two-state random sequence of the pulses of the fixed duration τ_p. Even condition (5) will still be stipulated. Only in one point our present problem is different from, and formulated more generally than, the problem solved in our former paper: we will not confine ourselves on the special function given by expression (4). Instead, we shall take a general function $\Phi(\tau, t_i)$ in the role of the probability for the waiting time τ between the pulses. We assume that this function is in conformity with the Markov property of the process $D(u)$: if t_i is any time instant when an n-th pulse is ended (i.e. when the Brownian particle begins moving in the D-state), the function $\Phi(\tau, t_i)$ defining the probability with which the D-state is to survive until the time instant $t_i + \tau$ (when the $(n+1)$-th pulse is switched on) must be independent of pulses preceding the n-th. We require four conditions which the function $\Phi(\tau, t_i)$ has to satisfy. The first three are these:

$$\Phi(\tau, t_i) > 0 \quad \text{for each} \quad \tau > 0,\ t_i > 0, \tag{6}$$

$$\partial\Phi(\tau, t_i)/\partial\tau \le 0 \quad \text{for each} \quad \tau > 0,\ t_i > 0, \tag{7}$$

$$\Phi(0, t_i) = 1. \tag{8}$$

The fourth is the requirement of the existence of a constant $\nu > 0$ so that

$$\lim_{\tau\to\infty} \exp(\nu\tau)\Phi(\tau) = c = \text{ const}, \ \Phi(\tau) = \Phi(\tau, 0), \tag{9}$$

where $0 < c < 1$. The last condition says, in fact, that the asymptotic behaviour of $\Phi(\tau, 0)$ for $\tau \to \infty$ resembles the function $c\exp(-\nu\tau)$. In the limiting case when the function $\Phi(\tau, 0)$ is reduced to the Poisson function (4), we have $c = 1$ and condition (8) is satisfied exactly for all values of τ. As the probability $\Phi(\tau, t_i)$ depends on t_i, the train of the M-pulses is not stationary in the statistical sense.

The mathematics of non-Poissonian point processes (or pulse processes) of the kind relevant to our problem is rather old (Domb [2] ; cf. also [3] , Chapter 2, § 12). Together with the function $\Phi(\tau, t_i)$, we shall also use the function

$$\varphi(\tau) = -\partial \ln \Phi(\tau, t_i)/\partial\tau. \tag{10}$$

Inversely,

$$\Phi(\tau, t_i) = \exp[-\int_{t_i}^{\tau} du\, \varphi(u)]. \tag{11}$$

We interpret

$$dP_D(\tau, t_i) = \Phi(\tau, t_i)\varphi(\tau)d\tau \tag{12}$$

as the joint probability that just after a certain M-pulse is applied at an arbitrary time instant t_i, the D-state lasts until the time instant $t_i + \tau$ and afterwards, within the time interval $(t_i + \tau, t_i + \tau + d\tau)$, the next M-pulse arrives with the probability $\varphi(\tau)d\tau$. Condition (9) implies that $\varphi(\tau)$ tends to some positive constant ν if $\tau \to \infty$. Clearly,

$$\int_0^{+\infty} d\tau\, \Phi(\tau, t_i)\varphi(\tau) = 1. \tag{13}$$

Eq. (1) suggests that we have to distinguish between the averaging $\langle\,\rangle_G$ with respect to the Gaussian white noise $f(u)$ and the averaging $\langle\,\rangle_p$ with respect to the process $D(u)$ (i.e. with respect to the pulse statistics). The total averaging is

$$\langle\,\rangle = \langle\langle\,\rangle_p\rangle_G = \langle\langle\,\rangle_G\rangle_p. \tag{14}$$

In § 2, we calculate the fundamental probability density

$$P(x, t|x_0, 0) = \langle\, \delta(x - \xi(t))\, \rangle \tag{15}$$

under the assumption that there are no boundaries in play. For brevity, we shall write $P(x, t|x_0, 0) \equiv P(x, t|x_0)$. However, if the initial time t_0 becomes shifted, then the function $P(x, t|x_0, t_0)$ is not equal to $P(x, t - t_0|x_0)$ (unless the function $\varphi(\tau)$ is reduced to some constant corresponding to the Poissonian case).

We use the same diagrammatic method here as in [1]. The function $P(x, t|x_0)$ is derived exactly in form of an infinite series. The derivation reveals explicitly the functional dependence of $P(x, t|x_0)$ on the function Φ. In § 3, we prove that $P(x, t|x_0)$ satisfies the Chapman-Kolmogorov equality. This verifies the Markov property of the process $\xi(u)$. In § 4, we calculate the cumulants of the end point x.

If the initial (probability) density (at time $t_0 = 0$) is defined as an arbitrary (non-negative) function $\psi_0(x)$, we can identify the density $\psi(x, t)$ at time $t > 0$ with the integral

$$\psi(x, t) = \int_{-\infty}^{+\infty} dx_0\, P(x, t|x_0)\psi_0(x_0). \tag{16}$$

§ 5 is devoted to derivations of three various forms of the evolution equation for $\psi(x,t)$. Two of these forms - the first being a functional differential equation and the second being an integro-differential equation - remind mathematically a "generalized Schrodinger equation". It is easy to show that these equations are equivalent. § 6 involves a brief discussion of how to incorporate boundary conditions into the theory. In § 7, we add a "potential-energy term" $-V(x,t)\psi(x,t)$ into these equations. Then the function $P(x,t|x_0)$ becomes a functional of the "potential energy" $V(x,t)$ and must be calculated anew. We can represent the function $P(x,t|x_0)$ in form of a product integral (§ 8). This is, in fact, a functional integral generalizing the concept of the Feynman-Kac integral (Feynman and Hibbs [4], Kac [5], Feynman [6]). Taking an infinitesimal time increment $\varepsilon \to +0$ and using the function $P(x,t+\varepsilon|x_0,t)$ in the role of a given kernel in the integral form of the evolution equation for $\psi(x,t)$, we can rederive the functional differential form of this equation.

In § 9, we discuss applications of our theory.

II. DERIVATION OF PROBABILITY DENSITY $P(x,t|x_0)$ BY A DIAGRAMMATIC METHOD

We define, at first, the Gaussian probability density

$$G(x,t|x_0) = \langle\, \delta(x-\xi(t))\,\rangle_G \tag{17}$$

so that

$$P(x,t|x_0) = \langle\, G(x,t|x_0)\,\rangle_p = \langle\, G(x,t|x_0)\,\rangle . \tag{18}$$

Evidently,

$$G(x,t|x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ \exp[ik(x-x_0)]\exp[-k^2\int_0^t d\tau\ D(\tau)]. \tag{19}$$

We write also the Fourier integral

$$P(x,t|x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ \exp[ik(x-x_0)]p(k,t). \tag{20}$$

Then

$$p(k,t) = \langle\, \exp[-k^2\int_0^t d\tau\ D(\tau)]\,\rangle_p . \tag{21}$$

We represent this function as the sum of terms corresponding to simple diagrams depicted in Fig. 1.

Fig. 1. The scheme of the summation over the diagrams contributing to the function $p(k,t)$.

Each diagram spans the time interval $(0,t)$. The diagrams involve the propagator lines (with the arrows indicating the time direction) and the M-pulse dots. We formulate three rules for reading the diagrams. The first says that we attribute the function

$$p_0(k,\Delta\tau,\tau_j) = \Phi(\Delta\tau,\tau_j)\exp(-k^2 D\Delta\tau) \tag{22}$$

to each propagator line of length $\Delta\tau$ starting at τ_j within the interval $(0,t)$. The second rule says that a dot the "coordinate" of which in the diagram is τ_j contributes by the factor $\varphi(\tau_j)\exp(-kM^2)$ to the corresponding term in the series representing the function $p(k,t)$. The third rule says that we must exhaust all possible intermediate time instants τ_j (of the dots), i.e. we must integrate over them. In this way we obtain the series, with $p_0(k,t) \equiv p_0(k,t,0)$,

$$p(k,t) = p_0(k,t)[1+\sum_{n=1}^{\infty}\exp(-nk^2M).\int_0^t d\tau_n\int_0^{\tau_n} d\tau_{n-1}\ldots\int_0^{\tau_2} d\tau_1\ \varphi(\tau_n)\varphi(\tau_{n-1})\ldots\varphi(\tau_1)].$$

However, it can easily been proved that

$$\int_0^t d\tau_n\ldots\int_0^{\tau_2} d\tau_1\ \varphi(\tau_n)\ldots\varphi(\tau_1) = \frac{1}{n!}[\int_0^t d\tau\ \varphi(\tau)]^n .$$

That is why we may write the result

$$p(k,t) = p_0(k,t)\exp[\int_0^t d\tau\ \varphi(\tau)\exp(-k^2M)] \tag{23}$$

or, more explicitly,

$$\begin{aligned} p(k,t) &= \exp\{-\int_0^t d\tau\ \varphi(\tau)[1-\exp(-k^2M)] - k^2Dt\} \\ &= \exp\{\ln\Phi(t)[1-\exp(-kM)] - kDt\} \end{aligned} \tag{24}$$

(where we have used relation (11)).

We rewrite once more the last expression as the sum

$$p(k,t) = \Phi(t)\sum_{n=0}^{\infty}\frac{1}{n!}(-\ln\Phi(t))^n\exp(-nk^2M) \tag{25}$$

where $\Phi(t) = \Phi(t,0)$.

Returning to the x-representation, we state that $P(x,t|x_0)$ can be written as a linear combination of an infinite number of Gaussians .h positive coefficients depending on the probability $\Phi(t)$,

$$P(x,t|x) = \Phi(t)\sum_{n=0}^{\infty}\frac{1}{n!}(-\ln\Phi(t))^n\Gamma(x-x_0, 2(nM+Dt)). \tag{26}$$

Here we have used the denotation

$$\Gamma(x,\sigma^2) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\exp(-\frac{x^2}{2\sigma^2}) \tag{27}$$

for the Gaussian with the dispersion (variance) σ^2.

III. THE MARKOV PROPERTY - THE CHAPMAN-KOLMOGOROV IDENTITY

Let t_1 be any intermediate time instant, $0 < t_1 < t$.

Since

$$p_0(k,t) = p_0(k,t-t_1)p_0(k,t_1) \tag{28}$$

and

$$\Phi(t) = \Phi(t-t_1,t_1)\Phi(t_1), \tag{29}$$

it is seen from formulae (23), (24) that

$$p(k,t) = p(k,t-t_1)p(k,t_1). \tag{30}$$

Then, after applying the convolution theorem, we arrive at the Chapman-Kolmogorov identity

$$P(x,t|x_0) = \int_{-\infty}^{\infty} \mathrm{d}x_1\, P(x,t|x_1,t_1)P(x_1,t_1|x_0,0) \tag{31}$$

expressing a semigroup composition and conforming the Markov character of our multiplicative process $\xi(u)$.

Let $P(x,t+\varepsilon|x_0,t)$ (for an infinitesimal time increment $\varepsilon > 0$) be known in advance (see § 7). Then, by dissecting the time interval $(0,t)$ on N equal subintervals and by applying identity (31) $N-1$ times, we get the product-integral representation of $P(x,t|x_0)$:

$$P(x,t|x_0) = \int_{-\infty}^{\infty}\mathrm{d}x_{N-1}\int_{-\infty}^{\infty}\mathrm{d}x_{N-2}\ldots\int_{-\infty}^{\infty}\mathrm{d}x_1$$
$$P(x,N\varepsilon|x_{N-1},(N-1)\varepsilon)\ldots P(x_2,2\varepsilon|x_1,\varepsilon)P(x_1,\varepsilon|x_0) \tag{32}$$

where $\varepsilon = t/N$. The multiple integral (32) becomes a functional integral if $N \to \infty$.

It should be pointed out that there would not be any essential hindrance to introduce a special mathematical measure for this functional integral. This measure would be dependent on the function $\Phi(\tau)$ and might be regarded as a generalization of the well-known Wiener measure (Yeh [7], Hida [8]).

IV. CUMULANTS OF THE END POINT x

Owing to the simplicity of the definition of our multiplicative stochastic process $\xi(u)$, we can derive all its correlation functions $\langle\xi(u_1)\xi(u_2)\ldots\xi(u_n)\rangle$. Here we will focus attention on the end point $\xi(t) = x$ which is a single random variable. We shall calculate all its cumulants $c_n(t)$. (The cumulants are linear combinations of the statistical moments $\langle x^m\rangle$, $m = 1,\ldots,n$.)

When reversing the Fourier transformation (20),

$$p(k,t) = \int_{-\infty}^{\infty}\mathrm{d}x\, \exp(-ikx)P(x,t|0), \tag{33}$$

we see that

$$p(k,t) = \langle\exp(-ikx)\rangle. \tag{34}$$

The cumulant $c_n(t)$ is defined by the formula

$$c_n(t) = i^n\frac{\partial^n \ln\langle\exp(-ikx)\rangle}{\partial k^n}\Big|_{k=0} \tag{35}$$

If we put expression (24) here, we find that all cumulants with odd indices are equal to zero. The first non-zero cumulant is:

$$c_2(t) = 2[-M\ln\Phi(t) + Dt]. \tag{36}$$

The higher-order non-zero cumulants are:

$$c_{2n}(t) = -(2n-1)!!\,(2M)^n\ln\Phi(t) \tag{37}$$

$(n = 2,3,\ldots)$, where $(2n-1)!! = (2n-1)(2n-3)\ldots 3.1$.

Formulae (36), (37) disclose that the time dependence of the cumulants is modelled by the waiting probability $\Phi(\tau)$. Clearly, $0 < \Phi(\tau) \leq 1$ and $\Phi(\tau) \to 0$ for $\tau \to \infty$. Let τ_c be a characteristic waiting time, chosen with respect to condition (9), allowing to discriminate whether the function $\Phi(\tau)$ is already resembling (*if* $\tau_c < \tau$), or not yet resembling (if $0 < \tau < \tau_c$), the exponential $c\exp(-\nu\tau)$. Obviously, $\varphi(\tau)$ may be approximated by the constant ν for $\tau \gg \tau_c$. We adopt a very natural definition of τ_c: we identify it with the average value $\langle\tau\rangle$ of the waiting time,

$$\langle\tau\rangle = \int_0^{\infty}\mathrm{d}\tau\,\tau\Phi(\tau)\varphi(\tau) = \int_0^{\infty}\mathrm{d}\tau\,\Phi(\tau). \tag{38}$$

For values of the time t less than $\langle\tau\rangle$, the multiplicative process $\xi(u)$ considered in the present paper is markedly distinct from the Poisson-modified process considered in our former paper [1]. For much longer times, on the other hand, all specific features resulting from the non-constancy of the function $\varphi(\tau)$ are lost. In the asymptotic approach, for $t \gg \langle\tau\rangle$, we may write down the former results [1] :

$$c_2(t) \cong 2(\nu M + D)t, \tag{39}$$

$$c_2(t) \cong (2n-1)!!\,(2M)^n \nu t,\ n = 2, 3, \ldots. \tag{40}$$

Therefore, for any Wiener process $\xi(u)$ modified by any non-Poissonian (but Markov) random train of the diffusion-enhancing M-pulses, the Poissonian modification does prove to be a good long-time approximation.

V. EVOLUTION EQUATION FOR THE GENERAL PROBABILITY DENSITY

In order to construct the evolution equation for the general probability density $\psi(x,t)$ (see eq. (16)), we formulate, at first, the evolution equation for the function $p(k,t)$. When putting together formula (24) and (34), we can at once state the validity of the equations

$$\ln\langle\exp(-ikx)\rangle = \int_0^t \mathrm{d}\tau\, \varphi(\tau)[\exp(-Mk^2) - 1] - Dtk^2, \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln p(k,t) = \varphi(t)\,[\exp(-Mk^2) - 1] - Dk^2. \tag{42}$$

(Here we have taken k as a constant.) Using eqs. (41), (42), we arrive at the equation

$$\frac{\mathrm{d}p(k,t)}{\mathrm{d}t} = \frac{1}{t}\{\ln\langle\exp(-ikx)\rangle + \int_0^t \mathrm{d}\tau\,[\varphi(t) - \varphi(\tau)].[\exp(-Mk^2) - 1]\}p(k,t). \tag{43}$$

Note that

$$\int_0^t \mathrm{d}\tau\,[\varphi(t) - \varphi(\tau)] = \int_0^t \mathrm{d}\tau\ \tau\frac{\mathrm{d}\varphi(\tau)}{\mathrm{d}\tau}. \tag{44}$$

Now we will show that the evolution equation for $\psi(x,t)$ can be written in three forms: in the functional differential form (which we call the cumulant-expansion form), in the integro-differential form and, finally, in the fully integral form with a kernel which we can give in advance. Actually, the last form is nothing but the integral equation common in the theory of propagators (Feynman 1948, [9]).

5.1. The cumulant - expansion form

According to the definition of the cumulants $c_n(t)$ we write the series

$$\ln\langle\exp(-ikx)\rangle = \sum_{n=1}^{\infty}\frac{c_{2n}(t)}{(2n)!}(-k^2)^n. \tag{45}$$

After inserting expressions (36), (37) into it, we obtain the equality

$$\ln\langle\exp(-ikx)\rangle = -\ln\Phi(t)\sum_{n=1}^{\infty}\frac{(-k^2)^n}{n!} - Dtk^2$$

so that, respecting relation (11), we may write

$$\ln\langle\exp(-ikx)\rangle = t\left\{\frac{1}{t}\int_0^t \mathrm{d}\tau\,\varphi(\tau)\,[\exp(-Mk^2) - 1] - Dk^2\right\}. \tag{46}$$

Thus the final form of eq. (43) for $p(k,t)$ is

$$\frac{\mathrm{d}p(k,t)}{\mathrm{d}t} = \{\varphi(t)\,[\exp(-Mk^2) - 1] - Dk^2\}p(k,t). \tag{47}$$

Now we must realize the transition from the k-representation to the x-representation. To do so we may replace $p(k,t)$ by $\psi(x,t)$ and the variable k by the operator $-i\partial/\partial x$. So we get the equation

$$\frac{\partial\psi(x,t)}{\partial t^2} = \left\{\varphi(t)\left[\exp\left(M\frac{\partial^2}{\partial x^2}\right) - 1\right] + D\frac{\partial^2}{\partial x^2}\right\}\psi(x,t). \tag{48}$$

The operator $\exp(M\partial^2/\partial x^2)$ has to be understood as the series:

$$\exp(M\frac{\partial^2}{\partial x^2}) = \sum_{n=0}^{\infty}\frac{M^n}{n!}\frac{\partial^{2n}}{\partial x^{2n}}. \tag{49}$$

The infinite number of the derivatives $\partial^{2n}/\partial x^{2n}$ in eq. (48) means that we have obtained in fact a functional differential equation.

5.2. The integro - differential form

For the Fourier original $P(x,t|x_0)$ to $p(k,t)$, we obtain directly, using the convolution theorem, the equation

$$\begin{aligned}\frac{\partial P(x,t|x_0)}{\partial t} &= \varphi(t)\int_{-\infty}^{+\infty}\mathrm{d}x'[\Gamma(x-x'), 2M) - \delta(x-x')]\,P(x',t|x_0) \\ &+ D\frac{\partial^2 P(x,t|x_0)}{\partial x^2}.\end{aligned} \tag{50}$$

Similarly,

$$\frac{\partial\psi(x,t)}{\partial t} \doteq \varphi(t) \int_{-\infty}^{+\infty} dx' \, [\Gamma(x-x',2M) - \delta(x-x')] \, \psi(x',t) + D\frac{\partial^2\psi(x,t)}{\partial x^2}. \quad (51)$$

(See the denotation for the Gaussian $\Gamma(x,\sigma^2)$, formula (27).)

5.3. The integral form

After rewriting eq. (16) with any initial function $\psi(x,t_0)$ instead of the function $\psi_0(x) = \psi(x,0)$ we obtain the equation

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx_0 \, P(x-x_0,t|0,t_0)\psi(x_0,t_0) \quad (52)$$

for $t > t_0 > 0$. What is known a priori here is the kernel $P(x-x_0,t|0,t_0) \equiv \equiv P(x,t|x_0,t_0)$ (cf. expression (26)) and the initial function $\psi(x,0) = \psi_0(x)$ (not the function $\psi(x,t_0)$ if $t_0 > 0$). We shall resume eq. (52) in § 8 where its meaning will be explained within a more general framework.

VI. A NOTE ON BOUNDARY CONDITIONS

If there is a boundary (or if there are two boundaries) we may still consider eq. (48) as validated but we must respect a boundary condition (or boundary conditions). We may take into account, e.g., an absorbing or reflecting boundary. As to eq. (51), a certain cautiousness is needed in interpreting the integral term. To elucidate this, let us consider the problem with one absorbing boundary, $x = 0$, for the process $\xi(u)$ running on the half-axis $x > 0$. Using the function

$$P_\infty(x,t|x_0) = \Phi(t)\sum \frac{1}{n!}(-\ln\Phi(t))^n\Gamma(x-x_0, 2(nM+Dt)) \quad (53)$$

(the solution for the free process without any confinement), we can easily construct the solution to eq. (50) satisfying the conditions

$$P(x,0|x_0) = \delta(x-x_0), \quad (54)$$

$$P(0,t|x_0) = 0. \quad (55)$$

Indeed,

$$P(x,t|x_0) = P_\infty(x,t|x_0) - P_\infty(x,t|-x_0) \text{ for } x > 0. \quad (56)$$

(Of course, also $x_0 > 0$. If the boundary $x = 0$ were reflecting, the sign minus on the r.h.s. of (56) would be replaced by plus. This result clearly reminds the possibility to use the well known method of images in problems with boundaries.)

From the point of view of physics, the solution $P(x,t|x_0)$ should vanish for $x < 0$. Nevertheless, when inserting the function $P(x,t|x_0)$ given by formula (56) into eq. (50), we have to use its analytical continuation in the variable x, i.e. not the solution with the zero value for $x < 0$. This follows from the stochastic equation

$$\frac{\partial G(x,t|x_0)}{\partial t} = D(t)\frac{\partial^2 G(x,t|x_0)}{\partial x^2}$$

and from its solution

$$G(x,t|x_0) = G_\infty(x,t|x_0) - G_\infty(x,t|-x_0)$$

satisfying the same conditions as those stipulated for the pulse-averaged Green's function $P(x,t|x_0)$ above (conditions (54), (55)). As $G_\infty(x,t|x_0)$, $G_\infty(x,t|-x_0)$ are defined as the functions due to the free diffusion from two sources, their Fourier components are equal and their pulse-averaging (cf. definition (18)), which may be done separately, is also equal. But we must stress it here that if the convolution theorem is to be applied separately with the functions $G_\infty(x,t|x_0)$, $G_\infty(x,t|-x_0)$, in order to derive the integral term in eq. (50), they have to be taken as analytical functions along the whole x-axis. Otherwise, if we defined already the function $G(x,t|x_0)$ with the cut-off for $x < 0$, we should obtain another integral term in eq. (50) .

As regards the integral equation (52), its formal validity is independent of absence or presence of any boundary condition(s). Of course, the boundary condition(s) must duly be taken into account in determining the kernel $P(x,t|x_0,t_0)$ which will be different from case to case.

VII. GENERALIZED EVOLUTION EQUATION RESPECTING THE POSSIBILITY OF ANNIHILATION AND/OR GENERATION

It is simple to consider the Brownian motion with a certain kind of annihilation. Let $V_a(x,t)dxdt$ be the probability that a Brownian particle, finding itself in the interval $(x, x+dx)$, may be removed, or shorn of its motion, during the time interval $(t, t+dt)$. The problem is to calculate the density $P(x,t|x_0,t_0)$ of the probability that the particle, finding itself in point x, is still moving at time $t > t_0$, provided that its position at time t_0 was x_0.

Similarly, we can discuss a situation when there is a large amount of equal diffusing particles not interfering with one another, each particle itself moving as the Brownian one. Then the problem is to calculate the evolution of their density $\psi(x,t)$ in time. In this case, we may even admit the possibility of their generation with a rate $V_b(x,t)$. Generation of conduction electrons diffusing in an intrinsic semiconductor is a good example: it can be produced by light (with photons of energy greater than the forbidden gap of the semiconductor).

That is why it is reasonable to define the rate function

$$V(x,t) = V_a(x,t) - V_b(x,t) \quad (57)$$

and to speak, in accordance with its sign, of the (prevailing) annihilation if

$V(x,t) > 0$ or generation if $V(x,t) < 0$. Then we can write down the equations

$$\frac{\partial\psi(x,t)}{\partial t} = \varphi(t)\,[\exp(M\frac{\partial^2}{\partial x^2} - 1]\,\psi(x,t) + D\frac{\partial^2\psi(x,t)}{\partial x^2} - V(x,t)\,\psi(x,t), \quad (58)$$

$$\frac{\partial\psi(x,t)}{\partial t} = \varphi(t)\int_{-\infty}^{\infty} dx'\,[\Gamma(x-x',2M) - \delta(x-x')]\psi(x',t) + D\frac{\partial^2\psi(x,t)}{\partial x^2} - V(x,t)\psi(x,t), \quad (59)$$

generalizing eqs. (48) and (51), respectively.

The integral equation (52) remains unchanged but we have to emphasize that the kernel P is a functional of the rate function $V(x,t)$, so we write

$$\psi(x,t) = \int_{-\infty}^{\infty} dx_0\, P_{\{V(\xi,u)\}}(x,t|x_0,t_0)\,\psi(x_0,t_0) \quad (60)$$

where $-\infty < \xi < \infty$, $0 \le t_0 \le u \le t$. In § 8, we give a construction of the kernel of eq. (60). Note that only in case of a time-independent rate function $V(x)$, we may write:

$$P_{\{V(\xi,u)\}}(x,t|x_0,t_0) \equiv P_{\{V(\xi,u)\}}(x,t-t_0|x_0). \quad (61)$$

VIII. THE FUNCTIONAL-INTEGRAL REPRESENTATION OF THE GREEN FUNCTION P

In order to prove the equivalence of eq. (60) with eqs. (58) and (59), we make the difference $t - t_0 = \varepsilon$ infinitesimal. So we write down eq. (60) in the form

$$\psi(x,t+\varepsilon) = \int_{-\infty}^{\infty} d\xi\, P_{\{V(x,t)\}}(x,t+\varepsilon|x+\xi,t)\,\psi(x+\xi,t), \quad (62)$$

$x_0 = x + \xi$, using the kernel defined by the expression

$$P_{\{V(x,t)\}}(x,t+\varepsilon|x+\xi,t) = \exp[-\varepsilon\varphi(t)]\,[\Gamma(\xi,2D\varepsilon) + \varepsilon\varphi(t)\Gamma(\xi,2M)] \cdot \exp[-\varepsilon V(x,t)]. \quad (63)$$

We may employ the developments

$$\psi(x,t+\varepsilon) = \psi(x,t) + \frac{\partial\psi(x,t)}{\partial t} + \ldots, \quad (64)$$

$$\psi(x+\xi,t) = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m\psi(x,t)}{\partial x^m}\xi^m. \quad (65)$$

After setting expressions (63), (65) into the integral in eq. (62) and performing the ξ-integration, we obtain the r.h.s. of eq. (62) as the following function of the variable ε:

$$\psi(x,t) + \varepsilon\left\{\varphi(t)[\exp(M\frac{\partial^2}{\partial x^2}) - 1]\,\psi(x,t) + D\frac{\partial^2\psi(x,t)}{\partial x^2} - V(x,t)\psi(x,t)\right\} + \ldots.$$

By equalizing the coefficients at ε in this function and in the r.h.s. of (64), we obtain exactly the functional differential equation (58). Eq. (59) is its equivalent because of the identity

$$\exp(M\partial^2/\partial x^2)\,\psi(x,t) = \int_{-\infty}^{\infty} dx'\,\Gamma(x-x',2M)\,\psi(x',t). \quad (66)$$

The short-time propagator (63) may be used repeatedly according to formula (32). When taking $\varepsilon = t/N$, $N \to \infty$, we can construct - for any rate function $V(\xi,\tau)$ - the functional integral

$$P_{\{V(\xi,\tau)\}}(x,t|x_0,0) = \exp[-\int_0^t d\tau\,\varphi(\tau)] \left(\prod_{j=0}^{N-1}\int_{-\infty}^{\infty} d\xi_j[\Gamma(\xi_{j+1}-\xi_j,2D\varepsilon) + \varepsilon\varphi(j\varepsilon)\,\Gamma(\xi_{j+1}-\xi_j,2M)].\exp[-\varepsilon V(\xi_j,j\varepsilon)]\right) .\delta(\xi_0 - x_0) \quad (67)$$

where $\xi_N = x$. (The points x, x_0 and the time $t > 0$ are fixed but the variables ξ, τ are running: $-\infty < \xi < \infty$, $0 < \tau < t$.)

Note that if $\varphi(\tau) \equiv 0$ then this functional integral is nothing but the well-known Feynman-Kac integral ([9], [5]). Actually, the same statement can be proved even if $M \to +0$. Then we may write:

$$\Gamma(\xi_{j+1}-\xi_j,2D\varepsilon) + \varepsilon\varphi(j\varepsilon)\,\Gamma(\xi_{j+1}-\xi_j,2M) \to \Gamma(\xi_{j+1}-\xi_j,2D\varepsilon)\,\{1 + \varepsilon\varphi(j\varepsilon)\,\delta(\xi_{j+1}-\xi_j)[\Gamma(\xi_{j+1}-\xi_j,2D\varepsilon)]^{-1}\} \cong \Gamma(\xi_{j+1}-\xi_j,2D\varepsilon)\,\exp[+\varepsilon\varphi(j\varepsilon)]$$

and

$$\prod_{j=1}^{N-1}\exp[\varepsilon\varphi(j\varepsilon)] \to \exp[\int_0^t d\tau\,\varphi(\tau)].$$

Thus the factor standing in front of the multiple integral (67) is accurately cancelled out in the limiting case when $M \to +0$ and we have again obtained the Feynman-Kac integral (with a general "potential energy" $V(\xi,\tau)$). The functional integral (67), valid for any $M > 0$, is its natural generalization.

IX. CONCLUDING REMARKS ABOUT POSSIBLE APPLICATIONS

The theory of the multiplicative stochastic processes described in the present paper (theory of the Wiener processes modified by random diffusion-enhancing, very short-lasting, pulses) can yield well motivated applications.

One of them is connected with the dechannelling kinetics of relativistic electrons in crystals (Laskin [10]). If a monochromatic beam of high-energy electrons is aimed into an "easy direction" in a crystal, their penetration may become considerably deep owing to the channelling phenomenon. Laskin has described the situation ve. clearly. According to his basic idea, there always exist intense, though extremely short, fluctuations in atomic vibrations along each channel in which the electrons are to travel. These fluctuations represent the most important factor influencing the kinetics of deflections of the electrons from the channels. The short, but intense, fluctuations along the channels are, in fact, analogues of what we have called the M-pulses. Moreover, he assumed that the random sequence of these events (i.e. pulses) should obey the Poisson statistics. Our present theory is a generalization of his : we admit an arbitrary, in general non-Poissonian, statistics of these pulses, stipulating only their Markov character. This generalization might be interesting, when, for instance, the crystalline lattice would be subjected to some inhomogeneous deformation. The statistical distribution of various defects leading to local deformations of the crystalline lattice may even imply *correlations* between the defects and, consequently, the random sequence of the pulses affecting the flight of the electrons in the channels really need not be Poissonian. Furthermore, our theory, when applied to the dechannelling kinetics, provides an additional term in Laskin's kinetic equation. It is the term with the diffusion constant D, whilst Laskin has taken $D = 0$ (cf. [1]).

In [1],we offered also another idea for consideration: to employ the concept of the Wiener process $\xi(u)$ accompanied by the random M-pulses in a theory of quasiparticles. Our idea was to vary the parameters defining the process $\xi(\tau)$, in order to model some interesting dispersion law in the corresponding Bloch equation for the canonical density matrix of the quasiparticles. When having this application in mind, the following remark is pertinent: in this application, we must *unequivocally* confine ourselves to the Poisson modification of the random train of the M-pulses. This simply follows from the physical meaning of the Bloch equation where the temperature parameter $\beta = 1/k_BT$ plays a similar role as the time variable t in the theory of the present paper. (If $\varphi(\beta)$ were not reduced to a constant, then there would actually be no place for such a function in the Bloch equation for the density matrix at all.)

The particular case when $D \to +0$ deserves a special attention. Indeed, this case is related to the notion of "diffusion processes with random intermittencies" (if we do not consider D_p as infinite and τ_p as zero). (If we do accept the formal condition of the present paper according to which $D_p \to \infty$ and $\tau_p \to +0$, then we have, in fact, a "degenerated class" of the diffusion processes with random intermittencies".) The problem of diffusions with randomly distributed intermittencies (a theoretical topic interesting in its own right) was put forward by Zeĺdovich et al. [11]. Applications of the problem of diffusions with random intermittencies were

discussed, e.g., by van Kampen [12] and by Balakrishnan et al. [13]. Recently also Luczka et al. analysed the case of randomly interrupted diffusion [14], [15].

The multiplicative stochastic processes $\xi(\tau)$ defined above are non-Gaussian (provided that $M \neq 0$ and $\varphi(\tau) \not\equiv 0$ irrespective of whether D does vanish or not). Because of their relatively simple definition, they can compete, as we hope, with other non-Gaussian stochastic processes. Examples of non-Gaussian stochastic processes for which cumulants and related other mean values can be derived in analytical forms do seem to be a rarity in the literature (cf. e.g. [16]).

Extensive calculations using the idea of pulse-modified Wiener processes are performable in solving the problem of the homogenization of random concentration profiles. This problem was recently solved with the Poissonian definition of the array of diffusion-enhancing pulses [17] . Clearly, the use of an arbitrary (non-Poissonian) array of the pulses in the calculations would equally be possible.

Our final example concerns the so-called superdiffusion in a randomly layered aquifier.Here we have in mind a theoretical model which was constructed by Matheron and de Marsily [18] and extended recently by Bouchaud et al. [18]. The model assumes a stationary (in time) and statistically uniform (in space) random velocity field $u_x(y)$. (For simplicity, they have taken $u_y \equiv u_z \equiv 0$ and $\langle u_x(y)\rangle_c \equiv 0$. Here $\langle\ \rangle_c$ is an average over all velocity "configurations".) The x-axis is directed along the layers and the y-axis perpendicularly to them. If there were no transfer of matter from layer to layer, then any diffusant, having its y-coordinate fixed in one of the layers, would simply flow with a constant velocity $u_x(y)$ along the x-axis (either forwards or backwards). However, if there is a migration mechanism in the y-direction, this automatically triggers off also a migration in the x-direction. In particular, Bouchaud et al. have defined the perpendicular migration by a Wienerian process $y(\tau)$ with a diffusion coefficient D. Then, having taken the autocorrelation function of the velocity field in the simplest form corresponding to a very short correlation length, $\langle u_x(y)u_x(y')\rangle_c = \sigma\,\delta(y-y')$, they were able to derive the superdiffusive behaviour of the process $x(\tau)$ (controlled by the process $y(\tau)$):

$$\langle x(t)^2\rangle = [4\sigma/3(\pi D)^{1/2}]t^{3/2} \tag{68}$$

Note that the superdiffusion means that $\langle x(t)^2\rangle/t$ tends to infinity if $t \to \infty$. To generalize the result expressed by formula (68), we propose to use $y(\tau)$ as the pulse-modified Wiener process in the sense of the theory presented in this paper. If we denote the average with respect to the process $y(\tau)$ as $\langle\ \rangle_{pW}$, we can write the mean square displacement of $x(t)$ as the integral

$$\begin{aligned}\langle x(t)^2\rangle &= \langle\langle x(t)^2\rangle_c\rangle_{pW}\\ &= (\sigma/\pi)\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\int_{-\infty}^{\infty} dk\,\langle\exp\{ik[y(\tau_1)-y(\tau_2)]\}\rangle_{pW}.\end{aligned} \tag{69}$$

We stop our discussion at this point since a further elaboration of formula (69) would require a new article. Anyway, we may in advance conclude that the expression $\langle\ \rangle_{pW}$ in formula (69) is a functional of the *arbitrary* probability function $\Phi(\tau)$ and thus formula (69) defines a whole class of superdiffusions. Obviously, the superdiffusion which was analysed by Bouchaud et al. [19] does belong to this class as a particular realisation.

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