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CONCENTRIC SHELLS OF MATTER ADMITTING SPIN GENERATED TORSION

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We study concentric shells of matter admitting a spin density and match the configuration to the exterior Schwarzschild solution. To render the problem solvable we allow the normal pressure to vanish in the outer shell and match the solution of the outer shell to the inner shell at the radius of the inner shell.

I. INTRODUCTION

After the discovery of general relativity, numerous generalizations to the theory have been developed. They have never been completely discarded because their predictions involve too small corrections to the usual predictions of general relativity. Weyl's gauge invariant geometry [1] includes a vector field in the geometry of space time, Moffat's theory [2] admitting a non-symmetric metric, Brans-Dicke theory [3] admitting a scalar field in addition to the metric and Einstein-Cartan theory (EC) [4] allowing for torsion and a nonsymmetric connection. All these theories represent generalizations of the original Einstein theory that lead to modifications in cosmology and astrophysics that are worthwhile considering. Actually numerous torsion theories exist with the E.C. theory being the most popular, however gauging the Poincare group leads to a tetrad e^{α}_{μ} and spin connection $\omega^{\alpha\beta}_{\mu}$ which in turn generate the metric and torsion and this theory is not necessarily equivalent to the E.C. theory [5]. In fact Weinberg has suggested that the conditions under which the Poincare theory is proved equivalent to the EC theory have never been fully understood [6]. In the original form of the EC theory, the Einstein-like equation relates the Einstein tensor to the canonical energy momentum tensor and the torsion is related to the spin density [7]. For spin density S and perfect fluid, the EC theory is equivalent to General Relativity with the replacement

$$P \to P - \frac{1}{4}\chi S^2, \quad \epsilon \to \epsilon - \frac{1}{4}\chi S^2,$$

where $\chi = 8\pi G/c^2$ [8].

The object of this paper is to study what effect a spin density has on the spherically symmetric distribution of matter which has two concentric layers. We first find the solution in the outer layer by setting the normal pressure equal to 0, we then find the solution in the inner layer with spin density and pressure. Finally, we match at the boundary of the two layers and match the solution in the outer layer to the exterior solution to obtain the complete solution.

II. SPHERICALLY SYMMETRIC DISTRIBUTION OF MATTER AND SPIN DENSITY IN TWO CONCENTRIC LAYERS

In [8] it was pointed out that for a spherical distribution of matter with the spins aligned in the radial direction, the E.C. theory can be described by allowing

$$P \to P - \frac{1}{4}\chi S^2,\ \epsilon \to \epsilon - \frac{1}{4}\chi S^2,\ \chi = \frac{8\pi G}{c^2} \tag{1}$$

with the usual spherically symmetric Einstein equations (S = spin density). The Einstein equations become with the above replacement and condition of spherical symmetry

$$\frac{d}{dr}(re^{-\lambda}) = 1 - \frac{8\pi G}{c^4}\left(\epsilon_0 - \frac{2\pi G S^2}{c^2}\right)r^2 \tag{2}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\frac{8\pi G}{c^4}\left(-P + \frac{2\pi G S^2}{c^2}\right) \tag{3}$$

$$e^{-\lambda}\left(\frac{\lambda''}{2} - \frac{\lambda'\nu'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r}\right) = -\frac{8\pi G}{c^4}\left(-P + \frac{2\pi G S^2}{c^2}\right) \tag{4}$$

where

$$T_4^4 = \epsilon,\ T_1^1 = -P = T_2^2 = T_3^3 \tag{5}$$

and ϵ and P are modified as in Eq.(1).

We consider a two layer configuration of matter: For $R_1 < r < R_2$, $\epsilon = \epsilon_{01}$, $S = 0$ and for $0 < r < R_1$, $\epsilon = \epsilon_{02}, S = S_0 =$ Constant.

For $R_1 < r < R_2$ we have

$$e^{-\lambda} = 1 - \frac{8\pi G}{3c^4}\epsilon_{01} r^2 + \frac{C_1}{r}, \tag{6}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0, \tag{7}$$

where we have set the radial pressure equal to 0 in the $\binom{1}{1}$ Einstein equation and $S = 0$. For $r > R_2$ we have

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2GM}{rc^2}. \tag{8}$$

¿From Eq. (6) we have at $r = R_2$ (matching to Eq. (8) at $r = R_2$)

$$1 - \frac{2GM}{R_2 c^2} = 1 - \frac{8\pi G}{3c^4}\epsilon_{01} R_2^2 + \frac{C_1}{R_2}$$

$$C_1 = \frac{8\pi G \epsilon_{01} R_2^3}{3c^4} - \frac{2GM}{c^2} \tag{9}$$

¿From Eq. (7) we have for $R_1 < r < R_2$

$$\nu' = \frac{1}{r}(e^{\lambda} - 1)$$

$$\nu = \int^r \frac{1}{r}(e^{\lambda} - 1)\, dr + \bar{C}_2. \tag{10}$$

From the relation

$$e^{\nu} = 1 - \frac{2GM}{R_2 c^2}$$

at $r = R_2$ we have matching Eq. (10) at $r = R_2$ to Eq. (8) at $r = R_2$

$$ln\left(1 - \frac{2GM}{R_2 c^2}\right) = \int^{R_2} \frac{1}{r}(e^{\lambda} - 1)\, dr + \bar{C}_2. \tag{11}$$

The solution for $R_1 < r < R_2$ is

$$\nu(r) = ln\left(1 - \frac{2GM}{R_2 c^2}\right) - \int_r^{R_2} \frac{1}{r}(e^{\lambda} - 1)\, dr \tag{12}$$

$$e^{-\lambda} = 1 - \frac{8\pi G}{3c^4}\epsilon_{01} r^2 + \frac{C_1}{r}$$

$$C_1 = \frac{8\pi G \epsilon_{01} R_2^3}{3c^4} - \frac{2GM}{c^2}. \tag{13}$$

$$P_{\text{radial}} = 0 \tag{14}$$

for $R_1 < r < R_2$.

We now study the set of Einstein equations for the region $0 < r < R_1$ for $0 < r < R_1$

$$\epsilon \to \epsilon_{02} - \frac{2\pi G S_0^2}{c^2},$$

$$P \to P_0 - \frac{2\pi G S_0^2}{c^2}$$

($S = S_0$, $\epsilon = \epsilon_{02}$; $\epsilon_{02}, S_0 =$ constant)
the equations read

$$\frac{d}{dr}(re^{-\lambda}) = 1 - \frac{8\pi G}{c^4}\left(\epsilon_{02} - \frac{2\pi G S_0^2}{c^2}\right)r^2 \tag{15}$$

$$\frac{d}{dr}\left(P_0 - \frac{2\pi G S_0^2}{c^2}\right) + \frac{1}{2}\left(P_0 + \epsilon_{02} - \frac{4\pi G S_0^2}{c^2}\right)\nu' = 0 \tag{16}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\frac{8\pi G}{c^4}\left(-P_0 + \frac{2\pi G S_0^2}{c^2}\right) \tag{17}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r}\right) = -\frac{8\pi G}{c^4}\left(-P_0 + \frac{2\pi G S_0^2}{c^2}\right). \tag{18}$$

Eq. (15) gives

$$e^{-\lambda} = 1 - \frac{8\pi G r^2}{3c^4}\left(\epsilon_{02} - \frac{2\pi G S_0^2}{c^2}\right) \tag{19}$$

Eq. (16) gives

$$\left(P_0 + \epsilon_{02} - \frac{4\pi G S_0^2}{c^2}\right) = \bar{C}_1 e^{-\nu/2} \tag{20}$$

and from Eqs. (15,17,19) and (20)

$$e^{\nu/2} = A - B\sqrt{1 - \frac{8\pi G}{3c^4}\left(\epsilon_{02} - \frac{2\pi G S_0^2}{c^2}\right)r^2} \tag{21}$$

(A, B = constants of integration).

To find A, B we express P in terms of e^ν and set it equal to 0 ar $r = R_1$, and then set e^ν equal to the value in the region $R_1 < r < R_2$ at $r = R_1$.
In calculating A, B we first find ν' from Eq. (21)

$$\alpha = \frac{8\pi G}{3c^4}\left(\epsilon_{02} - \frac{2\pi G S_0^2}{c^2}\right)$$

$$\nu' = \frac{+2\alpha B r}{(A - B\sqrt{1-\alpha r^2})\,(\sqrt{1-\alpha r^2})}$$

using Eq. (17), at $r = R_1$, ($P_0 = 0$) gives

$$(1 - \alpha R_1^2)\left[\frac{2\alpha B}{\left(A - B\sqrt{1-\alpha R_1^2}\right)\sqrt{1-\alpha R_1^2}} + \frac{1}{R_1^2}\right] = -\frac{16\pi^2 G^2 S_0^2}{c^6} - \frac{1}{R_1^2} \tag{22}$$

We next match Eq. (21) to Eq. (12) at $r = R_1$

$$2\ ln\left[A - B\sqrt{1-\alpha R_1^2}\right] = ln\left(1 - \frac{2GM}{R_2 c^2}\right) - \int_{R_1}^{R_2} \frac{1}{r}(e^\lambda - 1)\,dr. \tag{23}$$

Eq.(22) and Eq. (23) give for A, B

$$A = B\ \frac{\left[3\alpha(1-\alpha R_1^2)^{1/2} - \frac{16\pi^2 G^2 S_0^2}{c^6}(1-\alpha R_1^2)^{1/2}\right]}{\left(\alpha - \frac{16\pi^2 G^2 S_0^2}{c^6}\right)} \tag{24}$$

$$B = \frac{\left(\alpha - \frac{16\pi^2 G^2 S_0^2}{c^6}\right)\left(1 - \frac{2GM}{R_2 c^2}\right)^{1/2} e^{-\frac{1}{2}\int_{R_1}^{R_2}\frac{1}{r}(e^\lambda - 1)dr}}{2\alpha(1-\alpha R_1^2)^{1/2}}.$$

Thus for Region (A), for $r > R_2$

$$e^{-\lambda} = e^\nu = 1 - \frac{2GM}{rc^2}$$

Region (B) for $R_2 > r > R_1$

$$e^{-\lambda} = 1 - \frac{8\pi G\epsilon_{01}}{3c^4} + \frac{C_1}{r}\left(C_1 = \frac{8\pi G\epsilon_{01}R_2^3}{3c^4} - \frac{2GM}{c^2}\right)$$

$$\nu(r) = ln\left(1 - \frac{2GM}{R_2 c^2}\right) - \int_r^{R_2}\frac{1}{r}(e^\lambda - 1)dr,$$

$P_{radial} = 0$ for $R_1 < r < R_2$.
Region (C) for $r < R_1$

$$e^{-\lambda} = 1 - \frac{8\pi G}{3c^4}\left(\epsilon_{02} - \frac{2\pi G S_0^2}{c^2}\right)r^2 = 1 - \alpha r^2$$

$$P_0 + \epsilon_{02} - \frac{4\pi G S_0^2}{c^2} = \bar{C}_1 e^{-\nu/2}$$

(where $\bar{C}_1$ is found by setting $P_0 = 0$ at $r = R_1$ and using $e^{-\nu/2}$ at $r = R_1$ from Eq. (12)

$$\left[\bar{C}_1 = \frac{\epsilon_{02} - \frac{4\pi G S_0^2}{c^2}}{\left(e^{-\nu/2}\right)_{R_1}}\right]$$

and $(e^{\nu/2})_r = A - B\sqrt{1-\alpha r^2}$ where A, B are given by Eq. (24).

This represents a complete and exact solution for the restriction of vanishing normal pressure for $R_1 < r < R_2$ and constant energy density for $R_1 < r < R_2$, as well as constant spin density for $0 < r < R_1$ and constant energy density for $0 < r < R_1$.

III. CONCLUSION

This analysis has generated an exact solution for a configuration of spins confined within a spherical layer of matter with vanishing normal pressure. Actually to calculate the metric and pressure for $R_1 < r < R_2$ when the normal pressure does not vanish involves a set of approximations which makes a numerical analysis necessary. Since the mass of the two layer configuration will depend on the spin density we could expect that red-shifts from such condensed astrophysical objects in the cosmos would have an anomalous dependence on the spin density. This calculation also suggets that spin generated torsion would be influential in the early universe in generating stable bound condensation from an initially unpolarized medium. Cosmological signatures of such condensations might be unexplained structure formation near density voids [9] that cannot be explained by any other mechanism than that of spin generated torsion. In future works the question of stability of such a two layer configuration admitting spin density will be discussed.

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