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THE BIFURCATION IN A DIFFUSION–HYDRODYNAMICAL SYSTEM WITH O(3) SYMMETRY

JEX, I.[1], KREMPASKÝ, J.[1], Bratislava

In this paper we show that the bifurcation in some diffusion-hydrodynamical systems with O(3) symmetry does not start in general from a homogeneous but from a nonhomogeneous distribution of matter. It follows from this fact that the mean matter density in the structuralized system can be a certain function of the distance from the centre of the system. This function was found in an analytical form for an expanding-like material with diffusion and gravitation.

I. INTRODUCTION

During the last few years problems of structure formation in physical systems with diffusion and hydrodynamical flows were in the centre of interest of synergetics (a good review can be found in [1]).

The starting point in these problems is the continuity equation in the form:

$$\nabla \boldsymbol{J} + \frac{\partial \varrho}{\partial t} = 0, \qquad \boldsymbol{J} = -D\nabla\varrho + \varrho\boldsymbol{v} \tag{1}$$

where D is the diffusion coefficient, ϱ the mass density, $\boldsymbol{v}$ the "hydrodynamical" velocity caused by physical fields, as for instance the expansion of the system. The final form of the equation (1) can be written as:

$$\frac{\partial \varrho}{\partial t} = \nabla(D\nabla\varrho) - \varrho\nabla\boldsymbol{v} - \boldsymbol{v}\nabla\varrho. \tag{2}$$

Special cases of the equation (2) are systems with O(3) or O(2) symmetry. The growth of structures in such systems is studied in papers [2,3,4], with the difference that instead of diffusion thermal flow and rotation were considered.

In most cases of structure formation problems with diffusion the characteristic coefficient (the diffusion coefficient or in the case of heat conductivity the heat conductivity coefficient) is considered to be constant and practically in all cases it is supposed that the exhibited structure starts from a homogeneous state. The mean matter density in such systems is almost constant.

The systems observed in reality very often exhibit a nonuniform distribution of mean density. For example, we can mention the grandiose structuralisation of the structureless matter into galaxies and galaxies into stars. [5-9]. The measurements show that galaxies, as for instance Galaxy, exhibit a distribution of matter characterized with a decrease of the density from the centre to the periphery.

The nonhomogeneous distribution of matter in our Galaxy is most often explained as a result of the dynamics of stars [10]. It is possible to find a matter distribution function by fitting the relation for the angular velocity distribution of stars as a function of density. The starting model for this calculation is a model of a gas described by Boltzmann's equation, which cannot be expected as a totally adequate model.

It is possible to imagine that the cosmic cloud is nonhomogeneous before the fragmentation into stars [11]. Such a possibility follows naturally from the model presented in paper [9]. This model respects the gravitation, the diffusion and the expansion of the Universe and leads to the same results for the critical mass as Jeans' theory.

Under simplified assumptions used in [9] a structure with a homogeneous distribution of density was found, but it is possible to suppose that by solving the fundamental equation of the evolution without simplification a nonhomogeneous structure can be achieved. In this paper we will try to prove this.

The problem of structure formation is naturally actual not only in connection with astrophysical problems. All the gaseous and liquid states of matter with a micro- and macrodynamics can under certain circumstances exhibit a qualitative transition characterized by a discrete distribution of matter. In the case of a O(3) symmetry of the system such problems can be solved analytically. An example of such a system are spherical systems, expanding in the direction of the radius vector from the centre. This system can be characterized by a homogeneous chaotic microscopic motion or a macroscopic motion caused by radial symmetric forces (gravitational, electric). To be able to apply our results also to astrophysical problems we will suppose a linear type of expansion of the system.

II. THEORY

Taking into account the above mentioned assumptions the "hydrodynamical" velocity has two parts: the expansion part ($\boldsymbol{v}_e$) and the field part ($\boldsymbol{v}_f$). For the linear type of expansion we can set:

$$\boldsymbol{v}_e = H\boldsymbol{r},$$

[1] Institute of Physic, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 BRATISLAVA, CSFR

where H is a constant and in the case of gravitational field the field part is:

$$\boldsymbol{v}_f = \boldsymbol{v}_g = \tau \boldsymbol{E}.$$

$\boldsymbol{E}$ is the intensity of the gravitational field and τ the corresponding relaxation constant. The intensity of the field satisfies the equation:

$$\nabla \boldsymbol{E} = -4\pi\kappa\varrho,$$

where κ is the gravitational constant and ϱ the density. Under these conditions the equation (2) takes the form:

$$\frac{\partial \varrho}{\partial t} = K\nabla(\frac{1}{\varrho}\nabla\varrho) - H\boldsymbol{r}\nabla\varrho + (A - 3H)\varrho. \tag{3}$$

The parameters A and K are defined as:

$$A = \frac{2\pi\kappa m^{3/2}}{\sigma(5kT)^{1/2}}, \quad K = \frac{m}{3\sqrt{2}\sigma}.$$

Here m is the particle mass, σ the effective cross section of collisions, T the temperature and κ the gravitational constant. The parameters A and H change their values in general in time (e.g. in the case of the astrophysical structuralization H decreases and A increases), but these changes are so small that we can solve the equation (3) supposing A and H being constant. We suppose that the gas consists of 2-atomic molecules, but it is not important for the calculation. We took into account the dependence of the diffusion constant on the density ϱ in the form:

$$D = \frac{m}{3\sqrt{2}\sigma\varrho} = K\frac{1}{\varrho}.$$

In the spherically symmetric case we introduce new scaling variables $\varrho = \varrho_0 z$ and $r = r_0 x$, where ϱ_0 and r_0 are the characteristic parameters. The equation (3) can then be transformed into a dimensionless form:

$$\frac{\partial z}{\partial t} = \frac{D_0}{(r_0 x)^2}\frac{\partial}{\partial x}\left(\frac{x^2}{z}\frac{\partial z}{\partial x}\right) - Hx\frac{\partial z}{\partial x} + (A - 3H)z, \tag{4}$$

where

$$D_0 = \frac{m}{3\sqrt{2}\sigma\varrho_0}.$$

This equation is a special case of a general differential equation having the form:

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(F_1(t,x)\frac{\partial z}{\partial x}\right) + F_2(z). \tag{5}$$

Concrete cases of the equation (5) are analysed in many papers (see e.g. [12–19]). In some cases the solutions are spatially structuralized. The equation (4) was not analysed mathematically, but it is evident that we can expect a bifurcation point in its solutions. Monotonic solutions reaching this point change into solutions with spatial structure. Trying to prove it we use the "ansatz" ("automodel" solution) in the form

$$z = y_1(t) f(\xi), \qquad \xi = \frac{x^\alpha}{y_2(t)}. \tag{6}$$

Inserting (6) into the equation (4) we get the equation:

$$\dot{y}_1 f - \frac{y_1}{y_2^2}\dot{y}_2 x^\alpha f' = \frac{D_0}{r_0^2} x^\alpha \frac{\alpha(\alpha+1)}{y_2}\frac{f'}{f} + \frac{D_0}{r_0^2}\frac{\alpha^2 x^{(2\alpha-2)}}{y_2^2}\left[\frac{f'}{f}\right]' - H\frac{\alpha x^\alpha y_1}{y_2} f' + (A - 3H) y_1 f, \tag{7}$$

where the sign ($\dot{}$) is the derivative with respect to time and the sign ($'$) with respect to ξ. Using the relations:

$$\alpha = -1,$$

$$\begin{aligned} \dot{y}_1 &= -2Hy_1 \qquad (y_1 = \exp(-2Ht)), \\ \dot{y}_2 &= -Hy_2 \qquad (y_2 = \exp(-Ht)), \end{aligned} \tag{8}$$

the equation (7) can be transformed into the form:

$$\xi^4\left[\frac{f'}{f}\right]' + \beta f = 0, \tag{9}$$

with

$$\beta = \frac{r_0^2}{D_0}(A - H),$$

and after the substitution $f(\xi) = \exp[u(\xi)]$ the final form of the equation is:

$$\xi^4 u''(\xi) + \beta\exp[u(\xi)] = 0. \tag{10}$$

The approximation of small deviations $\exp[u(\xi)] \approx 1 + u(\xi)$ gives a simplified equation:

$$\xi^4 u''(\xi) + \beta(1 + u(\xi)) = 0, \tag{11}$$

with the analytical solutions (see e.g. [20]):

$$u(\xi) = \left\{C_1\exp(\sqrt{|\beta|}/\xi) + C_2\exp(-\sqrt{|\beta|}/\xi)\right\}\xi + u_p \qquad A - H < 0, \tag{12a}$$

$$u(\xi) = C_1 + C_2\xi + u_p \qquad A - H = 0, \tag{12b}$$

$$u(\xi) = \left\{C_1\cos(\sqrt{|\beta|}/\xi) + C_2\sin(\sqrt{|\beta|}/\xi)\right\}\xi + u_p \qquad A - H > 0, \tag{12c}$$

where u_p is a particular integral of (12). For further discussion it is important that the function for $A = H$ is an exact solution of the equation (10).

The functions (12) are one of the possible solutions of the equation (3). The advantage of this solution is that it fulfiles our initial condition and that it describes very well the evolution of the system from states, when $A < H$ through the bifurcation point ($A = H$) to states, where $A > H$. This evolution in the constallation of parameters A and H really took place in astrophysical systems.

The initial condition $\varrho = \varrho_0$ in time $t = 0$ can be fulfilled by choosing the constant $C_1 = 0$ in the function (12a). The singularity in the point $r = 0$ is of the type $\exp(-r)/r$ and corresponds to the fact that the density of matter in radial-symmetric bodies is not well definable for the point $r = 0$. (If a point like a particle, e.g. the electron, is placed in the centre then $\varrho(r = o) \to \infty$.) Everywhere except of the small neighbourhood of the point $r = 0$ in time $t = 0$ there is $u(\xi) \to 0$ and therefore $f(\xi) \to 1$ and $\varrho \approx \varrho_0$.

III. RESULTS

It is possible to interpret the solutions (12) in such a way that the system undergoes a change from the state described by a monotonic function to the state with a periodic spatial structure crossing the bifurcation point $A = H$. The possibility of such a structuralization from the state corresponding to the condition $A < H$ into the state with $A > H$ is discussed in [9].

The solution (12) shows a bifurcation point for $A = H$ with the corresponding exact solution. The matter density in the bifurcation point can be expressed by the function ($t = t_0$):

$$\varrho(r) = \varrho_0 \exp(-2Ht_0)\exp(C_1 + C_2\exp(Ht_o)/r) = A\exp(B/r), \qquad (12)$$

where A and B are constants determined by the boundary conditions. This solution is singular for $r = 0$. Astrophysical systems overcome this singularity by the formation of a central body (a black hole [10]?) characterized by a finite density.

The function (13) can be used for describing the distribution of the mass density in the time $t = t_0$, when the structuralisation started. One can suppose that this distribution does not change substantially in the process of further structuralization. This result shows that the structuralization of an expanding gaseous system with diffusion and gravitation starts in general from a nonhomogeneous distribution. This nonhomogeneous distribution can manifest itself in a nonhomogeneous distribution of matter in the system at later times. The measurements of our Galaxy (see for instance [12, 21]) can be fitted well with a function of the type (13). This fact can serve as an indication that at least partially the formation of stars in it from the gaseous state started from a nonhomogeneous state.

IV. CONCLUSION

A model of a diffusion-hydrodynamical system with O(3) symmetry can be used to explain the nonhomogeneous distribution of matter in different systems. This effect could be mainly caused by the dependance of the diffusion coefficient on the density of matter. An analytical function expressing the dependence of the mass density on the distance from the centre of the expanding system was found. A comparison with experimental values known for our Galaxy showed that the nonhomogeneity of matter distribution could be in the first step generated in the time before the structuralization.

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БИФУРКАЦИЯ В ДИФУЗИОННО-ГИДРОДИНАМИЧЕСКОЙ СИСТЕМЕ С O(3) СИММЕТРИЕЙ

В работе показано, что бифуркация в дифузионно-гидродинамических системах с O(3) симметрией неначинается как принято с однородного, но с неоднородного распределения материи, из чего следует, что средная плотность материи в структурных системах может быть некоторой функцией расстояния от центра системы. Такая функция в работе выражена в аналитической форме для экспандирующей материи с дифузией и гравитацией.