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ELASTIC COLLISION OF HARD SPHERES IN THE SPACE

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The study deals with the problems of elastic collision of two hard spheres in space providing that the spheres start from any arbitrary position in the space, either simultaneously or non- simultaneously. Spatial and time conditions of collisions are formulated and equations for the determination of the position of the spheres at the moment of the collision, velocities and directions of motion after the collision are derived. For the determination both analytical and graphical methods were used, the last one employing the hodograph of velocities. The submitted method of the hodographic analysis is relatively simple and illustrative. It can also be applied to the problems otherwise solved by the use of the Boltzmann equation. The method also enables to solve models involving multiple subsequent collisions. The derived relationships can be applied to the solution of the problem of the mean free path of molecules in ideal gas under anisotropic conditions.

I. INTRODUCTION

The paper submitted is the result of the study of a partial problem connected with the molecular distillation. It concerns the determination of the dependence of the rate of evaporation on the mean free path of the molecules in the space between the evaporator and the condenser [1]. The hitherto achieved results [2,3] are not quite satisfactory, because the anisotropical character of the mean free path in them is considered by means of empiric constants. Better results are expected when considering the mean values of all possible collisions in the space in question. The essence of this solution is the collision of two solid spheres in vacuum which also enables to program collisions among the already collided molecules. The result is of a general physical validity. The known methods of the determination of the free path based on the solution of the Boltzmann equation [4,5] provide a comprehensive solution of the problem; however, they are complicated and demanding as for their calculation.

II. BASIC MECHANICAL RELATIONS

Two hard spheres moving two given positions in space at given constant velocities in any direction can collide one with another only under certain conditions.

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This study is focused on the determination of the position of the sphere centres on the collision axis at the moment of the collision and to determine the velocities of both spheres after the collision.

Now, we shall suppose equal weights m and equal diameters d of both spheres. At the beginning, a simultaneous start of both spheres will be assumed ($\tau = 0$). A non-simultaneous start can be transferred by proper spatial transformation to the given special case.

At a collision of perfectly elastic spheres the laws of conversation are valid. If the initial velocities of the spheres are $\boldsymbol{v}_A(v_{Ax}, v_{Ay}, v_{Az})$ or $\boldsymbol{v}_B(v_{Bx}, v_{By}, v_{Bz})$, and the velocities after the collision are $\boldsymbol{v}'_A(v'_{Ax}, v'_{Ay}, v'_{Az})$ or $\boldsymbol{v}'_B(v'_{Bx}, v'_{By}, v'_{Bz})$, the law of conservation of momentum

$$m\boldsymbol{v}_A + m\boldsymbol{v}_B = m\boldsymbol{v}'_A + m\boldsymbol{v}'_B \tag{1a}$$

and the law of conservation of energy

$$m\frac{v_A^2}{2} + m\frac{v_B^2}{2} = m\frac{{v'_A}^2}{2} + m\frac{{v'_B}^2}{2} \tag{1b}$$

are valid.

Equations (1a) and (1b) provide a system of four scalar linear equations for six coordinates of the vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$, and therefore two other parameters are required for the solution of the collision. One can prove [6] that (1a) and (1b) remain valid if they are solved as follows:

$$\boldsymbol{v}'_A = \boldsymbol{v}_A + (\boldsymbol{v}_R \cdot \boldsymbol{o}_1)\boldsymbol{o}_1 \tag{2a}$$

$$\boldsymbol{v}'_B = \boldsymbol{v}_B - (\boldsymbol{v}_R \cdot \boldsymbol{o}_1)\boldsymbol{o}_1 \tag{2b}$$

where $\boldsymbol{v}_R = \boldsymbol{v}_B - \boldsymbol{v}_A$ is the relative velocity of the motion of both spheres and $\boldsymbol{o}_1$ is the unit vector of the collision axis o. The vectors $\boldsymbol{v}'_A$ and $\boldsymbol{v}'_B$ can be determined at the given position of the collision axis o by quite a simple geometric construction (Fig. 1), where $\boldsymbol{u} = (\boldsymbol{v}_R \cdot \boldsymbol{o}_1)\boldsymbol{o}_1$. It is evident from Fig. 1 that the components of the velocities perpendicular to the collision axis do not change during the collision, and the components parallel to the collision axis are exchanged. The point C in the hodograph of the velocities is lying on the circle $\boldsymbol{k}$ circumscribed above the vector of the relative velocity $\boldsymbol{v}_R$.

Vectors $\boldsymbol{v}_A$, $\boldsymbol{v}_B$ and $\boldsymbol{o}$ need not be in one plane. Vectors $\boldsymbol{v}'_A$, $\boldsymbol{v}'_B$ can always be determined geometrically by means of the hodograph in Fig. 2, where the vectors $\boldsymbol{v}_A$, $\boldsymbol{v}_B$ and $\boldsymbol{o}$ are plotted from an optional point S. Point V_A is the terminal point of the vector $\boldsymbol{v}_A$ and point V_B is the terminal point of the vectors $\boldsymbol{v}_B$ and $\boldsymbol{v}_R$. The sphere k constructed above the diameter $\boldsymbol{v}_R$ crosses the parallel line with the collision axis o_1 in the points V_A, C_A and V_B, C_B, respectively. The line segments

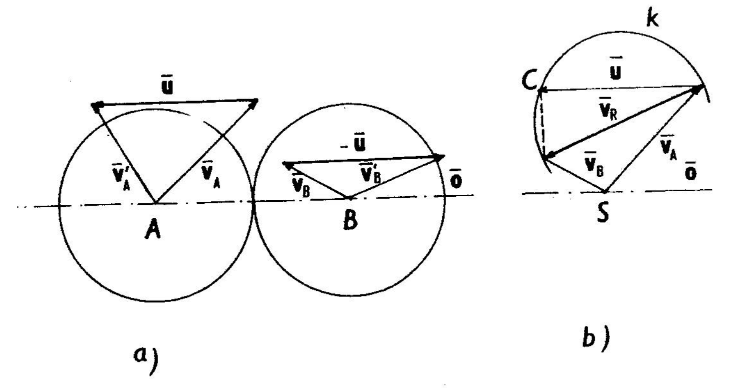

Fig. 1. Collision of two spheres on the collision axis.a) in space; b) hodograph of velocities.

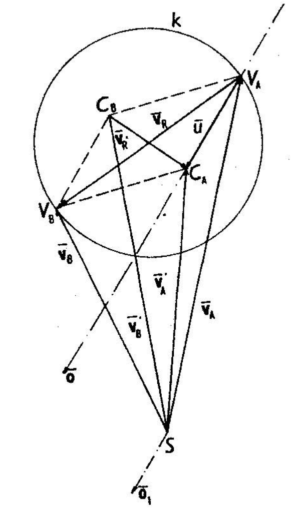

Fig. 2. Hodograph of the velocities at collision in the space.

V_AC_A and V_BC_A form a right angle in the point C_A, because it is the angle above the circle diameter on the sphere $\boldsymbol{k}$. The connecting line V_AC_A determines the line segment $\boldsymbol{u} = (\boldsymbol{v}_R \cdot \boldsymbol{o}_1)\boldsymbol{o}_1$ a therefore — according to (2a) — the vector defined by the connection line SC_A equals the required vector $\boldsymbol{v}'_A$.

When applying the previous consideration to the point V_B and the point of intersection C_B with the sphere, it is possible to determine the final velocity $\boldsymbol{v}'_B$ as a line SC_B, as well as the relative velocity of both spheres after collision $\boldsymbol{v}'_R = \boldsymbol{v}'_B - \boldsymbol{v}'_A$, which is illustrated by the line C_BC_A. Thus, the geometric solution of the problem is given if we know the collision axis.

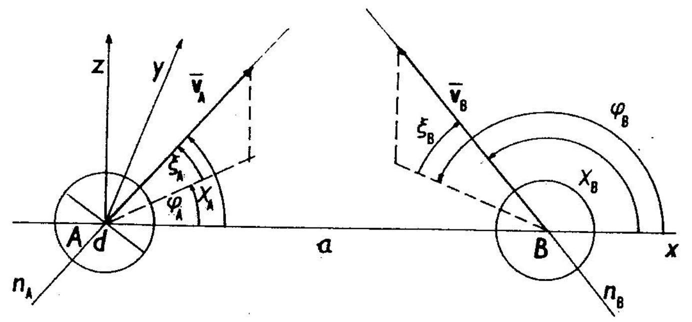

Fig. 3. Input data and the angles ξ_A and ξ_B.

The determination of the direction of the collision axis and the positions of the vectors of velocities v'_A and v'_B in the space requires analytical solution of the problem of collision by means of spatial representation whose determining elements are presented in Fig. 3. Let the centre of the sphere $\mathbb{A}$ at the start be given in the orthogonal system of coordinates by the coordinates of the point $A(x_A, y_A, z_A)$, and the centre of the sphere $\mathbb{B}$ by the point $B(x_B, y_B, z_B)$. Let the initial distance between the spheres $\mathbb{A}$ and $\mathbb{B}$ be $a = AB$. The coordinates system is chosen so that the xy plane is fixed to the given experimental equipment — the beginning of the coordinate system is in the point A and the point B is on the axis x. Then, $x_A = y_A = z_A = 0$ and $x_B = a$, $y_B = z_B = 0$. Let the directions of velocities be given by the angles φ_A, φ_B, which are formed by projections of the velocities v_A, v_B into the plane xy, and angles ξ_A, ξ_B, which are formed by these velocities with the plane xy. Let the angles formed by the velocities with the axis x be χ_A, χ_B whereas $\cos\chi_A = \cos\varphi_A \cos\xi_A$ and $\cos\chi_B = \cos\varphi_B \cos\xi_B$. The carrying line of the vector v_A is $n_A(\varphi_A, \xi_A)$ and the carrying line of the vector v_B is $n_B(\varphi_B, \xi_B)$.

III. SPATIAL CONDITION OF COLLISION

Since the lines n_A and n_B are, in general, oblique lines, only such input data can be considered for which the shortest distance d_o of both skew lines is smaller than the diameter of the spheres d. Calculation of the shortest distance of these skew lines for our conditions will be performed in accordance with Fig. 4. If the connecting line of the points A_o, B_o on the straight lines n_A and n_B determines the shortest distance of both skew lines, then — based on the definition of the shortest connection line — the angles at the apexes A_o and B_o of the triangles AA_oB_o and BB_oA_o are rectangular, which results in the following relationships:

$$d_o^2 = b_A^2 - c_A^2 = b_B^2 - c_B^2, \tag{3}$$

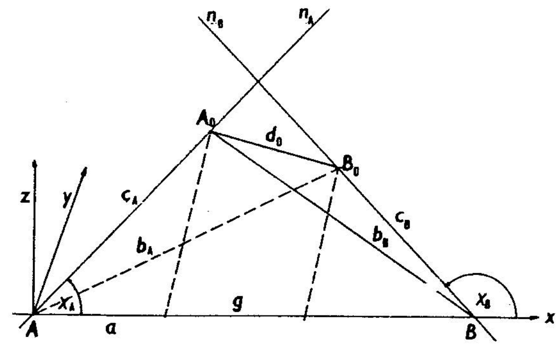

Fig. 4. The shortest connection d_o of the carrying lines n_A and n_B.

where $d_o = A_oB_o$, $b_A = AB_o$, $c_A = AA_o$, $b_B = BA_o$ and $c_B = BB_o$.

By means of the triangles ABA_o and ABB_o we can get the following:

$$c_A^2 + c_B^2 - 2c_Ac_B\cos\vartheta - 2a(c_A\cos\chi_A - c_B\cos\chi_B) + a^2 - d_o^2 = 0, \tag{4}$$

where

$$\cos\vartheta = \sin\xi_A \sin\xi_B + \cos\xi_A \cos\xi_B \cos(\varphi_B - \varphi_A) \tag{5}$$

and ϑ is the angle of the vectors v_A and v_B in the hodograph of velocities. For the shortest connection line d_o one gets

$$d_o^2 = \frac{a}{\sin^2\vartheta}(\sin^2\vartheta - n_{AB}^2) \tag{6}$$

where $n_{AB}^2 = \cos^2\chi_A + \cos^2\chi_B - 2\cos\chi_A\cos\chi_B\cos\vartheta$. The spatial condition for collision is

$$d_o < d. \tag{7}$$

The data which do not satisfy the condition (7) exclude the possibility of a collision. In the case of a planar collision when the vectors v_A and v_B are in one plane, and $d_o = 0$, the condition (7) is always satisfied, however, collision does not necessarily need to take place if also the time condition of the collision is not satisfied.

IV. TIME CONDITION OF COLLISION

If the sphere $\mathbb{A}$ moves at the velocity v_A and the sphere $\mathbb{B}$ at the velocity v_B, and both spheres meet at the time τ from their start, then the first sphere passed the path l_A and the second one the path l_B, whereas

$$l_A = v_A\tau \qquad l_B = v_B\tau. \tag{8a,b}$$

For the calculation of the paths l_A and l_B Fig. 4 can be taken into consideration in which the quantities c_A, c_B and d_o are substituted by the quantities l_A, l_B and d, and the conditions of perpendicularity given by (3) will not be required. In accordance with (4) and (5) the following equation will be valid:

$$l_A^2 + l_B^2 - 2l_A l_B \cos\vartheta - 2a(l_A \cos\chi_A - l_B \cos\chi_B) + a^2 - d^2 = 0 \qquad (9)$$

where d is given. When substituting the corresponding values from (8a,b) into (9) a quadratic equation for τ will be obtained whose root with physical significance has a shape as follows:

$$\tau = \frac{a}{v_R^2}\left(-v_{Rx} - \sqrt{v_{Rx}^2 - D^2 v_R^2}\right). \qquad (10)$$

Here the following symbols have been introduced: $D^2 = (a^2 - d^2)/a^2$

$$v_R^2 = v_A^2 + v_B^2 - 2v_A v_B \cos\vartheta$$
$$v_{Rx} = v_A \cos\chi_A - v_B \cos\chi_B$$

and where $\boldsymbol{v}_R = \boldsymbol{v}_B - \boldsymbol{v}_A$ is relative velocity, and v_{Rx} is the coordinate of the vector $\boldsymbol{v}_R$ projection into the x axis and the angle ϑ is the angle of vectors, introduced in Eq.(5). The period of collision depends only on the relative velocity. The time condition of collision requires that the time τ is a real quantity, and therefore it must be valid that

$$v_{Rx}^2 - D^2 v_R \geqq 0. \qquad (11)$$

This condition of inequality limits a certain interval of permitted values for $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ and also a certain time interval. We also allow for non-central collisions. In the extreme case if the spheres just touch without any collision, the equality is valid in (11), which gives for the ratio of the values of vectors $\rho = v_B/v_A$ the equation

$$\rho^2 \sin^2\chi_B - 2\rho(\cos\chi_A \cos\chi_B - \cos\vartheta) + \sin^2\chi_A = 0. \qquad (12)$$

The roots of (12), $\rho_{\max}$ and $\rho_{\min}$ determine the maximum and minimum ratio v_B/v_A at which the spheres just touch without collision. Thus, for any given velocity $\boldsymbol{v}_A$ the interval of the allowed values of the velocities between $\boldsymbol{v}_{B\max}$ and $\boldsymbol{v}_{B\min}$ at which a collision can take place, is determined. To these velocities, proper times of the collision, $\tau_{\max}$ and $\tau_{\min}$, correspond. The collision can take place only if τ is, according to (8a,b), within the limits

$$a\left(\frac{v_{Rx}}{v_R^2}\right)_{\min} < \tau < a\left(\frac{v_{Rx}}{v_R^2}\right)_{\max} \qquad (13)$$

For τ calculated from (10) it is possible to determine the paths l_A and l_B from (8a,b), and from them also the coordinates of the positions of the spheres $\mathbb{A}$ and $\mathbb{B}$ at a collision given by points A_o and B_o.

$$x_{AO} = l_A \cos\xi_A \cos\varphi_A \qquad x_{BO} = a + l_B \cos\xi_B \cos\varphi_B$$
$$y_{AO} = l_A \cos\xi_A \sin\varphi_A \qquad y_{BO} = l_B \cos\xi_B \sin\varphi_B$$
$$z_{AO} = l_A \sin\xi_A \qquad z_{BO} = l_B \sin\xi_B \qquad (14)$$

V. DETERMINATION OF THE COLLISION AXIS

The collision axis $\boldsymbol{o}$ passes the points A_o, B_o with coordinates in accordance with (14) (Fig. 5). Its position in our coordination system can be determined by

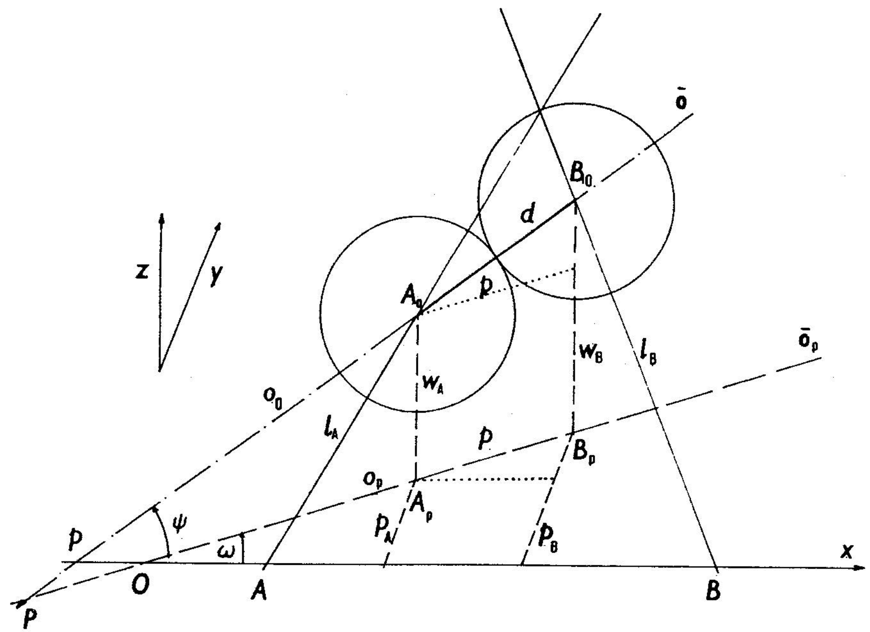

Fig. 5. Position of the collision axis.

the position of its point of intersection $P(p_x, p_y)$ by the plane xy, by the angle ψ, which is formed with the plane xy and by the angle ω formed by its projection o_p with the axis x. The projection o_p intersects the x-axis in the point O. A_p is the projection of the point A_o and B_P is the projection of the point B_o. Let $o = PB_o$, $o_p = PB_p$, $p = A_pB_p$, $w_A = A_pA_o = l_A \sin\xi_A$ and $\omega_B = B_pB_o = l_B \sin\xi_B$.

Then,

$$o_o = d\frac{w_B}{w_B - w_A}, \qquad o_P = p\frac{w_B}{w_B - w_A} \tag{15}$$

$$p^2 = d^2 - (w_B - w_A)^2 = d^2 - (l_B \sin\xi_B - l_A \sin\xi_A). \tag{16}$$

From that

$$\sin\psi = \frac{1}{d}(l_B \sin\xi_B - l_A \sin\xi_A) \tag{17a}$$

$$\cos\psi = \frac{p}{d} \tag{17b}$$

$$\sin\omega = \frac{1}{p}(l_B \cos\xi_B \sin\varphi_B - l_A \cos\xi_A \sin\varphi_A) \tag{18a}$$

$$\cos\omega = \frac{1}{p}(a - l_A \cos\chi_A - l_B \cos\chi_B). \tag{18b}$$

For a unit vector $\boldsymbol{o}_1$ of the collision axis in the vector form the following relationship is valid

$$\boldsymbol{o}_1 = \cos\psi\cos\omega\boldsymbol{i} + \cos\psi\sin\omega\boldsymbol{j} + \sin\psi\boldsymbol{k}. \tag{18c}$$

By substituting into (2a) and (2b) it is possible to determine $\boldsymbol{v}'_A$ and $\boldsymbol{v}'_B$. A detailed analysis of the results obtained in such a way is based on the collision hodograph (Fig. 2).

VI. HODOGRAPHIC ANALYSIS

The analytical analysis is performed by putting the hodograph in Fig. 2 into the coordinate system xyz used in Fig. 3, whereas the directions of the velocities remain, and the point S is chosen on the x-axis. The situation is evident from Fig. 6, where the collision axis o_B crosses the point V_B. A relative velocity vector $\boldsymbol{v}_R = v_{Rx}\boldsymbol{i} + v_{Ry}\boldsymbol{j} + v_{Rz}\boldsymbol{k}$ has the form

$$\boldsymbol{v}_R = (v_B\cos\varphi_B - v_A\cos\varphi_A)\boldsymbol{i} + (v_B\sin\varphi_B - v_A\sin\varphi_A)\boldsymbol{j} + (v_B\sin\xi_B - v_A\sin\xi_A)\boldsymbol{k} \tag{19}$$

or

$$\boldsymbol{v}_R = v_R(\cos\rho_z\cos\rho_p\boldsymbol{i} + \cos\rho_z\sin\rho_p\boldsymbol{j} + \sin\rho_z\boldsymbol{k}), \tag{20}$$

where the angles ρ_z and ρ_p are shown in Fig. 7, and using (18c) and (20) the scalar product in (2a,b) can be expressed.

For the determination of the vectors $\boldsymbol{v}'_A$ and $\boldsymbol{v}'_B$ by means of a hodograph the angle η formed by the relative velocity $\boldsymbol{v}_R$ with the collision axis $\boldsymbol{o}$ must be

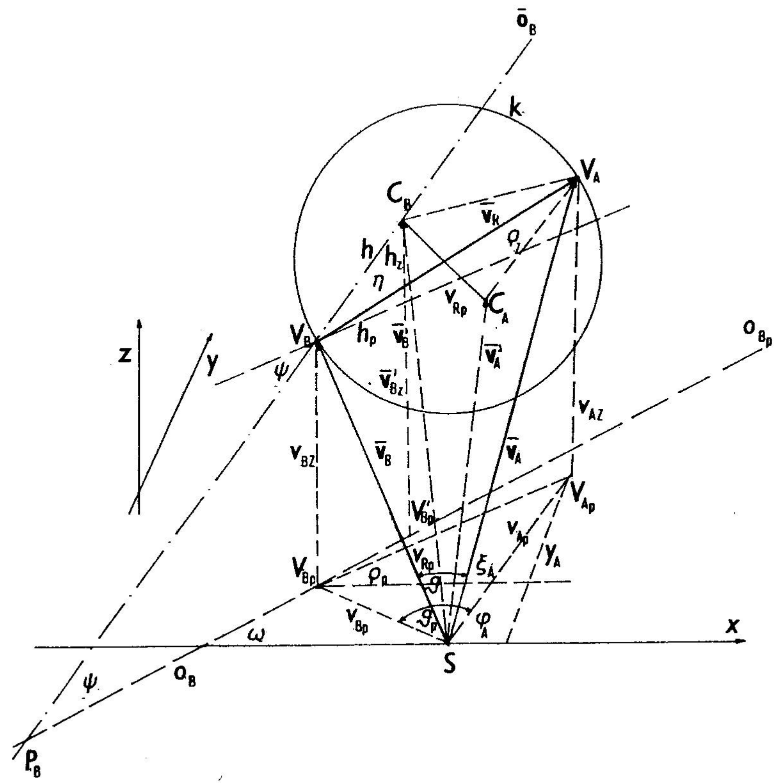

Fig. 6. Coordinate system of the hodograph.

known. This angle can be determined graphically from Fig. 7, where the unit length $l = SE$, the line segment J is perpendicular to S, the line segment IL is perpendicular to SL and EI is perpendicular to SI. Then, $EI = \sin\eta$, $SI = \cos\eta$, $EK = \sin\psi$, $SK = \cos\psi$, $IL = \cos\eta\sin\rho_z$; $SL = \cos\eta\cos\rho_z$, $\delta = \omega - \rho_p$. Then, from the triangles SKL and EIJ one gets

$$\cos\eta = \sin\psi\sin\rho_z + \cos\psi\cos\rho_z\cos(\omega - \rho_p). \tag{21}$$

The position C_B (Fig. 6) is determined from the condition that the triangle $V_AC_BV_B$ has a right angle at the apex C_B and V_B is on the o_B axis. If $h = V_BC_B$, then $h = v_R\cos\eta$. Simultaneously, $h_z = h\sin\psi$ and $h_p = h\cos\psi$. If projections of the vectors v'_A and v'_B are marked analogically the those of the vectors v_A and v_B, then — according to Fig. 6 — the following is valid for the triangle SC_BC_{BP}:

$$v'_{Bz} = v_{Bz} + h_z = v_B\sin\xi_B + v_R\sin\psi\cos\eta; \tag{22}$$

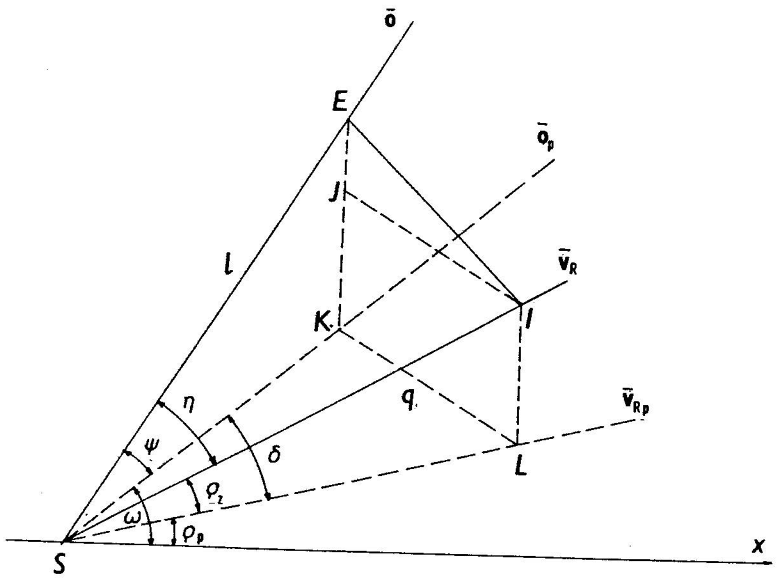

Fig. 7. Angle of the collision axis o with the relative velocity v_R.

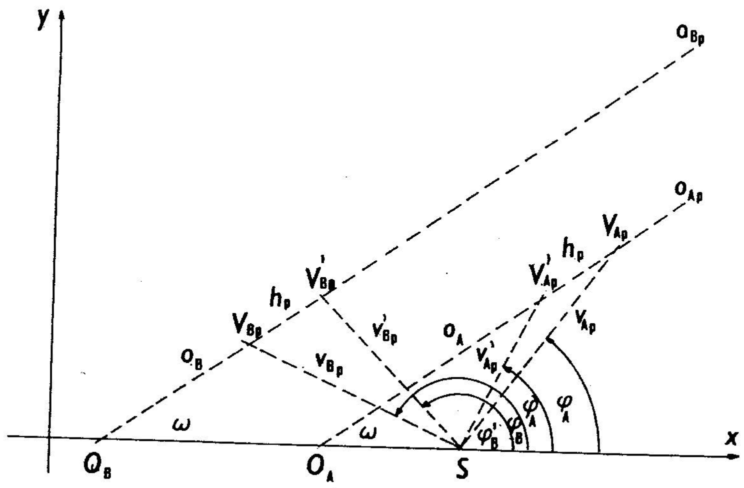

Fig. 8. Plan of the hodograph.

and in the triangle $SV'_{Bp}V_{Bp}$ according to Fig. 8:

$$v'^2_{Bp} = v^2_{Bp} + h^2_p - 2v_{Bp}h_p\cos(180 - \varphi_B + \omega) \tag{23}$$

and therefore

$$v'^2_B = v^2_B + v^2_R\cos^2\eta + 2v_Bv_R[\sin\xi_B\sin\psi\cos\eta + \\ + \cos\xi_B\cos\psi\cos\eta\cos(\omega - \varphi_B)]. \tag{24}$$

The square root of this expression determines v'_B. In a similar way also v'_A can be determined. Then, $h = V_AC_A$,

$$v'_{Az} = v_{Az} - h_z = v_A\sin\xi_A - v_R\sin\psi\cos\eta \tag{25}$$

and from triangle $SV'_{Ap}V_{Ap}$ (Fig. 8)

$$v'_{Ap}{}^2 = v^2_{Ap} + h^2_p - 2v_{Ap}h_p\cos(\omega - \varphi_A) \tag{26}$$

and thus

$$v'^2_A = v^2_A + v^2_R\cos^2\eta - 2v_Av_R[\sin\xi_A\sin\psi\cos\eta + \\ + \cos\xi_A\cos\psi\cos\eta\cos(\omega - \varphi_A)]. \tag{27}$$

From (24) and (27) v'_A and v'_B can be determined using the given parameters and the calculated axis of collision.

For angles of the vectors v'_A and v'_B with the plane xy the following relationships are valid:

$$\cos\xi'_B = \frac{v'_{Bp}}{v'_B} \qquad \sin\xi'_B = \frac{v'_{Bz}}{v'_B} \tag{28}$$

$$\cos\xi'_A = \frac{v'_{Ap}}{v'_A} \qquad \sin\xi'_A = \frac{v'_{Az}}{v'_A} \tag{29}$$

where expressions from (22) through (27) are substituted for the corresponding vector functions.

Calculation of the angles formed by the projections of the vectors v'_A and v'_B with the x-axis are determined from Fig. 8, which is a projection of the hodograph in Fig. 6. If $o_B = O_BV_{Bp}$ and $o_A = O_AV_{Ap}$, then $o_B : v_{Bp} = \sin(180 - \varphi_B) : \sin\omega$ and $(o_B + h_p) : v'_{Bp} = \sin(180 - \varphi'_B) : \sin\omega$ from which

$$\sin\varphi'_B = \frac{1}{v'_{Bp}}(v_{Bp}\sin\varphi_B + v_R\cos\psi\sin\omega\cos\eta) \tag{30a}$$

and

$$\sin\varphi'_A = \frac{1}{v'_{Ap}}(v_{Ap}\sin\varphi_A - v_R\cos\psi\sin\omega\cos\eta) \tag{30b}$$

and thus the directions of vectors $v'_A v'_B$ in the coordinate system xyz are determined.

The solution of the problem of elastic collision of two spheres given by (2a) and (2b) required first to determine the spatial condition of collision in (7), followed by the determination of the time of collision which is in the case of a simultaneous start of both spheres given by (10). The paths l_A and l_B according to (8ab) were determined, as well as the position of the spheres at the instant of collision in agreement with (14). During the calculation the direction of the axis of collision was determined in (17ab) and (18ab). The velocities of the spheres after the collision starting from points A_o and B_o are given by (24) and (27) and their directions by (28) and (30ab).

The data obtained allow to calculate secondary collisions, i.e. collisions between already collided spheres provided that they are transformed into the coordinate system according to Fig. 3, after arrangement to the simultaneous start.

VII. NON-SIMULTANEOUS START

In case of secondary collisions of two spheres M and N these either need do not start simultaneously, or from the xy plane. Let the sphere M (Fig. 9) start

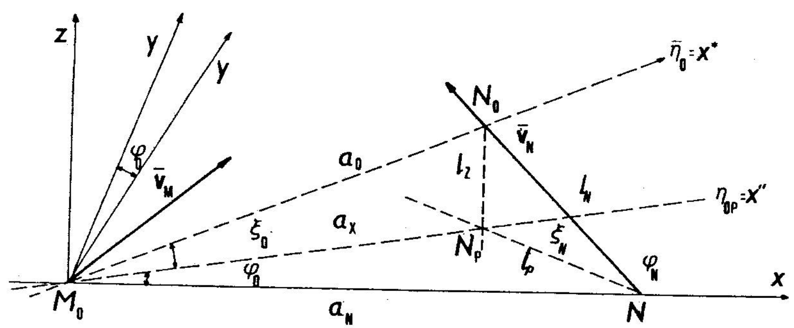

Fig. 9. Non-simultaneous start.

from the point $M_o(x_M, y_M, z_M)$ at the time τ_o with velocity v_M, and the sphere N from the point $N(x_N, y_N, z_N)$ at the time τ_N with velocity v_N at a distance $a_N = M_oN$. Let $\tau_o > \tau_N$. The sphere N will pass a path $l_N = v_N(\tau_o - \tau_N)$, within the time interval $\tau_o - \tau_N$ and will reach the point $N_o(x_o, y_o, z_o)$, and thus we can consider a simultaneous start of both spheres M and N from the points M_o and N_o on their connecting line n_o at a distance $a_o = M_oN_o$ with original velocities v_M and v_N. Introducing the proper orthogonal transformation which transforms the given coordinate system xyz into a new one $x^*y^*z^*$ in which the x^* axis would be

on the line n_o, the calculation of a simultaneous start can be applied to the case solved in the previous chapters.

The presented transformation is carried out around the point M_o at two stages:

a) by turning the xyz coordinate system around the z axis passing through the point M_o by the angle φ_o formed by the projection $n_{op} = x''$ with the axis x. The y axis will thus turn in the xy plane by an equivalent angle φ_o into a new axis y'' and the axis $z = z''$ remains unchanged;

b) by turning the coordinate system $x''y''z''$ around the $y'' = y^*$ axis by the angle ξ_o which is formed by n_o with its projection n_{op} one gets the final coordinate system $x^*y^*z^*$.

Parameters of these transformations are derived from Fig. 9

$$\sin\varphi_o = \frac{l_N}{a_x}\cos\xi_N\cos\varphi_N$$
$$\sin\xi_o = \frac{l_z}{a_x} = \frac{l_N}{a_x}\sin\xi_N.$$

The velocity directions are additively changed with these transformations. Usually it is necessary to transform the calculated collision parameters (position, velocity after collision) back into the original coordinate system xyz so that they can be used for further secondary, tertiary, etc. collisions.

For properly chosen models of a great number of starting molecules (for example, from the evaporating area) mean values of the distance of molecules (l_A, l_B) between two subsequent collisions can be calculated using the above presented method, and to approximate the calculation of the mean free path in ideal gas under anisotropic conditions.

VIII. EXAMPLE: BACK-REBOUND ON A SERIES OF MOLECULES

As an illustration let us take a probable portion of molecules issuing from the point source on the evaporation surface with a constant velocity with direction distribution in agreement with the cosine law, which move back towards the evaporation surface after the first collision.

Let us consider molecules B_n with a diameter d at the distance l from the surface of evaporation, which are equidistantly located by the length a on the right side of the axis x (Fig. 10). Providing that the molecules start from the point $A(0;0)$ at a constant rate with a distribution of direction according to the cosine law, only that part of released molecules will be back-rebounded which collides with the molecule B_i in the bow $B_{i1}B_{i2}$. If the evaporated molecule moves in the direction providing an angle ξ with the axis z, then — in agreement with the cosine law —

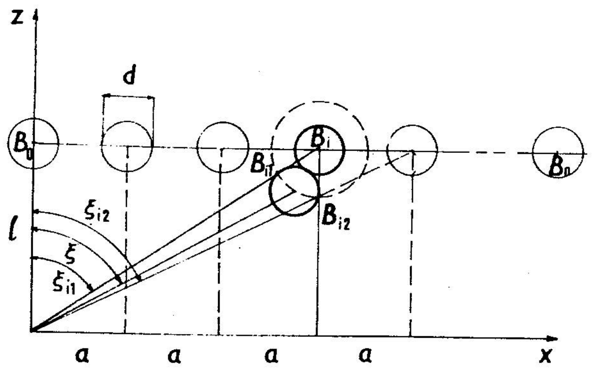

Fig. 10. Back-rebound from a series of molecules.

the propability of leaving the molecule in the angle $d\xi$ is as follows:

$$dP = \frac{1}{2}\cos\xi d\xi.$$

Then, the probability of impact on the bow $B_{i1}(\xi_{i1})B_{i2}(\xi_{i2})$ is

$$P_i = \frac{1}{2}\int_{\xi_{i1}}^{\xi_{i2}} \cos\xi d\xi = \frac{1}{2}(\sin\xi_{i2} - \sin\xi_{i1}).$$

In agreement with Fig. 10 the following relationships are valid for $\sin\xi_{i2}$ or $\sin\xi_{i1}$

$$\sin\xi_{i2} = \frac{ia}{s_i}; \qquad \sin\xi_{i1} = \frac{ia}{t_i}$$

where

$$s_i = \sqrt{i^2a^2 + (l-d)^2}; \qquad t_i = \sqrt{i^2a^2 + l^2}.$$

At a sufficiently large angle ξ_k not a single molecule can pass between the molecules B_k and B_{k+1} and all of them will rebound back. This will be in the case if $k = (l-d)/d$, where k does not depend on a. If k is not an integer it is necessary to consider the nearest integer m for the value of k. In the example illustrated in Fig. 10 $i = m$.

It results for the probability of the back-rebound of the molecules A from the series of the molecules B_n that

$$P = \frac{a}{2}\sum_{i=1}^{i=m}\left(\frac{i}{s_i} - \frac{i}{t_i}\right) + \frac{1}{2}\int_{\xi_m}^{\pi} \cos\xi_m d\xi_m =$$

$$= \frac{a}{2}\sum_{i=1}^{i=m}\left(\frac{i}{s_i} - \frac{i}{t_i}\right) + \frac{1}{2}(1 - \sin\xi_m).$$

For $\sin\xi_m$ the following relationship can be derived

$$\sin\xi_m = \frac{(m+1)a}{\sqrt{(m+1)^2a^2 + l^2}}.$$

If the unit diameter of the molecules is $d = 1$, then $m = l - 1$ and

$$\sin\xi_k = \frac{a}{\sqrt{a^2+1}},$$

and thus the angle ξ_k does not depend on l. There will never be any back-rebound from the molecule B_o in the case of the coordinate $x = 0$, $z = l$.

The probability values P in dependence on the coordinate z of the series of the molecules B_n are shown in Fig. 11 for different distances a between the molecules.

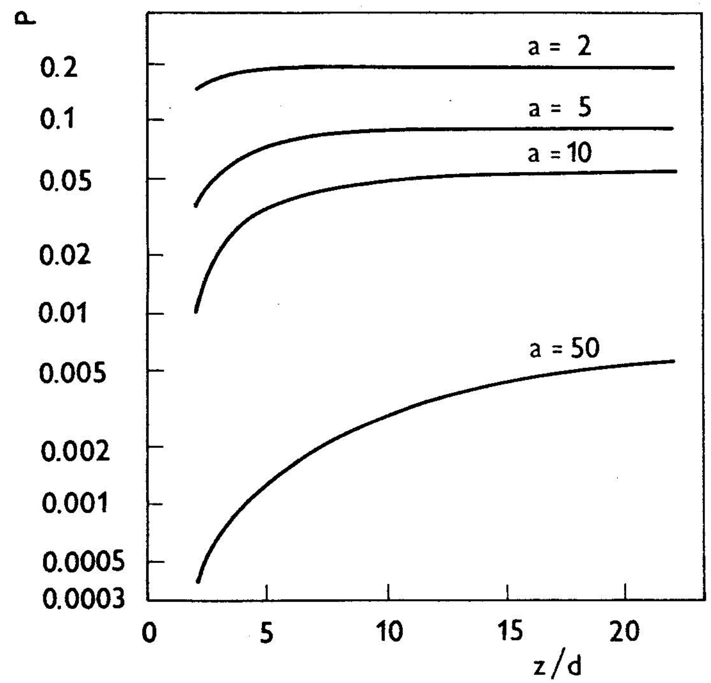

Fig. 11. The probability P dependence of the rebound on the coordinate z of the series of molecules for different distances a for $d = 1$.

It is evident from Fig. 11 that for x larger than 20 d and for a larger than 50 the probability of the back-rebound is smaller than 5 promile.

The relationships presented in the paper enable probability calculations for the back-rebound even after multiple collisions.

IX. CONCLUSION

Molecular distillation is connected with the idea that the molecules of the distilled substance are passing undisturbed the distillation space between evaporator and condenser with a minimum of collisions. Frequently emphasized in this connection is the condition that the mean free path of molecules of the distilled substance must be larger than, or a comparable with, the width of the distillation gap. For the mean free path $\overline{\lambda}$ the classic relationship is being presented that has been derived from the kinetic theory of gases

$$\overline{\lambda} = \frac{1}{\sqrt{2}\pi d^2 n} = \frac{kT}{\sqrt{2}\pi d^2 p} \tag{31}$$

where n is the number of particles within the volume unit. If values typical of molecular distillation are substituted in relation (31) ($M = 200\,\mathrm{g.mol^{-1}}$, $p = 1\,\mathrm{Pa}$, $d = 9 \times 10^{-10}\,\mathrm{m}$, $T = 400\,\mathrm{K}$), the obtained $\overline{\lambda}$ values is approximately 1 mm. Here a discrepancy arises from practical experience because even at a condenser-evaporator distance of up to 50 mm in large production molecular evaporators no significant drop of the distillation rate versus theoretical values is observed.

In molecular distillation the mean free path of distilled molecules obviously plays a significant role and affects the process; its effective value, however, is to be determined for this case from other relations. Equation (31) has been derived for nondirectional chaotic molecular motion with an equal probability of all directions. Conditions in the distillation space in molecular distillation, however, are different. The molecules of the evaporated substance are departing from the evaporation surface with a preference of directions according to the cosine law, whereby most of the molecules depart in a direction perpendicular to the evaporating surface. It follows, for example, from the cosine law that 50% of evaporated molecules from the point source on the evaporation surface depart in a space angle around the normal that represents less than 30% out of the whole semi-space above the evaporation surface. Thus what prevails here is the directional motion of particles in the direction from the evaporator toward the condenser and relation (31) loses its justification.

At the directional motion of molecules in the distillation space their free path is prolonged because of the reduced relative speed of particles versus each other and the proportion of molecules emitted from the evaporator under large angles from the normal is small. Few are also the particles which, after collision, take the course back to the evaporation surface. If collisions occur at the motion of molecules with a preference of directions, these collisions need not have a fatal consequence for the process. What will probably prevail are collisions among molecules flying roughly in the same directions, faster molecules catch up with the slower ones and particles will retain the preferred direction of motion also after collision. Deflections from

directions of the greatest probability will thus be relatively small, at least after the first collisions.

The relationships derived in the present paper allow to analyse the situation at the collision of molecules as flexible spheres and to draw conclusions about the distribution of directions, frequency of collisions and the effective free path under conditions of directed substance transport.

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УПРУГОЕ РАССЕЯНИЕ ТВЕРДЫХ ШАРОВ В ПРОСТРАНСТВЕ

В работе исследованы вопросы связанные с упругим рассеянием твердых шаров в пространстве, где шары стартуют одновременно, или в разное время с любой точки пространства. Пространственные, или временные условия соударения выражены уравнениями определяющими координаты соударения, скорость и направление движения после соударения. Вычисление проводится в аналитической и также графической форме с применением годографических скоростей. Предложеный метод оказывается простым и показательным. Может быть также применен при решении других проблем, где используется уравнение Больцана. Метод позволяет решать проблему многократных последующих соударении. Полученые уравнения применимы в вычислении среднего свободного расстояния молекул идеального газа в условиях анизотропии.