acta physica slovaca 42 (1992), No. 1, 3–8

A DIRECT ALGEBRAIC METHOD OF SOLVING A NONLINEAR EQUATION

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The method is proposed for deriving soliton solutions of the sine-Gordon equation and a continuous nonstationary model of the Peierls dielectric connected with the Dirac operator, which does not use the inverse scattering method.

I. INTRODUCTION

The main purpose of the present paper is the description, in a form acceptable to nonspecialists, of the method of constructing localized solutions of some class (the definition is given below) of nonlinear models that have been successfully applied in different fields of solid state physics [1]. In our opinion, the developed approach to constructing multisoliton solutions is one of the simplest approaches, and the fact that it apparently cannot be simplified for the sine Gordon equation (SG) is likely.

For the first time a similar approach to the nonlinear Schroedinger equation and its vector generalizations for periodic solutions has been suggested in [2].

Two models are used as an example. The first is the wellknown (SG) model. The second is a nonstationary Peierls model in the approximation satisfying the "smallness" of the forbidden band when a nonstationary Schroedinger operator transforms into a nonstationary Dirac operator (7). The choice of these continuous dynamic models is not accidental. Notably, in spite of the fact that both these models are solved by the same scheme, there is a strong difference between them. It consists in that the model connected with the Dirac operator (7) is solved immediately, whereas the SG equation is to be transformed to an equivalent system of equations and only then a general method can be applied.

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The examples to be considered below belong to a class of models possessing a remarkable property i.e. a complication of a spectral parameter (in our case, a transition from the plane of variable λ to the Riemann surface of the function $(k+1/k)$ simplified the scheme of the solution. Especially, for the eigenvalue problem $L\psi = \lambda\psi$ connected with the SG equation the operator L is of the fourth order [3]. Upon changing the spectral parameter $(\lambda \to k + 1/k)$ the order of L decreases to two (see for instance [4]). It is to be noted that there are other models with the same property. For example, for the relativistic-invariant Thirring model it is convenient to pass from the complex plane λ to be Riemann surface of the the function $k^2 + 1/k^2$.

The main point of the construction expounded below is the use of the fact that the solution of the equations studied are to be searched in a special form representing the "degeneracy" of the Baker—Akhiezer functions depending on the spectral parameter k and also on the auxiliary variables κ_i^+, κ_i^- $(i = 1, 2, \dots M)$[6]:

$$\psi(x,t,k) = Q_N(x,t,k)\psi_0(x,t,k),$$

where $\psi(x,t,k)$ see (3) and $Q_N(x,t,k) = a_0(x,t)k^N + a_1(x,t)k^{N-1} + \dots + a_N(x,t)$ is the polynomial of a certain degree N. Moreover, the common feature of the considered class of models is that $\psi(x,t,k)$ is a meromorphic function of the variable having two essential singular points $P_0(k=0)$ and $P_\infty(k=\infty)$.

There are two reasons why the SG equations are used as an example to consider the method in detail. First, we want to show how to proceed in the cases when the method cannot be applied immediately to the initial equations. Secondly the SG equation does not encounter additional technical difficulties connected with the solution of self-consistency equations (7) and (8).

In the coordinates of the light cone the SG equation $u_{\xi\eta}(\xi,\eta) = 4\sin u(\xi,\eta)$ is equivalent to the condition of compatibility of the problems $\partial_\xi\psi = U\psi$, $\partial_\eta\psi = V\psi$, $\partial_\eta U - \partial_\xi V + [U,V] = 0$ where $\psi = (\psi_1,\psi_2)^T$ and matrices U, V have the form:

$$U = \begin{bmatrix} -\frac{i}{2}u_\xi & 1 \\ 1/\lambda & \frac{i}{2}u_\xi \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & \lambda\exp(-iu) \\ \exp(iu) & 0 \end{bmatrix} \tag{1}$$

with $u_\xi = \partial_\xi u(\xi,\eta)$.

Using (1) we get for the functions ψ_1 and ψ_2 the following equations:

$$\partial_\xi\psi_1 = -\frac{i}{2}u_\xi\psi_1 + \psi_2 \qquad \partial_\xi\psi_2 = \frac{1}{\lambda}\psi_1 + \frac{i}{2}u_\xi\psi_2,$$
$$\partial_\xi\psi_2 = \psi_1\exp(iu) \qquad \partial_\eta\psi_1 = \lambda\psi_2\exp(-iu) \tag{2}$$

Now we introduce, according to [6], the functions ψ_1 and ψ_2, admitting in the vicinity of two singular points P_0 and P_∞ expansions in the form: (for the point

P_0 the local parameter k is chosen as $k = \lambda^{-1/2}$)

$$\psi_1(\xi,\eta,k) = \left[1 + \sum_{s=1}^{\infty}\xi_s^{10}(\xi,\eta)k^{-s}\right]\exp(\xi k + \eta/k) \tag{3a}$$

$$\psi_2(\xi,\eta,k) = k\left[1 + \sum_{s=1}^{\infty}\xi_s^{20}(\xi,\eta)k^{-s}\right]\exp(\xi k + \eta/k). \tag{3b}$$

Analogously, in the vicinity of P_∞ (the local parameter $k = \lambda^{1/2}$) we have:

$$\psi_1(\xi,\eta,k) = c_1(\xi,\eta)\left[1 + \sum_{s=1}^{\infty}\xi_s^{100}(\xi,\eta)k^{-s}\right]\exp(\xi k + \eta/k) \tag{4a}$$

$$\psi_2(\xi,\eta,k) = c_2(\xi,\eta)\frac{1}{k}\left[1 + \sum_{s=1}^{\infty}\xi_s^{200}(\xi\eta)k^{-s}\right]\exp(\xi k + \eta/k). \tag{4b}$$

Substituting expansions (3) and (4) into (2) and equating the coefficients at the same degrees of k we get the recurrence relations for the functions $\xi_s^{ij}(x,t)$ $(i,j = 1,2;\ j = 0,\infty,\ s = 1,2,\dots)$

$$\xi_1^{10} - \xi_1^{20} = -\tfrac{i}{2}u_\xi, \qquad \xi_{s+1}^{10} - \xi_{s+1}^{20} + \frac{\partial}{\partial\xi}\xi_s^{10} = -\tfrac{i}{2}u_\xi\xi_s^{10},$$
$$\xi_{s+1}^{20} - \xi_{s+1}^{10} + \frac{\partial}{\partial\xi}\xi_s^{10} = -\tfrac{i}{2}u_\xi\xi_s^{20}, \qquad s \geq 1. \tag{5}$$

According to the general scheme [6], to derive an M soliton solution one should define M pairs of auxiliary parameters κ_i^+ and κ_i^- (owing to the presence of symmetry we can choose $\kappa_i^+ = -\kappa_i^-$) and M conditions of the bond:

$$\psi(\kappa_i^+) = \sum_{j=1}^{M}\alpha_{ij}\psi(\kappa_j^-). \tag{6}$$

where α_{ij} is the matrix $M \times M$. To determine $u(\xi,\eta)$ we use the first relation in (5) in which $\xi_s^{10}(\xi,\eta)$ and $\xi_s^{20}(\xi,\eta)$ are determined from a system of linear equations (6) corresponding to each solution. We can immediately verify that for $\alpha_{ij} = \alpha_{11} = i\alpha_0$ (α_0 is the real number) we get a one-soliton solution; for $\alpha_{11} = i\alpha_1$, $\alpha_{22} = i\alpha_2$ $(\alpha_{12} = \alpha_{21} = 0)$ $\kappa_1^+ = -\kappa_1^- = \kappa_1$, $\kappa_2^+ = -\kappa_2^- = \kappa_2$ we get a two soliton solution, and assuming $\alpha_{12} = \alpha$, $\alpha_{21} = -\alpha$, $\alpha_{11} = \alpha_{22} = 0$; $\kappa_1^+ = \alpha + i\beta$, $\kappa_1^- = -(\kappa_1^+)$, $\kappa_2^+ = (\kappa_1^+)$, $\kappa_2^- = -\kappa_1^+$ we get a breather:

$$u(\xi,\eta) = 4\operatorname{arctg}\left\{\frac{\alpha}{\beta}\,\frac{\sin\left[\dfrac{2\beta\xi}{(\alpha^2+\beta^2)} + 2\beta\eta + \varphi_0\right]}{\cosh\left[\dfrac{2\alpha\xi}{(\alpha^2+\beta^2)} + 2\alpha\eta + \varphi_1\right]}\right\}.$$

As a second example we consider the nonstationary Peierls problem (for a more concrete statement see [5]) connected with an operator of the Dirac type. The problem implies the search for a self-consistent state of electrons and lattice. Determination of the ground state is reduced to the solution of the equation [5]:

$$\partial_t\psi - \sigma_3\partial_x\psi + [\sigma - \Delta^*(x,t) - \sigma_+\Delta(x,t)]\psi = 0 \qquad (7)$$

with the equation of the self-consistency

$$\mathrm{i}\int \mathrm{d}\Gamma\psi\sigma_-\psi^+ = \lambda\Delta - \mu\partial_t^2\Delta + \lambda_2\partial_x^2\Delta + 2i\lambda_1\partial_x\Delta + \lambda_s\Delta|\Delta|^2. \qquad (8)$$

Here $\psi^+(x,t,k,) = (\psi_1^*,\psi_2^*)$ is the two-component spinor; σ_i is the ith Pauli matrix $(\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2)$ and λ, λ_1, λ_2, λ_3, μ are phenomenological constants, $\mathrm{d}\Gamma = vL\mathrm{d}p[2\pi\langle\psi,\psi^*\rangle_x]^{-1}$ is a density of states, $\mathrm{d}p$ a differential of the quasi-moment, v is an occupation number and the angle brackets $\langle\ldots\rangle_x$ here denote averaging over the length of the system $\langle f\rangle_x = L^{-1}\int_0^L f(x)\mathrm{d}x$ [6]. L is the system's length and $\Delta(x,t)$ is the order parameter (gap parameter). To find soliton solutions of eq. (7) with condition (8) it is convenient to renormalize the function $\psi(x,t,k)$:

$$\Phi(x;t,k) = \frac{\psi(x,t,k)}{(k-\kappa_1)\ldots(k-\kappa_N)} = \qquad (9)$$
$$= \left\{r_\infty + \sum_{i=0}^{N}\frac{r_i(x,t)}{k-\kappa_i}\right\}\exp\left\{i\left[kz_+ + \frac{z_-}{k}\right]\right\},$$

where $|\kappa_0|^2 = 1$, $z_\pm = \Delta_0(t \pm x)/2$, r_∞ see below. The conditions (6) for the renormalizes function (9) can be written in the form:

$$\mathrm{i}\,\mathop{\mathrm{res}}_{\kappa_i}\ \phi(x,t,k)(k-\kappa_0)(k-\kappa_0^*)\frac{1}{k^2} = \sum_{j=1}^{N} c_{ij}\Phi(x,t,\kappa_j^*) \qquad (10)$$

Denote by $\Phi_1(x,t,k)$ the function of the form (9) satisfying the condition (10) and normalized so that $\Phi(k=0) = 0$, $(r_\infty = 1)$:

$$\Phi_1(x,t,k) = \frac{k}{k-\kappa_0} + \sum_{i=1}^{N}\frac{\tilde r_i(x,t)k}{(k-\kappa_i)(k-\kappa_0)}\exp\left\{\mathrm{i}\left[kz_+ + \frac{z_-}{k}\right]\right\}. \qquad (11)$$

By analogy we normalize $\Phi_2(x,t,k)$ by the condition $\psi(k=0) = 1, (r_\infty = 0)$:

$$\Phi_2(x,t,k) = -\frac{\kappa_0}{k-\kappa_0} + \sum_{i=1}^{N}\frac{\tilde r_i(x,t)k}{(k-\kappa_i)(k-\kappa_0)}\exp\left\{\mathrm{i}\left[kz_+ + \frac{z_-}{k}\right]\right\}. \qquad (12)$$

$$\Delta(x,t) = \frac{\mathrm{i}}{\kappa_0}\left[-1 + \sum_j \frac{\tilde r_j(x,t)}{\kappa_j}\right], \quad \Delta^*(x,t) = \mathrm{i}\left[\kappa_0 - \sum_j \tilde r_j(x,t)\right]. \qquad (13)$$

For further consideration it is important that the function of the type (9) constructed by us apart from the poles at the points $k = \kappa_i (i = 1,2,\ldots,N)$ has singularities also at $k = 0$ and $k = \infty$. In the vicinity of these points the functions $\Phi(x,t,k)$ determined by (11) and (12) can be represented as

$$\Phi_1(x,t,k) = \left\{1 + \sum_{s=1}^{\infty}\xi_s^{11}(x,t)k^{-s}\right\}\exp[ikz_+]$$
$$\Phi_2(x,t,k) = \left\{\sum_{s=1}^{\infty}\xi_s^{21}(x,t)k^{-s}\right\}\exp[ikz_+] \qquad (14)$$

Analogously, as $k \to 0$:

$$\Phi_1(x,t,k) = \left\{\sum_{s=1}^{\infty}\xi_s^{12}(x,t)k^{s}\right\}\exp\left[\mathrm{i}\frac{z_-}{k}\right] \qquad (15a)$$

$$\Phi_2(x,t,k) = \left\{1 + \sum_{s=1}^{\infty}\xi_s^{22}(x,t)k^{s}\right\}\exp\left[\mathrm{i}\frac{z_-}{k}\right]. \qquad (15b)$$

For the quantities $\xi^{ij}(x,t)$, using (7) we get

$$\begin{gathered}\mathrm{i}\xi_1^{21} = -\Delta^*(x,t),\ \partial_-\xi_1^{11} = \mathrm{i}(|\Delta|^2-1),\ \mathrm{i}\xi_1^{12} = \Delta(x,t),\\ \mathrm{i}\xi_{s+1}^{21} + \partial_+\xi_s^{21} = -\Delta^*\xi_s^{11},\ \mathrm{i}\xi_{s+1}^{12} + \partial_-\xi_s^{12} = \Delta\xi_s^{22},\\ \partial_+\xi_1^{22} = \mathrm{i}(|\Delta|^2-1),\quad s = 1,2,\ldots\ldots,\end{gathered} \qquad (16)$$

where the notation $\partial_\pm = (\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x})$. The relations (16) are used as a basis for constructing solutions (7) with the equations of self-consistency (8).

We shall not consider this model any more as the derivation of the solutions $\Delta(x,t)$ and the wave functions $\Phi_1(x,t,k)$ and $\Phi_2(x,t,k)$ of electrons in the presence of a kink (topological solitons) and a polaron can be found in detail in [5].

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Received July 25th, 1990
Accepted for Publication October 22nd, 1990

ПРЯМИЙ АЛГЕБРАИЧЕСКИЙ МЕТОД РЕШЕНИЯ НЕЛИНЕЙНЫЦХ УРАВНЕНИЙ

Метод предназначен к вычислению солитонных решений уравнения Син-Гордона и непрерывной нестационарной модели диелектрика Пайерлса связанного с оператором Дирака, что не позволяет применение метода обратного рассеяния.