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CHAOS IN A SYSTEM OF COUPLED OSCILLATORS

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The paper present on the basis of numerical methods the transition from periodic to chaotic orbits in a system of coupled nonlinear oscillators. The periodic orbits have been traced by solving a boundary value problem while the chaotic motions have been examined by means of standard methods based on the solution of the initial value problem.

INTRODUCTION

The answer to the question about the possible ways from regular to irregular chaotic motion in real physical systems is one of the important problems of chaotic dynamics of nonlinear systems. In the case of simple sinusoidally driven anharmonic oscillators the problem seems to be to a great extent explained (for reviews on the subject see e.g. [1—3]).

A particularly effective method which makes it possible to trace the transition from regular to chaotic or quasiperiodic orbits is the numerical method based on the solution of the boundary value problem with simultaneous observation of characteristic multipliers. The latter are decisive for the stability and the bifurcation of the considered periodic orbit [4]. In the case of simple oscillators with harmonic forcing chaos can occur after the saddle-node bifurcation (the characteristic multiplier crossing $+1$) or by a cascade of bifurcations doubling the period of solution which is connected with the multipliers crossing -1) [5].

We shall now follow the situation leading to chaos in a system of coupled nonlinear oscillators subject to harmonic forcing.

II. THE MODEL AND THE METHOD

The calculation model is an unbalanced rotor with a rectangular cross-section supported in a harmonically forced frame [6]. The dynamics of the considered system is governed by the following equations

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$$M\ddot{x}_1 + c_1\dot{x}_1 + k_0x_1 + k_1x_1^3 + x_1\left(\frac{k_\xi + k_\eta}{2} + \frac{k_\xi - k_\eta}{2}\cos 2\omega t\right) +$$
$$+ x_2\left(\frac{k_\xi + k_\eta}{2} + \frac{k_\xi - k_\eta}{2}\cos 2\omega t\right) + x_3\frac{k_\xi - k_\eta}{2}\sin 2\omega t = F\cos\omega_0 t,$$
$$m\ddot{x}_2 + c_2\dot{x}_2 - \left(\frac{k_\xi + k_\eta}{2} + \frac{k_\xi - k_\eta}{2}\cos 2\omega t\right)x_1 +$$
$$+ \frac{k_\xi + k_\eta}{2} + \frac{k_\xi - k_\eta}{2}\cos 2\omega t)\,x_2 + x_3\frac{k_\xi - k_\eta}{2}\sin 2\omega t = ma\omega^2\sin(\omega t + \varphi_0), \qquad (1)$$
$$m\ddot{x}_3 + c_3\dot{x}_3 + x_3\frac{k_\xi - k_\eta}{2}\sin 2\omega t - x_2\frac{k_\xi - k_\eta}{2}\sin 2\omega t +$$
$$+ x_3\frac{k_\xi + k_\eta}{2} - \frac{k_\xi - k_\eta}{2}\cos 2\omega t = ma\omega^2\cos(\omega t + \varphi_0) + mg,$$

where while x_1 is the horizontal displacement of the frame; x_2, x_3 is the horizontal and vertical displacement of the concentrated mass m found in the middle of the rotor length; M is the frame mass; k_0, k_1 are rigidities connecting the frame with the base; k_ξ and k_η are rigidities of the transverse section of the rotor; ω — is the frequency of the rotor revolutions; F and ω_0 are the amplitude and the external forcing frequency; a and φ_0 is the unbalance position; c_1, c_2, c_3 is viscotic damping; g is the acceleration of gravitation.

Thanks to the relations given below

$$\tau = (k_0M^{-1})^{1/2}t,\ y_1 = (k_1k_0^{-1})^{1/2}x_1,\ x_2 = \frac{2mgy_2}{k_\xi + k_\eta},\ x_3 = \frac{2mgy_3}{k_\xi + k_\eta}, \qquad (2)$$
$$z = \frac{k_\xi + k_\eta}{2k_0},\ b = \frac{k_\xi - k_\eta}{2k_0},\ d_1 = c_1(k_0M)^{-1/2},\ e = mg(k_1k_0^{-3})^{1/2},$$
$$v = \omega/M/k_0)^{1/2},\ \mu = \frac{m}{M},\ q = F(k_1k_0)^{-31/3},\ v_0 = \omega_0(M/k_0)^{1/2},$$
$$q_1 = \omega^2ag^{-1}\cos\varphi_0,\ q_2 = \omega^2ag^{-1}\sin\varphi_0,\ d_2 = 2c_2\left(\frac{k_0}{M}\right)^{1/2}\frac{1}{k_\xi + k_\eta},$$
$$d_3 = 2c_3\left(\frac{k_0}{M}\right)^{1/2}\frac{1}{k_\xi + k_\eta}$$

the following dimensionalless set of equations is obtained

$$\ddot{y}_1 + d_1\dot{y}_1 + (1+z)y_1 + y_1^3 + by_1\cos 2v\tau - ey_2 - \varepsilon ey_2\cos 2v\tau +$$
$$+ \varepsilon ey_3\sin 2v\tau = q\cos v_0\tau;$$
$$\frac{\mu}{z}\ddot{y}_2 + d_2\dot{y}_2 - \frac{z}{e}y_1(1 + \varepsilon\cos 2v\tau) + (1 + \varepsilon\cos 2v\tau)y_2 - \varepsilon y_3\sin 2v\tau =$$
$$= q_1\sin v\tau + q_2\cos v\tau, \qquad (3)$$
$$\frac{\mu}{2}\ddot{y}_3 + d_3\dot{y}_3 + \frac{z}{e}\varepsilon y_1\sin 2v\tau - \varepsilon y_2\sin 2v\tau + y_3(1 - \varepsilon\cos 2v\tau) - 1 =$$
$$= q_1\cos v\tau + q_2\sin v\tau,$$

where: $\varepsilon = \frac{b}{z} = \frac{k_\xi - k_\eta}{k_\xi + k_\eta}$ and $' = \frac{d}{d\tau}$.

The approximate fixed point $y_F^{(k)}$ near the unknown "true" one is assumed and numerical integration using the Gear method is carried out. A point mapping $M(y_F^{(k)}) = M_F^{(k)}$ is defined then. The error $E = y_F^{(k)} - M_F^{(k)}$ shows the accuracy of the estimation (k). Thanks to the shooting method and the Newton—Raphson procedure we can look for zeros of the error function E (in the considered case calculations were interrupted if the following norm

$$\|E\| = |E|^2 \leqslant 10^{-5} \text{ occurs}).$$

The characteristic equation

$$\chi(\sigma) = \det(H_F - \sigma 1) = 0, \qquad H_F = \frac{\partial M_F}{\partial y_F} \qquad (4)$$

yields the characteristic multipliers, where H_F is already known from the above-mentioned iteration of the point mapping at the fixed point.

III. RESULTS

Calculations have been performed at the following constant parameters: $z = 0.1$, $d_1 = 0.4$, $d_2 = d_3 = 0.2$, $e = 1.0$, $\mu = 0.1$, $v = 0.8$, $q = 0.4$, $v_0 = 1.0$, $q_1 = 0.15$, $q_2 = 0.0$. b has been assumed as bifurcation parameter. For $b = 0.001$ a periodic solution has been found with the period 10π on the left and on the right of the point 0. They are not symmetrical, however, as it can be proved by the time histories shown in Fig. 1a and the exemplary projection of the solution on the plane $y_1(\dot{y}_1)$ — Fig. 1b. The analysis of the other projections of the two

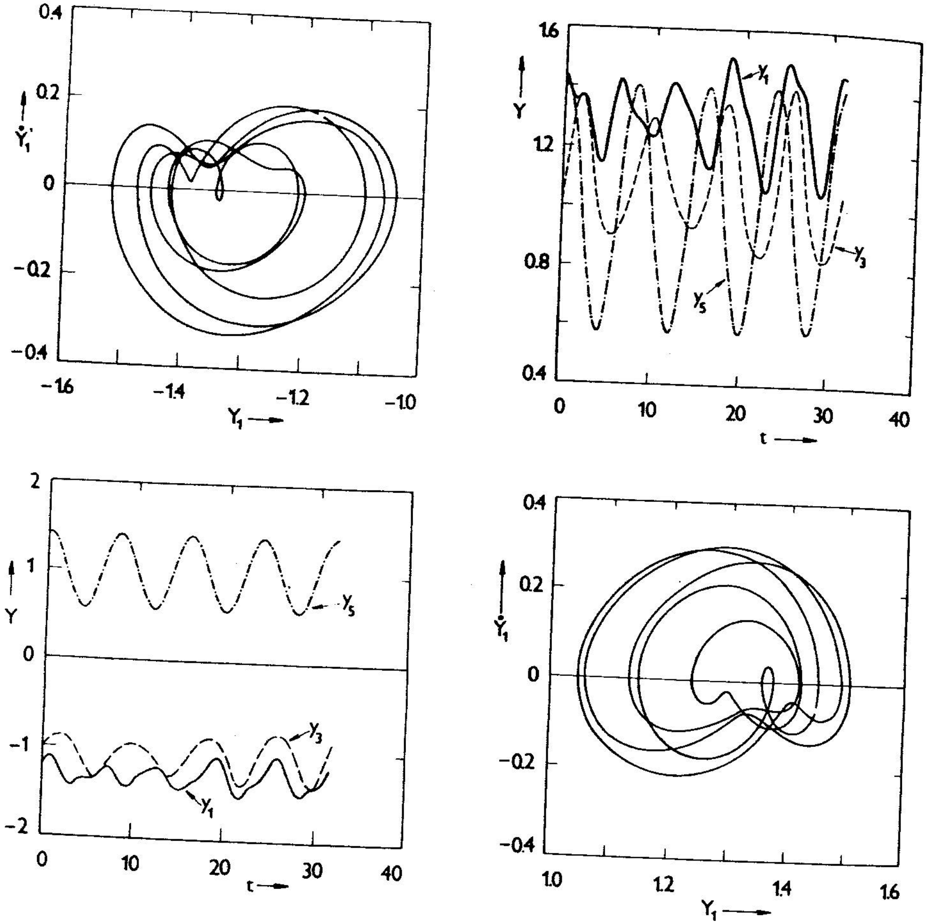

Fig. 1. Time histories and projections of the 10-π periodic orbits on the left (a) and on the right (b) for b = 0.001.

considered periodic orbits makes it clear that symmetry in relation to zero is slightly unbalanced for both solutions. It appears that the lack of complete symmetry has a considerable influence on the latter behaviour of the considered orbits with the increase of b. This situation is shown in Table 1, where the characteristic multipliers corresponding to the chosen values of b are given (see also Fig. 2). When observing the orbit on the right it is clearly visible that one of the characteristic multipliers increases with the increase of b. For $b = 0.006$ it reaches the value of 0.9 and it has been found impossible to further trace the stability of the orbit retaining the accuracy of calculations. As it can be supposed

Table 1
The investigated period solutions with the corresponding characteristic multipliers

	b	Characteristic multipliers (Re, Im)					
		1	2	3	4	5	6
	0.001	−0.13; 0.13	−0.13; −0.13	0.0; 0.0	0.0; 0.0	0.21; 0.01	0.21; −0.01
	0.003	−0.10; 0.15	−0.10; 0.15	0.0; 0.01	0.0; −0.01	0.21; 0.01	0.21; −0.01
	0.005	−0.06; 0.17	−0.06; −0.17	0.0; 0.0	0.0; −0.0	0.21; 0.01	0.21; −0.01
	0.009	0.37; 0.0	0.08; 0.0	0.0; 0.0	0.0; −0.0	0.21; 0.02	0.21; −0.02
	0.011	−0.04; 0.17	−0.04; −0.17	0.21; 0.02	0.21; −0.02	0.0; 0.0	0.0; −0.0
	0.012	−0.54; 0.0	0.21; 0.03	0.21; −0.03	−0.6; 0.0	0.0; 0.0	0.0; 0.0
	0.013	−1.0; 0.0	0.20; 0.03	0.20; −0.03	−0.03; 0.0	0.0; 0.0	0.0; 0.0
	0.014	−1.61; 0.0	0.20; 0.04	0.20; −0.04	0.0; 0.0	0.0; −0.0	−0.02; 0.0
	0.0144	−1.56; 0.0	0.20; 0.04	0.20; −0.04	0.0; −0.04	0.0; 0.0	−0.2; 0.0
	0.0146	−0.11; 0.13	−0.11; −0.13	0.20; −0.05	0.20; 0.05	0.0; 0.0	0.0; 0.0
The orbit to the right of the origin	0.001	−0.13; 0.13	−0.13; −0.13	0.00; 0.0	0.0; 0.0	0.21; 0.1	0.21; −0.01
	0.03	0.01; 0.26	0.01; −0.26	0.01; 0.0	0.01; −0.0	0.28; 0.01	0.28; −0.01
	0.004	0.08; 0.24	0.08; −0.24	0.01; 0.0	0.01; −0.0	0.28; 0.01	0.28; −0.01
	0.005	0.20; 0.16	0.20; −0.16	0.01; 0.0	0.01; −0.0	0.28; 0.01	0.28; −0.01
	0.0054	0.36; 0.0	0.18; 0.0	0.26; 0.0	0.01; 0.0	0.01; 0.0	0.31; 0.0
	0.0056	0.50; 0.0	0.13; 0.0	0.27; 0.0	0.01; 0.0	0.01; 0.0	0.30; 0.0
	0.0058	0.68; 0.0	0.10; 0.0	0.27; 0.0	0.01; 0.0	0.01; 0.0	0.30; 0.0
	0.006	0.90; 0.0	0.07; 0.0	0.28; 0.0	0.29; 0.0	0.01; 0.0	0.01; 0.0

that a slight increase of b causes the increase of the multiplier to the value of $+1$, and a solution with the period 5π can be found. No such solution has been found which makes it possible to draw the calculation that the analysed solution has left the real space.

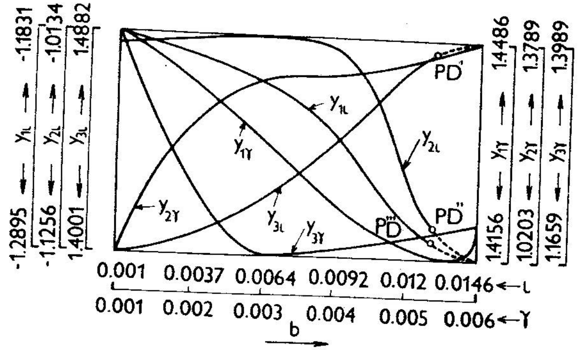

Fig. 2. Fixed points (periodic-orbits) against the change of b (l denotes the orbit to the left of the origin, and r that to the right). Dashed line denotes unstable solutions.

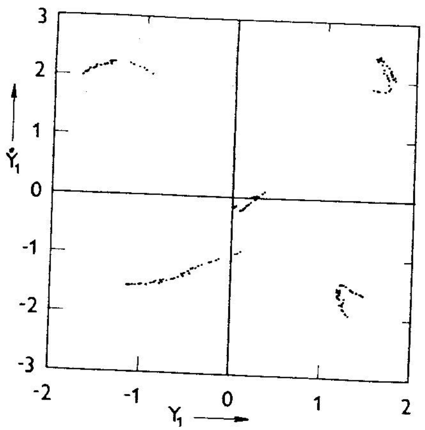

Fig. 3. One projection of the Poincaré map of the strange chaotic attractor presented for $b = 0.02$.

In the case of the orbit on the left it can be seen that for $b = 0.013$ it loses stability. It regains stability for $b = 0.0146$. A further increase of b results in the fact that this solution leaves the real space. It has been replaced by a chaotic orbit with its projection shown in Fig. 3 (for $b = 0.02$).

IV. CONCLUSIONS AND DISCUSSION

In complex physical systems it is possible to observe transitions from periodic to chaotic orbits which have not yet been found in systems of the dimension 3 such as the harmonically driven oscillators. The upsetting of the symmetry of both 10π — periodic orbits causes their different behaviour with the increase of b. First the right ($b > 0.006$) and then the left ($b > 0.0146$) orbit leaves the real space. The void in the phase space is filled by a chaotic attractor. The occurrence of the chaotic orbit is not accompanied by a bifurcation of the left periodic orbit connected with the passage of the characteristic multipliers through a unitary circle of the complex plane.

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ХАОС В СИСТЕМЕ СВЯЗАННЫХ ОСЦИЛЛЯТОРОВ

В работе приведены основные нумерические методы для перехода от периодической к хаотической орбитам в системе связанных нелинейный осцилляторов. В периодических орбитах характерно решение граничных условий проблемы, пока что хаотическое движение исследуется с применением стандартных методов с решением первичной величины исследуемой проблемы.