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SCALING RELATION FOR THE WIDTHS OF ARNOLD TONGUES

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The scaling relation for the widths of the Arnold tongues is introduced together with the method of estimation of the scaling exponents. The accuracy of estimation is discussed.

INTRODUCTION

Dynamical regimes on circle maps together with the corresponding scaling relations have been widely studied in recent years [1—5]. Here we introduce a method of estimation of the scaling exponents, which scale the widths of mode locked intervals in parameter space. As a model of circle map we used the well-known two parametrical sine map

$$f_\Omega(x, k) = x + \Omega - \frac{k}{2\pi}\sin(2\pi x) \qquad (1)$$

with frequency parameter $\Omega \in (0, 1)$ and nonlinearity parameter $k \geq 0$.

I. CIRCLE MAPS

The phenomenon of mode locking is not rare in systems with two competing frequencies. When two oscillators are coupled together, they influence each other and under certain conditions their combined motion becomes periodic (locked). Sometimes it does not happen and the resulting motion has two independent frequencies (unlocked motion). The dynamics of such coupled oscillators and the structure of locked and unlocked regions is succesfully modelled by so-called circle maps.

Circle maps in general are defined by the relation

$$f(x) = x + \Omega + g(x), \qquad (2)$$

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where $g(x)$ exhibits a nonlinear function which fulfils the identity

$$g(x) = g(x+1).$$

For the function $f(x)$ the circle map identity is valid

$$f(x+1) = f(x) + 1. \qquad (3)$$

The functions

$$\begin{aligned} f^0(x) &= x \\ f^1(x) &= f(x) \\ f^2(x) &= f(f(x)) \\ &\vdots \\ f^n(x) &= \underbrace{f(f\ldots(x))\ldots)}_{n_x} \end{aligned} \qquad (4)$$

exhibit the zero-th, first, second ... n-th iteration of $f(x)$. The dynamics of circle maps such as (1) is well described by the so-called winding number

$$W = \lim_{n\to\infty} \frac{f^n(x, k) - x}{n}. \qquad (5)$$

The most frequently used representative of circle maps is a sine map (1). For the nonlinearity parameter k less than or equal to 1 ($k \leq 1$) and W rational $\left(W = \frac{P}{Q}\right)$, there exists a nonzero interval of frequencies $\Delta\Omega_W(k)$ in which

$$f^{n+Q} = f^n (\text{mod } P). \qquad (6)$$

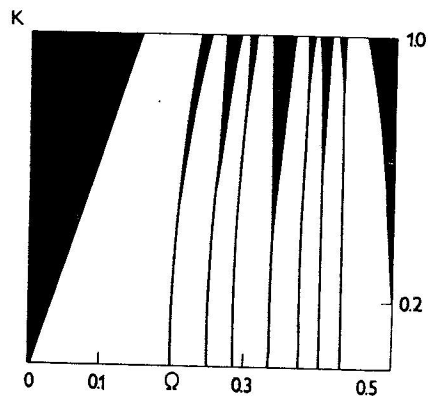

Fig. 1. The Arnold tongues for a map (1). The black areas exhibit parameter values of periodic regime.

This is called "mode locking" and the map has a periodical dynamical regime (periodical cycle). The intervals of Ω exhibit the Arnold tongues (ATS) in the parameter space of (1) (Fig. 1). If W is irrational, the dynamical regime on a map is quasiperiodical.

As k grows, ATS become wider and at $k = 1$ they cover the entire Ω axis excluding a set of points (Cantor set) where the winding numbers are irrational. The mode locked intervals at $k = 1$ develop a nice self similar structure called the "devil staircase" (Fig. 2). For $k > 1$ the ATS overlap and even chaotic orbits could arise. The parameter value $k = 1$ is therefore a critical value of chaotic motion.

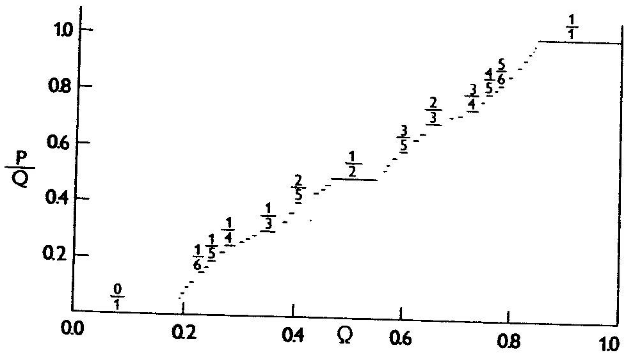

Fig. 2. "Devil staircase" of a critical sine map ($k = 1.0$). Winding number does not change for an interval of Ω. The structure is self similar.

The periodical cycle (6) with the rational winding number $W = P/Q$ is stable until

$$\left|\frac{\mathrm{d} f_\Omega^Q(x)}{\mathrm{d}x}\right|_{x=x_i} \leq 1 \qquad x_i \text{ — point of the cycle.} \tag{7}$$

The parameter $\Omega = \Omega_n$ for which the equality holds is called the maximally stable point of $\Delta\Omega_W(k)$ and is always located very closely to the centre of the periodical interval.

II. SCALING RELATIONS

The scaling of the Arnold tongues is a problem, which interests many scientists [1—5]. In [2] Shenker studied the scaling relations connected with the winding numbers converging to the "golden mean"

$$W^{gm} = \cfrac{1}{1 + \cfrac{1}{1 + \dots}} = [1, 1, 1, \dots]. \tag{8}$$

One easily finds that

$$W^{gm} = \lim_{n \to \infty} (F_n / F_{n+1}) \qquad \begin{matrix} F_n = F_{n-1} + F_{n-2} \\ F_0 = 0,\ F_1 = 1 \end{matrix} \tag{9}$$

F_n are the Fibbonacci numbers.

Shenker's results were extended by Alström et al. [3] to ATS having winding numbers converging to periodic irrationals

$$W = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_1 a_2, \dots a_n \dots a_{n+s}, a_n \dots, a_{n+s}]. \tag{10}$$

The focus of our interest is the subcritical region of the map (1) ($k \leq 1$) [5]. We have demonstrated numerically that there exists a scaling law for the widths of ATS is this region of parameter space.

We chose ATS having the winding numbers

$$\begin{matrix} W(Q) = P_0/Q \\ Q = Q_0 + mP_0 \qquad m = 0, 1, 2, \dots, \end{matrix} \tag{11}$$

where $W(Q_0) = P_0/Q_0$ represents the first and biggest winding number of all the set.

We studied numerically the Q-dependency of the widths of ATS having W given by (11) for a fixed P_0/Q_0 and k. All the numerical calculations were carried out for a sine map (1). The edges of ATS were calculated by a two-dimensional Newton iteration method proposed by Jensen et al. [1]. The condition of a stability of a certain Arnold tongue is

$$\begin{matrix} f_\Omega^Q(x) = x + P \\ \dfrac{\mathrm{d} f_\Omega^Q(x)}{\mathrm{d}x} = 1. \end{matrix} \tag{12}$$

That means that at the edges of the ATS vector $\boldsymbol{g}(x, \Omega)$

$$\boldsymbol{g}(x, \Omega) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_\Omega^Q(x) - x - P \\ \dfrac{\mathrm{d} f_\Omega^Q x}{\mathrm{d}x} - 1 \end{pmatrix} \tag{13}$$

is equal to zero. Expanding $\boldsymbol{g}^* = \boldsymbol{g}(x^*, \Omega^*)$ around the initial point of iteration, $\boldsymbol{g}(x_0, \Omega_0) = \boldsymbol{g}_0$

$$\boldsymbol{g}^* \simeq \boldsymbol{g}_0 + \boldsymbol{\Delta}\mathcal{M}, \tag{14}$$

where

$$\Delta = (x^*, \Omega^*) - (x_0, \Omega_0) \tag{15a}$$

and

$$\mathcal{M} = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial \Omega} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial \Omega} \end{bmatrix} \tag{15b}$$

we find that (for $\boldsymbol{g}^* = \boldsymbol{0}$)

$$\Delta \simeq -\mathcal{M}^{-1} \boldsymbol{g}_0 \tag{16}$$

and so the first approximation will be

$$(x^*, \Omega^*) \simeq (x_1, \Omega_1) = -\mathcal{M}^{-1} g_0 + (x_0, \Omega_0). \tag{17}$$

Iterating (16), (17) it is possible to locate the endpoints of the P/Q interval. The accuracy of our calculation was tested by calculating symmetric ATS ($\tilde{W} = 1 - W$). All boundaries of ATS were determined to an accuracy of at least 10^{-12}.

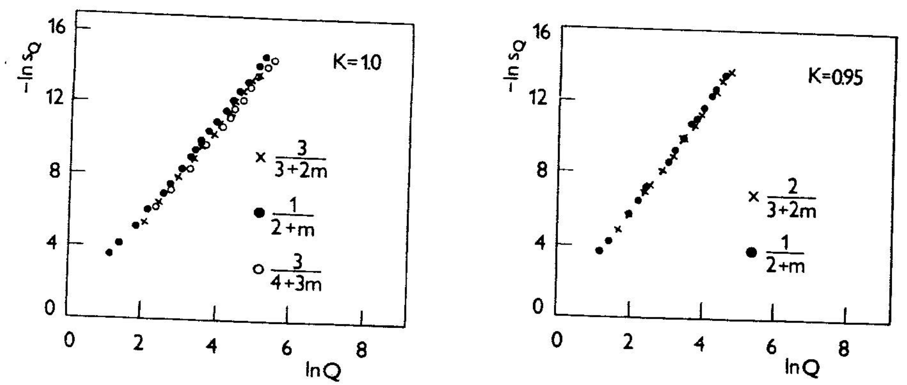

Fig. 3. Logarithmic plot of denominator Q of winding number against the width of corresponding Arnold tongue. The dependences are asymptotically linear.

As it is seen from Fig. 3, all logarithmic Q-dependencies of $\Delta\Omega_W(k) = s_Q(k)$ are asymptotically ($Q \to \infty$) linear. This linearity indicates scaling relation for a great Q

$$s_Q = CQ^{-\tilde{\alpha}}. \tag{18}$$

There are two possibilities of calculating the scaling exponent $\tilde{\alpha}$:
1) to calculate $\tilde{\alpha}$ as a slope of a line (Fig. 3) by a linear regression from a few points having a great Q.
2) to find a method of estimation of the exponent $\tilde{\alpha}$.

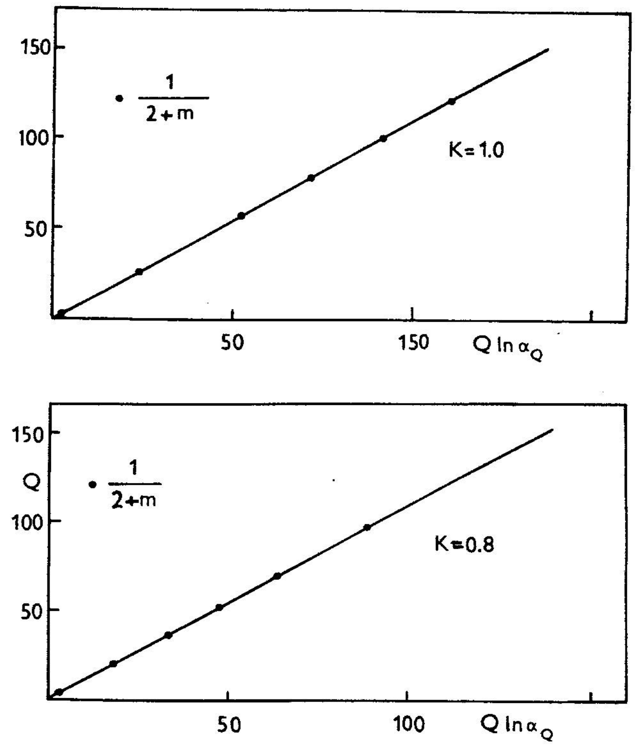

Fig. 4. Dependences of α_Q on Q.

The first method is rather time consuming, because we need to penetrate sufficiently into asymptotic regime. Moreover we have no guarantee that our approach is successful enough. Therefore we have used a second way. By a linear regression we calculated an approximate $\tilde{\alpha}_Q$ from five neighbour points of ($\ln Q$, $-\ln s_Q$). We repeated this process several times, for several Q-s in a certain sequence of ATS (11) and for a defined parameter k. This way we mapped how the scaling exponent changes with Q (Q was taken as the smallest one of the five points). The dependences (Q, $Q \ln \tilde{\alpha}_Q$) (Fig. 4) are linear. Therefore we concluded that

$$Q \ln \tilde{\alpha}_Q = u + vQ, \tag{19}$$

which means that

$$\tilde{\alpha}_Q = e^{\frac{u}{Q}} + v \quad (20)$$

and because

$$\lim_{Q \to \infty} \tilde{\alpha}_Q = \tilde{\alpha}$$

from (20) we get

$$v = \ln \tilde{\alpha} \quad (21)$$

The parameters u, v were calculated by a linear regression method. The results for a few investigated sequences of ATS are listed in tab. 1.

Table I

The list of scaling exponents estimated for various parameters k and various sequences of Arnold tongues.

k	$\frac{1}{2+m}$	$\frac{2}{3+2m}$	$\frac{3}{4+3m}$	$\frac{3}{5+3m}$
1.0	2.999179	3.000157	2.999366	3.000048
0.97	2.996893	2.991927	2.9870187	2.992142
0.95	2.996388	2.988935	2.983508	2.9942489
0.9	2.993215	2.978083		

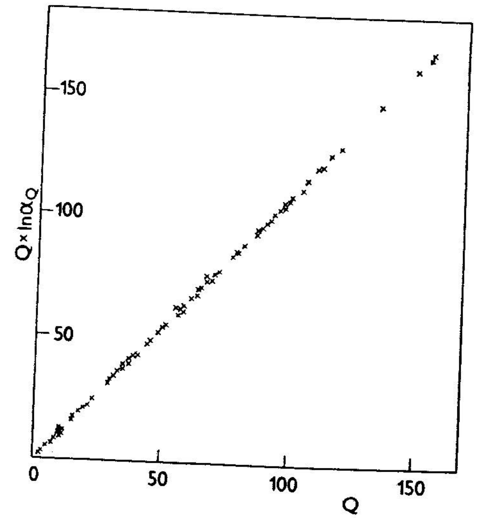

Fig. 5. Dependences of α_Q on Q plotted together for all values of k which were investigated and various sequences of Arnold tongues. The linearity indicates the smallness of the higher order contributions in (19).

A similar problem for a subcritical circle map was investigated by Ecke et al [4]. They plot the logarithm of s against the logarithm of Q for a constant P. Q was not objected the by (11). They found the same scaling relation as we (18)

with $\tilde{\alpha} = 3.0$. They calculated the scaling exponent $\tilde{\alpha}$ by measuring the asymptotic slope of lines ($\ln s_Q$, $\ln Q$).

To summarize our results we found the scaling relation for the widths of Arnold tongues having fixed P and Q given by (11). In order to avoid cumbersome consuming direct calculations of the scaling exponent $\tilde{\alpha}$, we developed a method to estimate it. The results given by our method agree with those of Ecke [4] for a critical circle map ($k = 1.0$) very well. For a subcritical map ($k < 1.0$) the higher order contributions in (19) should probably be taken into account. These contributions will be small, because a plot of (Q, $Q \ln \tilde{\alpha}_Q$) for all values of k and all investigated sequences of ATS makes a line with a slope (Fig. 5)

$$\ln \tilde{\alpha} = 1.092 \pm .0026,$$

which means that $\tilde{\alpha}$ is equal to 2.9802....

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ВЫРАЖЕНИЕ СКЕЙЛИНГА ДЛЯ ШИРИНЫ ЯЗЫКА АРНОЛЬДА.

Выражение скейлинга для ширины языка Арнольда введено вместе с методом оценки эксионентов скейлинга. Обсуждается точность оценки.