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PATH INTEGRAL OF A HARD CORE POTENTIAL

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In order to determine the propagator of a quantum particle moving on an axis under the constraint of a hard core, we first introduce a second particle, identical to the first and moving on the same axis. This fictitious particle is eliminated afterwards, with boundary conditions determining the fermionic nature of the two particles.

I. INTRODUCTION

In the framework of a path integral, we propose a method of calculating the propagator of a particle in a constant potential limited by a hard core. Choose the potential to be equal to zero, for $x > 0$, and infinite, otherwise:

$$V(x) = \begin{cases} 0, & \text{for } x > 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

To solve the problem we introduce a fictitious particle identical to the first, moving along the axis $0y$, and thus extending the space from the axis $0x$ to the plan $0xy$. The physical nature of the two particles is determined by the boundary conditions; in the present case we consider them to be fermions.

The propagator of the real particle is obtained by the elimination of the auxiliary fictitious fermion. This expression of the propagator has already been obtained through the image method [1], a method which has been used for a long time in electromagnetism, and which is independent of the path integrals calculus principle, in the same way as the Pauli exclusion principle is not a result of the Schrödinger equation.

II. PROPAGATOR

The propagator relative to potential (1) taken into consideration can be written formally as follows is standard notation [2]:

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$$K(x_b, x_a; T) = \int \mathscr{D}x(t) \exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\frac{m\dot{x}^2}{2} - V(x(t))\right]\right\}, \tag{2}$$

or, in discretized form,

$$K(x_b, x_a; T) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{1/2} \prod_{j=1}^{N-1} dx_j$$

$$\prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\varepsilon}(x_j - x_{j-1})^2 - \varepsilon V\frac{x_j + x_{j-1}}{2}\right]\right\},$$

where

$x_j = x(t_j)$; $\varepsilon = t_j - t_{j-1} = \dfrac{T}{N}$; $x_a = x(t_a)$; $x(t_b) = x(t_b)$; $T = t_b - t_a$, are the usual notations.

It is obvious from this discrete form of the propagator that if x_a or $x_b \in]-\infty, 0]$, then $K = 0$ because

$\exp\left[-\dfrac{i}{\hbar}\varepsilon V\left(\dfrac{x_a + x_1}{2}\right)\right]$ or $\exp\left[-\dfrac{i}{\hbar}\varepsilon V\left(\dfrac{x_b + x_{N-1}}{2}\right)\right]$ are oscillating with a high frequency.

Thus, the propagator to be calculated is

$$K(x_b, x_a; T) = \int \mathscr{D}x(t) \exp\left[\frac{i}{\hbar}\int_0^T \frac{m\dot{x}^2}{2} dt\right] \tag{3}$$

for x_b and $x_a \in]0, \infty[$.

Let us now insert the identity

$$\int_{-\infty}^{+\infty} dy_b \left[\int \mathscr{D}y(t) \exp\left\{\frac{i}{\hbar}\int_0^T \frac{M}{2}\dot{y}^2 dt\right\}\right] = 1$$

into equation (3). It yields:

$$K(x_b, x_a; T) \int_{-\infty}^{+\infty} dy_b \bar{K}(x_b, y_b, x_a, y_a; T), \tag{4}$$

where

$$\bar{K} = \int \mathscr{D}x(t) \mathscr{D}y(t) \exp\left\{\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\dot{x}^2 + \frac{M}{2}\dot{y}^2\right) dt\right\}, \tag{5}$$

is the propagator of two particles moving along the axes $0x$ and $0y$, their mass being respectively m and M. The dimension of space has thus been increased by the introduction of this second auxiliary particle. The elimination of the fictitious particle with a mass M is obtained through the integration on the

variable y_b, in equation (4). Let us now identify the masses m and M with the reduced mass and with the total mass of two identical particles of equal mass $m_1 = m_2 = 2m$,

$$m = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad M = m_1 + m_2 = 4m. \tag{6}$$

Let us now suppose that these two particles are travelling on the same straight line, taken to be the x axis, and similarly, let us identify the position $x(t)$ with the relative distance of the two particles located at the points $x_1(t)$ and $x_2(t)$, and let us identify $y(t)$ with the position of the centre of mass of these two particles:

$$\left.\begin{aligned} x(t) &= x_1(t) - x_2(t), \\ y(t) &= \frac{x_1(t) + x_2(t)}{2}. \end{aligned}\right\} \tag{7}$$

The Jacobian of this transformation (7) being equal to 1, the kernel (5) describing the motion on the same axis — of the system of two identical particles, becomes:

$$\bar{K}(x_{1b}, x_{2b}, x_{1a}, x_{2a}; T) = \int \mathscr{D}x_1(t) \mathscr{D}x_2(t) \exp\left[\frac{i}{\hbar}\int_0^T m(\dot{x}_1^2 + \dot{x}_2^2) dt\right]. \tag{8}$$

Since

$$-\infty < y < +\infty \text{ and } 0 < x < +\infty,$$

it is easy to show by means of the transformation (7) that the motion of the two particles, described by (8), obeys the condition

$$-\infty < x_2(t) < x_1(t) < +\infty. \tag{9}$$

However, the two particles are strictly undiscernible (that is to say, there is no way they can be tagged). The quantum mechanics thus compells us to take into consideration the exchange of the two identical quantum particles and therefore the condition:

$$-\infty < x_1(t) < x_2(t) < \infty \tag{10}$$

when one calculates the propagator (8).

Then the transition amplitude (8) is the algebraic sum of the amplitudes characterizing the two processes, the direct one and the exchange one [2].

The kernel (8) is thus written:

$$\bar{K} = \bar{K}_{direct}(x_{1b}, x_{2b}, x_{1a}, x_{2a}; T) \pm \bar{K}_{exch.}(x_{2b}, x_{1b}, x_{1a}, x_{2a}; T). \tag{11}$$

The sign of the exchange term $\bar{K}_{exch}$ is plus or minus, depending on whether these two particles are bosons or fermions. One can specify the nature of the two particles as follows: For x_b or $x_a = 0$, $K = 0$, and following statement (4) $\bar{K} = 0$. This condition is only satisfied by fermions, i.e. by the minus sign in eq. (11). The kernel (11) is therefore given by:

$$\bar{K} = \left(\frac{m}{\mathrm{i}\,\pi\hbar T}\right)\left\{\exp\left[\frac{\mathrm{i}}{\hbar}\frac{m}{T}[(x_{2b}-x_{2a})^2+(x_{1b}-x_{1a})^2]\right] - \exp\left[\frac{\mathrm{i}}{\hbar}\frac{m}{T}[(x_{1b}-x_{2a})^2+(x_{2b}-x_{1a})^2]\right]\right\}, \tag{12}$$

and, in the former coordinates,

$$\bar{K} = \frac{m}{\mathrm{i}\,\pi\hbar T}\exp\left(\frac{2im}{\hbar t}(y_b-y_a)^2\right)\left[\exp\left(\frac{im}{2\hbar t}(x_b-x_a)^2\right) - \exp\left(\frac{im}{2\hbar T}(x_b+x_a)^2\right)\right]. \tag{13}$$

Finally, we get by the equation (4),

$$K(x_b, x_a; T) = \left(\frac{m}{2\mathrm{i}\,\pi\hbar T}\right)^{1/2}\left[\exp\left(\frac{im}{2\hbar T}(x_b-x_a)^2\right) - \exp\left(\frac{im}{2\hbar T}(x_b+x_a)^2\right)\right]$$

which is the result already obtained through the image method [1]. This result can also be obtained through direct calculation: one utilizes the following integral, given by Peak and Inomata [3]:

$$\int \mathscr{D} r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\left[\frac{m}{2}\dot{r}^2 - \frac{1}{2}m\omega^2 r^2 - \frac{g}{r^2}\right]\right\} =$$

$$= \frac{m\omega}{\mathrm{i}\hbar\sin(\omega T)}(r_a r_b)^{1/2}\exp\left[\frac{\mathrm{i}m\omega}{2\hbar}(r_b^2+r_a^2)\cot(\omega T)\right] I_\gamma\left(\frac{m\omega r_b r_a}{\mathrm{i}\hbar\sin(\omega T)}\right),$$

where $\gamma = \frac{1}{2}\left[1+\frac{8mg}{\hbar^2}\right]^{1/2}$.

One has to set $\omega = g = 0$ and to use $I_{1/2}(z) = (2\pi z)^{-1/2}\sinh z$.

III. CONCLUSION

In this calculation of the propagator, we have shown that the Dirichlet boundary conditions are satisfied by fermions.

The + sign for the bosons in equation (11) takes into account the Von Neuman boundary conditions.

On the other hand, it is easy to show that the kernel (12) can be interpreted in the following manner:

(i) it governs the movement of two identical particles (fermions) moving on the same axis, so that the first term of the equation (12) satisfies condition (9) for a and b, and so that the second term of this equation (12) satisfies conditions (9) and (10) for a and b, respectively.

(ii) it governs the motion of one single particle of mass $2m$, which is submitted to the following potential:

$$V(x_1, x_2) = \begin{cases} 0, & \text{if } x_2 < x_1, \\ \infty, & \text{if } x_2 \geqslant x_1. \end{cases}$$

The action of the first term of the kernel (12) is evaluated on the classical path: The particle starts from the point (x_{1a}, x_{2a}) of the plane (x_1, x_2) and arrives at the point (x_{1b}, x_{2b}) after a certain time T. The action of the second term is estimated according to the direct path leading from the point (x_{1a}, x_{2a}) either to the point (x_{1b}, x_{2b}) after bouncing on the $x_1 = x_2$ wall, or to the image (x_{2b}, x_{1b}). These two methods are equivalent.

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КОНТИНУАЛЬНЫЙ ИНТЕГРАЛ ПОТЕНЦИАЛА ТВЕРДОГО ЯДРА

С целью определения пропагатора квантовой частици, которая движется по оси под влиянием твердого ядра. Впервые введения идентична с первой вторая частица, движущаяля по той же оси. Такая фиктивная частица, при учете граничных условий опредляющих их фермионную природу, позже исключается.