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THE IMPORTANCE OF INTEGRABLE SYSTEMS

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Integrable systems of classical mechanics and nonlinear field theory are studied. It is shown that for special problems of classical mechanics it is preferable to characterize the system instead of a Hamiltonian by a certain $(f \times f)$ matrix where f is the number of degrees of freedom. If the time evolution is now given by a similarity transformation, then the system is integrable and the conserved quantities appear as the invariants of this matrix. If all particles are interacting with each other, then those systems are integrable, where the interaction potentials obey the addition formula of the Weierstraß $\wp$-function.

In the case of nonlinear evolution and wave equations in $(1 + 1)$ dimensions the conditions of integrability are discussed and it is shown that the nonlinear superposition principle plays an important role. In integrable systems only elastic scattering of solitary waves is found.

I. INTRODUCTION

In the last 10—20 years a number of rather complicated problems in the framework of nonlinear (nl.) differential equations were successfully approached and methods developed to treat them systematically. These are essentially equations of motions and nl. classical field equations. However, a greater number of interesting physical problems, e. g. turbulence, still resisted to be handled in a satisfactory way. It has now become customary to divide the problems into "integrable" and "nonintegrable systems", meaning that in the first case they can be solved systematically in principle, while in the second case there are at the moment no reliable analytical or numerical methods available to treat them.

When new exact solutions or even methods of solution are found the question naturally arises, if we can give some explanation of this fact. Are there special underlying mathematical structures connected with them? To answer this question we have to differentiate between integrable systems of "finite order", being represented by ordinary differential equations and those of "infinite order", meaning nl. partial differential equations. As it will be explained below

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the essential ingredients in the first case are the "conserved quantities", while in the second case this role is taken over by the "nonlinear superposition principle", representing the combination of excitations or their scattering. To be brief this means that for integrable systems we can either give a sufficient number of conserved quantities or formulate the nl. superposition principle [1].

It is worthwhile to look for the general properties of integrable systems and to try with their help to construct further examples or even classes of examples and to search if they have interesting physical applications.

II. GENERAL METHODS OF INTEGRABLE SYSTEMS OF CLASSICAL MECHANICS

II. 1 General considerations

Let us start with "integrable systems" well defined in Hamiltonian classical mechanics [2], [3]. It is a system described by a Hamiltonian H depending on $2f$ canonically conjugate variables (q_i, p_i) where f is the number of degrees of freedom and possessing f conserved quantities $I_i(p, q)$:

$$H(p_1 \dots p_f, q_1 \dots q_f), \quad \text{with } \dot{I}_i(p, q) = 0, \quad i = 1 \dots f. \tag{1}$$

In the formalism of poisson brackets this means

$$\{H, I_i\} = 0, \quad i = 1 \dots f,$$

but in addition the conserved quantities have to be in involution

$$\{I_i, I_j\} = 0, \quad i, j = 1 \dots f, \tag{2}$$

in order to be independent. Then one can find a canonical transformation from the (p, q) to (I, ϑ) with a Hamiltonian depending now only on the I's providing

$$H(p, q) \to H(I), \quad \dot{I}_k = 0$$

$$\dot{\vartheta}_k = \frac{\partial H}{\partial I_k} = \omega_k(I), \; \vartheta_k(t) = \omega_k t + \vartheta_{k0}, \quad k = 1 \dots f, \tag{3}$$

where (I, ϑ) are called action-angle variables describing the time evolution of the system as a motion on an f-dimensional torus in the $2f$-dimensional phase space. The essential meaning of the integrability is that one can find in connection with the f conserved quantities a set of variables admitting to formulate the general solution.

Now we will look for general conditions of systems having f conserved quantities. For this purpose it is preferable to characterize the system instead of by a Lagrangian or Hamiltonian by an $(f \times f)$ — matrix L, where the f^2 matrix

elements L_{ij} depend on the canonically conjugate variables $L = (L_{ij}(p, q))$. If we find now such a form of L that the time evolution is given by a similarity transformation

$$L(t) = B(t)L(0)B^{-1}(t), \tag{4}$$

then all the f different traces

$$I_n = \frac{1}{n}\operatorname{Tr} L^n(t) = \frac{1}{n}\operatorname{Tr} L^n(0), \quad n = 1 \dots f, \tag{5}$$

are constant with respect to time and independent, because they are homogeneous functions

$$\operatorname{Tr}\sum_i L_{ii}, \; \operatorname{Tr} L^2 = \sum_{i,j} L_{ij}L_{ji}, \; \operatorname{Tr} L_3 = \sum_{i,j,k} L_{ij}L_{jk}L_{ki}, \tag{6}$$

of the degrees $1 \dots f$ of the matrix elements. These expressions can serve as f independent conserved quantities. The question is, however, to determine the physical systems obeying the condition (4). The corresponding equation of motion is given by the matrix equation

$$\dot{L} = LA - AL = [L, A], \tag{7a}$$

where the matrix A is given by

$$\dot{B}B^{-1} = -A \quad \text{or } \dot{B} + AB = 0. \tag{7b}$$

If we demand L to be hermitian $L^+ = L$ and B to be unitary $B^+B = I$, then $A^+ = -A$.

As the first step we try to connect the trace of the matrix L^2 directly with the Hamiltonian in the simple form

$$H = \frac{1}{2}\operatorname{Tr} L^2 = \frac{1}{2}\sum_{i,j} L_{ij}L_{ji} = T + V, \tag{8}$$

selecting a special class of problems. It is advantageous to choose as diagonal elements the momenta p_i, resp. p_i/m, providing the kinetic energy

$$T = \frac{1}{2}\sum_i L_{ii}^2, \quad L_{ii} = p_i, \quad i = 1 \dots f,$$

while the nondiagonal elements should give the interaction energy

$$V = \sum_{i>j} L_{ij}^2. \tag{9}$$

II. 2. Interaction between nearest neighbours

Let us start with a matrix L of nearly tridiagonal form with rows of the type

$$(0 \dots L_{ii-1} L_{ii} L_{ii+1} \dots 0) \tag{10a}$$

with

$$L_{ii-1} = iv_{i-1i}, \; L_{ii+1} = iv_{ii+1} \quad i = 2, \dots f-1, \tag{10b}$$

incorporating $L^+ = L$, where v_{ij} are some real functions of the coordinates.

The question is now the form of the first and the last row. Considering a system with f particles on a circle, i.e. with periodic boundary conditions $q_{f+1} = q_1$, we choose for the nondiagonal elements of these rows

$$L_{12} = iv_{12}, \; L_{1f} = -iv_{f1} \text{ and } L_{f1} = iv_{f1}, \; L_{ff-1} = -iv_{f-1f}, \tag{10c}$$

and zero otherwise. Due to (8) we obtain for the Hamiltonian

$$H = \frac{1}{2} \operatorname{Tr} L^2 = \sum_1^f \left(\frac{1}{2} p_i^2 + v_{ii+1}^2 \right), \tag{11}$$

with the convention $v_{ff+1} = v_{f1}$, a system of f particles interacting with both its next neighbours on a circle by the same potential

$$V_{ii+1} = v_{ii+1}^2, \tag{12}$$

where translational invariance requires

$$v_{ii+1} = v(q_i - q_{i+1}). \tag{13}$$

The $2f$ equations of motion for the Hamiltonian system (11)

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}, \tag{14}$$

are of first order and should now be compatible with the f^2 equations of first order due to (7) containing still the undetermined elements of the matrix A. Giving A the same form as L a straightforward calculation provides us with a differential condition for the potential. Its solution gives the Toda potential [4], [5]

$$V(q) = a^2/b^2 \, e^{-bq} + cq + d, \tag{15}$$

representing the interaction between neighbouring particles. There are two limiting cases:

i. Choosing $a^2 = k$, $c = a^2/b$, $d = -a^2/b^2$, $b \to 0$:

$$V(q) = \frac{1}{2} k \left(q^2 - \frac{1}{3} b q^3 + \dots \right) \tag{16a}$$

gives a weakly anharmonic coupling.

ii. Taking $a = b \exp bd/2$, $c = 0$, $d = 0$: $V(q) = \exp b(d - q)$ (16b)

gives for $b \to \infty$ the two limiting values ∞ for $q < d$ and 0 for $q > d$, representing a hard sphere potential of diameter d.

It is very instructive to consider in detail the Toda system consisting of 3 particles in one dimension of the type [2]

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{(q_1 - q_2)} + e^{(q_2 - q_3)} + e^{(q_3 - q_1)}, \tag{17a}$$

which can also be represented by a $(3 + 3)$ matrix of the form

$$L = \begin{pmatrix} p_1 & ie^{\frac{1}{2}(q_1 - q_2)} & -ie^{\frac{1}{2}(q_3 - q_1)} \\ -ie^{\frac{1}{2}(q_1 - q_2)} & p_2 & ie^{\frac{1}{2}(q_2 - q_3)} \\ ie^{\frac{1}{2}(q_3 - q_1)} & -ie^{\frac{1}{2}(q_2 - q_3)} & p_3 \end{pmatrix} \tag{17b}$$

possessing the 3 conserved quantities

$$\begin{aligned} I_1 &= \operatorname{Tr} L = p_1 + p_2 + p_3, \\ I_2 &= \frac{1}{2} \operatorname{Tr} L^2 = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + e^{(q_1 - q_2)} + e^{(q_2 - q_3)} + e^{(q_3 - q_1)}, \\ I_3 &= \frac{1}{3} \operatorname{Tr} L^3 = \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + (p_1 + p_2) e^{(q_1 - q_2)} \\ &\quad + (p_2 + p_3) e^{(q_2 + q_1)} + (p_3 + p_1) e^{(q_3 - q_1)}, \end{aligned} \tag{18}$$

which are clearly independent. They are polynomials of first, second and third degree in the momenta p, where I_1 represents the total linear momentum due to translational invariance, I_2 is just the total energy H, while I_3 has no "evident" interpretation. They are not uniquely determined because instead of I_3 we can use a more elegant form, containing only differences in momenta and coordinates [2]

$$I_3' = \frac{1}{9}(p_1 + p_2 - 2p_3)(p_2 + p_3 - 2p_1)(p_3 + p_1 - 2p_2)$$

$$- (p_1 + p_2 - 2p_3)\,e^{(q_1 - q_2)} - (p_2 + p_3 - 2p_1)\,e^{(q_2 - q_3)}$$
$$- (p_3 + p_1 - 2p_2)\,e^{(q_3 - q_1)}, \tag{18'}$$

being related with the I_n in the following way

$$I_3' = -3I_3 + 2I_2 I_1 - \frac{2}{9} I_1^3. \tag{19}$$

All the conserved quantities are in involution

$$\{I_n, I_m\} = 0, \quad n, m = 1, 2. 3.$$

This demonstrates clearly the advantage of the matrix representation of the system in comparison with the Hamiltonian formulation, because with the construction of the matrix L it is a straightforward procedure to get the conserved quantities. The essential point is that the nondiagonal matrix elements of L are proportional to the square root of the potential, which are such an essential ingredient of the conserved quantity. This fact is not obvious in the Hamiltonian formalism.

If the number f is arbitrary, then the independent conserved quantities are polynomials of degrees 1 to f, the coefficients containing powers of the square root of the potential. They are listed here in a convenient form [4]

$$I_1 = \sum_k p_k, \quad I_2 = \sum_k \left(\frac{1}{2} p_k^2 + e^{(q_k - q_{k+1})}\right) = H$$

$$I_3 = \sum_k \left(\frac{1}{3} p_k^3 + (p_k + p_{k+1})\,e^{(q_k - q_{k+1})}\right)$$

$$I_4 = \sum_k \left(\frac{1}{4} p_k^4 + (p_k^2 + p_k p_{k+1} + p_{k+1}^2)\,e^{(q_k - q_{k+1})} + \frac{1}{2} e^{2(q_k - q_{k+1})} + \right.$$
$$\left. + e^{(q_k - q_{k+1})} e^{(q_{k+1} - q_{k+2})}\right),$$

$$\vdots$$

$$I_n = \sum_k \left(\frac{1}{n} p_k^n + \dots\right) \qquad 1 \leqslant n \leqslant f. \tag{20}$$

There is at the moment no interpretation of those polynomials of the momenta having a degree higher than 2. So far there are only variational principles used, where the essential functions are polynomials of second degree in the momenta. The question is now, as regards integrable systems, what will be the general structure of the conserved quantities which are in involution, how can they be interpreted and used to derive new variational principles to characterize these systems.

II. 3. Interaction among all particles

Now we shall consider systems, where all particles are interacting with each other. Then it is preferable to take a matrix with the elements

$$L_{ii} = p_i, \quad L_{ij} = i v_{ij} \quad (i \neq j) i, j = 1 \dots f, \tag{21a}$$

where the v_{ij} are real odd functions of the type

$$v_{ij} = v(q_i - q_j) \text{ with } v(q_i - q_j) + v(q_j - q_i) = 0. \tag{21b}$$

The relation to the Hamiltonian is again given by the simple relation

$$H = \frac{1}{2} \operatorname{Tr} L^2 = \sum_i \frac{1}{2} p_i^2 + \sum_{i>j} v_{ij}^2 = T + V, \tag{22a}$$

the first term being the kinetic energy while the second is the interaction energy

$$V(q) = \sum_{i>j=1}^{f} v^2(q_i - q_j). \tag{22b}$$

For the equation of motion (7) we need the matrix A, containing now f^2 elements. They will be chosen in such a way that the f^2 equations of motion (7) shall be compatible with the $2f$ Hamiltonian equations (14). After a lengthy calculation [5], [6] we end up with a functional equation which the admitted interaction potentials have to obey. It can be written in the form

$$\begin{vmatrix} 1 & V(q) & V'(q) \\ 1 & V(Q) & V'(Q) \\ 1 & V(q+Q) & -V'(q+Q) \end{vmatrix} = 0, \tag{23}$$

where q and Q are independent sets of coordinates.
This relation is, however, identical with the addition theorem for the Weierstraß $\wp$-function meaning

$$V(q) \approx \wp(q; w, w'), \tag{24}$$

where $\wp$ is a doubly periodic function and w and w' are connected with the periods [7]. So we come to the important statement that all interaction potentials which obey the addition theorem of the Weierstraß $\wp$ function lead us to integrable many-body systems.

II. 4. Survey of integrable many-body systems

It is worthwhile to survey integrable systems of an arbitrary number of degrees of freedom which can be derived from the fundamental condition that the time evolution is given by a similarity transformation (4).

i. In the case of an interaction between next neighbours only, the potential has the form

$$V(q_1 \dots q_f) = \sum_{i=1}^{f} V(q_i - q_{i+1}) \quad \text{with } V(q) = e^{cq}\text{: Toda system,} \qquad (25)$$

where c is a real parameter.

ii. In the class of rational functions for $V(q)$ there are

$$V(q_1 \dots q_f) = \sum_{i>j=1} \frac{a^2}{(q_i - q_j)^2}, \text{ i.e. } V(q) = \frac{a^2}{q^2}\text{: Calogero-Moser system} \qquad (26)$$

and $V(q) = a^2q^2$, the harmonic coupling, where a is a real parameter.

iii. Looking at the class of simply periodic functions possible $V(q)$ are:

$$a^2/\sin^2 cq,\ a^2\cot^2 cq,\ a^2/\sinh^2 cq,\ a^2\coth^2 cq. \qquad (27)$$

iv. Admitting doubly periodic functions with module k, $0 = k \leqslant 1$, we have, e.g., for $V(q)$:

$$a^2/\text{sn}^2(cq, k),\ a^2\text{cn}^2(cq, k)/\text{sn}^2(cq, k),\ a^2\text{dn}^2(cq, k)/\text{sn}^2(cq, k), \qquad (28)$$
$$a^2\text{cn}^2(cq, k)/\text{sn}^2(cq, k)\text{dn}^2(cq, k),\ a^2\text{cn}^2(cq, k)\text{dn}^2(cq, k)/\text{sn}^2(cq, k),$$
$$a^2\text{dn}^2(cq, k)/\text{sn}^2(cq, k)\text{cn}^2(cq, k),$$

where sn(x, k), cn(x, k), dn(x, k) are the Jacobian elliptic functions [8], which in the case of a vanishing modulus, approach $\sin x$, $\cos x$ and 1. These elliptic functions can be used to approximate a wide class of potential shapes by varying the modulus k from 0 to 1. It is worthwhile to study in more detail these integrable systems and their applications in statistical and quantum physics, because they admit now exact statements for many-body systems with genuine interaction, which is of great importance, since no perturbation theoretical methods need to be involved [4].

III. INTEGRABLE FIELD EQUATIONS

Nonlinear evolution and wave equations play nowadays a very important role in many different fields of physics. Because of nonlinearity the standard methods of the linear field theory for solving field equations exactly or approximately cannot usually be applied. Surprisingly, however, in the last 10—20 years it was possible to develop methods to treat certain classes of nl. evolution and wave equations in (1 + 1)-dimensions systematically. Very often the equations are called "soliton equations", because they admit solitary waves [9] as solutions. Again it has become customary to divide nonlinear field equations into socalled "integrable" and "nonintegrable" partial differential equations, i.e. to consider them as an extension of integrable systems with an infinite number of degrees of freedom. The question naturally arises, how such an extension will be formulated and what the analogies are between the integrable systems with a finite and an infinite order.

So far no rigorous definition could be given to characterize an "integrable partial differential equation", but a number of conditions evolved describing such systems. They can be formulated for partial differential equations in (1 + 1)-dimensions in the following way:

i. Existence of solitary solutions.
ii. Existence of an infinite number of conservation laws.
iii. Existence of a set of nonlinear superposition functions.

These conditions require already a certain knowledge of the solutions of the equations, so there are no criteria which can be applied "beforehand" to decide about integrability. The existence of solitary solutions means that localized stable excitations can propagate through the system without any deformations, which are asymptotically constant. If the two asymptotical values are identical, they are called "bell shaped", otherwise "kink shaped". These solutions have more or less a particle-like behaviour [9]. Because these systems are considered as an extension of a finite-dimensional case, they have an infinite number of conservation equations. But so far nothing has been found that they are "complete" or "independent", i.e. no extension of the notion of "in involution" or "complete" is proposed. Most important is the condition iii., requiring to formulate the combination of two or more solitary excitations leading to the "nonlinear superposition principle". The nl. superposition means a dramatic change with respect to the linear superposition.
While the linear superposition can be formulated quite generally, meaning that a sum of two solutions or excitations is again a solution independent of the type of the equation and form of the solution, this is not at all the case for nl. superosition. There may occur different superpositions depending on the type of the equation and of the form of the excitation. Looking at solitary waves there may occur an elastic scattering leading to "solitons" or inelastic processes, where additional solitary excitations and decaying wave tracks may appear. Now the concept has arisen that in integrable evolution and wave equations only an elastic scattering of solitary waves occurs leading to N-soliton solutions. The existence of N-soliton solutions is connected with the infinite number of conservation laws [10]. A certain class of integrable evolution and wave equations in (1 + 1)-dimensions exists but no general idea analogous to (4) has yet been found. Extensively studied are the Korteweg-de Vries equation [9]

$$\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial x} + vaU\frac{\partial U}{\partial x} + b\frac{\partial^3 U}{\partial x^3} = , \qquad (29)$$

and its modifications and hierarchies,
the sine-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \sin kU = 0, \qquad (30)$$

and its modifications and hierarchies and
the nl. parabolic equation (misleadingly called "nl. Schrödinger equation")

$$\mathrm{i}\frac{\partial U}{\partial t} - b\frac{\partial^2 U}{\partial x^2} + c|U|^2 U = 0 \qquad (31)$$

and its hierarchies.
All admit solitary waves of the form

$$U = U(x - vt), \qquad (32)$$

where v is a parameter representing the velocity of the excitation [9]. Many other integrable field equations are found and discussed [11]. An infinite number of densities $D_n(U)$ and flows $F_n(U)$ can be constructed in the way obeying

$$\frac{\partial D_n}{\partial t} + \frac{\partial F_n}{\partial x} = 0, \quad n = 1, 2 \dots . \qquad (33)$$

providing us with the required infinite number of conservation laws. For all these equations the Cauchy problem can be solved with the "method of spectral transformation" (MST), which is an extension of the method of the Fourier transformation to nl. partial differential equations [12]. Sometimes the statements are reversed by saying that if the Cauchy problem for a nl. field equation can be solved with the help of the method of spectral transformation, then the equation is integrable. But how does one know beforehand that this method can be applied to the equation?

The essential properties of nl. field equations seem to be connected with the nl. superposition principle. As a first orientation it is of great value to search e.g., with the help of computer experiments, if an elastic scattering of solitary waves will show up [11]. If it does not occur, it is an indication that the equation is not integrable. A more conclusive argument is to look for an analytic formulation of higher solitary excitations which are more complicated than the standard solitary waves of the form (32). If it can be found, then one can write the so-called "N-soliton" solutions, incorporating a very special but interesting rule of superposition. Here a correlation matrix (K_{ij}) plays a fundamental role being

constructed from one-soliton solutions. Let us start with the sine-Gordon equation, where the N-soliton solution can be written in the form

$$\cos U(x, t; 1 \dots N) = 1 - 2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\ln\det(K_{ij}), \qquad (34)$$

where the elements of the correlation matrix are given by

$$K_{ij} = \frac{2}{a_i + a_j}\cosh\frac{1}{2}(z_i + z_j), \qquad (35)$$

with

$$a_i = \frac{1 - v_i/c}{1 + v_i/c'}\ \gamma_i = \left[1 - \left(\frac{v_i}{c}\right)^2\right]^{\frac{-1}{2}}, \ z_i = \gamma_i(x - v;\ t - x_{i0}),$$

where v_i is the velocity of the ith soliton $(v_i \neq v_j)$ and x_{i0} is some constant. In the case of the KdV equation (29) the N-soliton solution looks similar

$$U(x, t; 1 \dots N) = -2\frac{\partial^2}{\partial x^2}\ln\det(K_{ij}) \qquad (36)$$

with

$$K_{ij} = \frac{(2k_ik_j)^{1/2}}{k_i + k_j}\exp\left[\frac{1}{2}(k_i + k_j)x - 2(k_i^3 + k_j^3)t + \beta_{ij}\right], \qquad (37)$$

where k_i is the wave number of the ith soliton and β_{ij} are some constants. So the elements K_{ij} are constructed out of the ith and jth solitary wave representing in some way the correlation between these two. Let us assume for the moment that the correlation between the different excitations should be negligible, meaning that the correlation matrix becomes diagonal

$$K_{ij} \to K_i^0\delta_{ij},$$

then the determinant of the K_{ij} is the product of the K_i^0 and we obtain finally

$$\ln\det(K_{ij}) \to \ln \Pi K_i^0 \to \sum_{i=1} \ln K_i^0, \qquad (38)$$

which resembles the linear superposition law, saying that a certain combination of N weakly interacting solitary waves gives a higher wave excitation. Looking now at the so far known N-soliton solutions, they are all governed by the expression

$$\ln\det(K_{ij}), \qquad (39)$$

which might therefore be interpreted as a rather general nl. superposition rule for solitary waves in integrable nl. evolution and wave equations in (1 + 1)-dimensions. An extension to higher dimensions is very problematic, because up to now it is not clear what the solitary excitations in two or more space dimesions are hence all investigations about integrability in this case become very speculative.

IV. CONCLUSIONS

In the first moment it seems to be a very formal aspect to divide nl. differential equations, representing physical problems, into integrable and nonintegrable systems. However, the corresponding investigation leads us to a deeper understanding of the conditions of integrability and provided us with a prescription for constructing a great variety of such systems from fundamental principles. In the case of Hamiltonian systems with a finite degree of freedom it was demonstrated how the condition of the time evolution (4) of the corresponding matrix guided us directly to the conserved quantities, which are the backbone of such systems providing us with the action-angle-variables permitting to write down the formal solutions. In the case of nl. field equations the infinte number of conservation equations is a plausible extension, but the essential features seem to be incorporated in the nl. superposition principle, representing a dramatic change with respect to the linear case. Now two different excitations can no longer propagate independently through the system but certain correlations appear, being represented by the correlation matrix (K_{ij}) or equivalently by nl. superposition functions. It is now necessary that they exhibit properties which can be interpreted as the elastic scattering of solitary excitations, but a list of properties determining fully integrable systems is still lacking. Since integrable systems permit exact statements they are a very important tool for statistical and quantum physics. They allow us to study nonlinear models having a manageable thermodynamical and quantum mechanical treatment, e.g., soliton gas or a quantum Toda system [4].

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ВАЖНОСТЬ ИНТЕГРИРОВАТЕЛЬНЫХ СИСТЕМ

В работе изучаются интегрировательные системы классической механики и нелинейной теории поля. Показано, что специфические проблемы классической механики преимущественно характеризует замена гамильтониана матрицей $(f \times f)$, где f — число степеней свободы. Если временное развитие дается с подобной трансформацией, система будет интегрировательной. Сохранение количества появится тогда в форме инварианта матрицы. В случае взаимодействия всех частиц между собой показано, что такие системы будут тоже интегрировательные, где отенциаль взаимодействия обоснован на дополнительной формуле $\wp$ функции Вайерштрасса. В случае нелинейной эволюции и волновых уравнений с размерами (1 × 1) условие интегрировательности изучается. Показано, что правило нелинейной суперпозиции играет важную роль. В интегральных системах обсенованы только упругое рассеяние и волны солитонов.