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HYDRODYNAMIC FLOW IN A POROUS CHANNEL WITH VOLUME SOURCES OR SINKS OF MASS

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A similarity solution is presented for the steady flow of an incompressible viscous fluid through a porous medium in a channel with volume sources or sinks of mass. It is shown that the problem can be solved by the method of matched asymptotic expansions for small values of the permeability K of the medium corresponding with any value of the source of sink parameter N. The pressure drop required to maintain a given flow rate decreases with K. For fixed K, the pressure drop decreases with the increase in N for $N \gg 0$ (volume sinks of mass) and increases with the increase in $|N|$ for $N < 0$ (volume sources of mass).

I. INTRODUCTION

Berman [1, 2) studied the problem of the steady two-dimensionnal flow of an incompressible viscous fluid through a porous channel when the fluid is withdrawn from the channel walls. Such studies gained importance in the problem of lubrication, transpiration cooling, boundary layer control and gaseous diffusion. The similarity solution for the flow as obtained by Berman for the small suction Reynolds number was extended for the large suction Reynolds number by Sellers [3] and for the large blowing Reynolds number by Yuan [4]. This problem was discussed from a different aspects also by Yuan and Finkelstein [5], Donoughe [6] and Morduchow [7].

There is also another class of problems in the flow through a channel which admits similarity solutions. Aladiev and Zaichik [8] have shown that the similarity solutions of the Navier-Stokes equations exist for a non-porous channel when there is a uniform volume distribution of sources or sinks of mass in the flow. Na, Gupta and Nanda [9] extended this problem to include the effect of a transverse magnetic field when the fluid is electrically conducting.

The purpose of the present paper is to consider the steady flow of an incompressible viscous fluid through a porous medium in a channel with volume

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sources or sinks of mass and to show that a similarity solution can be found for the velocity field. This poses a single perturbation problem and a solution is obatined by the method of matched asymtotic expansions. It is shown that when the strength of the volume sources or sinks of mass is small, the determination of the velocity field involves a regular perturbation problem. The motivation of the present analysis is that it has bearing in flows in a channel with evaporation or condensation in the movement of underground water resources, for the filtration of natural gases and oil through oil resevoirs and so on. The method of analysis in the present investigation is similar to that of Na, Gupta and Nanda [9].

II. SOLUTION FOR SMALL PERMEABILITY

Let us consider the steady flow of an incompressible fluid through a porous medium of constant permeability K in a horizontal channel. We choose the x-axis along the central axis of the channel and the y-axis normal to the plates of the channel. The governing equations are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\nu}{K}u, \tag{1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\varrho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\nu}{K}v, \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{S}{\varrho}, \tag{3}$$

where u, v are the velocity components along the x, y axes, p is the pressure, ϱ the density, ν the kinematic viscosity and S is the capacity of the volume sources or sinks of mass. It may be noted that the presence of S in equation (3) is due to evaporation or condensation taking place in the channel. Clearly $S > 0$ corresponds to the sinks and $S < 0$ corresponds to the source. It is assumed that S is constant in the present investigation.

It can be shown by group-theoretic methods [10] that the above equations admit similarity solutions of the form

$$u=U\left(1-\frac{NX}{2}\right)f'(\xi),\quad v=\frac{LS}{2\varrho}[f(\xi)-\xi], \tag{4}$$

where L is the distance between the plates of the channel, U is the average velocity at the inlet $x = 0$ and

$$X=\frac{2\nu x}{UL^2},\quad \xi=\frac{2y}{L},\quad N=\frac{SL^2}{\nu\varrho}. \tag{5}$$

Here N is a non-dimensional quantity expressing the intensity of sources or sinks. It is evident that the velocity components u and v are consistent with (3).

Using (4) in (2) we find that $\frac{\partial p}{\partial y}$ is independent of x so that

$$\frac{\partial^2 p}{\partial x\,\partial y}=0. \tag{6}$$

From (4) and (1) we find that

$$f'''-n^2f'+\frac{N}{4}[f'^2-(f-\xi)f'']=P, \tag{7}$$

where a prime denotes differentiation with respect to ξ, P is the dimensionless pressure gradient and n is the dimensionless parameter defined by

$$P=\frac{L^2}{4\nu\varrho U\left(1-\frac{NX}{2}\right)}\frac{\partial p}{\partial x},\quad n=\frac{L}{2\sqrt{\varrho K}}, \tag{8}$$

P, n being constants.

Noting that the flow through the porous medium of the channel is symmetrical about the central axis $y = 0$, the boundary conditions are

$$\frac{\partial u}{\partial y}=0,\quad v=0\quad \text{at}\quad y=0 \tag{9}$$

and the no slip conditions are

$$u=0,\ v=0\quad \text{at}\quad y=\frac{L}{2}. \tag{10}$$

Then the equations (4), (9) and (10) give the following boundary conditions for $f(\xi)$:

$$f(0)=0,\quad f''(0)=0,\quad f(1)=1,\quad f'(1)=0. \tag{11}$$

We shall solve the equation (7) satisfying the boundary conditions (11) when $K \ll 1$, i.e. when $n \gg 1$. One would expect that for large values of n, i.e. for small values of K, viscous forces and the forces arising from the presence of the porous matrix of the material would be of comparable magnitude in a thin layer near the wall $\xi = 1$. There would, of course, be a similar layer on the lower wall $\xi = -1$. This would clearly involve a singular perturbation problem and therefore we emloy the method of matched asymptotic expansions to solve the problem. This Method has been discussed successfully by Van Dyke [11]. It may be noted in this connection that although the differential equation (7) is of order three, P is unknown and this may be found by solving the equation (7) subject to the boundary conditions (11).

Let us write the equation (7) in the following form:

$$\varepsilon^2 f''' - f' + \frac{N\varepsilon^2}{4}[f'^2 - (f-\xi)f''] = P\varepsilon^2 \tag{12}$$

where $\varepsilon = n^{-1} \ll 1$. Outside the layer $\xi = 1$, we take the outer expansion for f as follows:

$$f^{(ou)} = f_0^{(ou)} + \varepsilon f_1^{(ou)} + \varepsilon^2 f_2^{(ou)} + \ldots \tag{13}$$

and the pressure gradient P in the form

$$P = \frac{c_0}{\varepsilon^2} + \frac{c_1}{\varepsilon} + \frac{c_2}{1} + c_3\varepsilon + \ldots, \tag{14}$$

where the constants $c_0, c_1, c_2, \ldots$ are to be determined. The outer boundary conditions are obtained from (11) as

$$f_i^{(ou)}(0) = 0, \quad f_i^{(ou)''}(0) = 0, \qquad (i = 0, 1, 2, \ldots). \tag{15}$$

Note that in view of the symmetry, we are considering the flow through the porous medium only in the upper half of the channel. Substituting (13) and (14) in (12) and equating different powers of ε we get

$$f_0^{(ou)'} = -c_0, \quad f_1^{(ou)'} = -c_1, \quad -f_0^{(ou)'''} + f_2^{(ou)'} - \frac{N}{4}[f_0^{(ou)'2} - (f_0 - \xi)f_0^{(ou)''}] = -c_2. \tag{16}$$

The solutions of (16) satisfying the boundary conditions (15) are

$$f_0^{(ou)} = -c_0\xi, \quad f_1^{(ou)} = -c_1\xi, \quad f_2^{(ou)} = \left(\frac{Nc_0^2}{4} - c_2\right)\xi. \tag{17}$$

To derive the inner expansion valid for the layer $\xi = 1$, we rescale the variables in the following forms:

$$\Phi = \frac{1-f}{\varepsilon}, \qquad \eta = \frac{1-\xi}{\varepsilon}. \tag{18}$$

Substituting (18) into (12) we get

$$\frac{d^3\Phi}{d\eta^3} - \frac{d\Phi}{d\eta} + \frac{N\varepsilon^2}{4}\left[\left(\frac{d\Phi}{d\eta}\right)^2 + (\eta - \Phi)\frac{d^2\Phi}{d\eta^2}\right] = c_0 + c_1\varepsilon + c_2\varepsilon^2 + \ldots. \tag{19}$$

The corresponding inner boundary conditions are derived from (11) as

$$\Phi(0) = 0, \quad \Phi'(0) = 0. \tag{20}$$

Note that the inner expansion is required to satisfy only the no slip conditions at the channel wall. We expand Φ as

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots \tag{21}$$

Using (21) in (19), we get on equating different powers of ε

$$\frac{d^3\Phi_i}{d\eta^3} - \frac{d\Phi_i}{d\eta} = c_i \qquad (i = 0, 1) \tag{22}$$

and

$$\frac{d^3\Phi_2}{d\eta^2} - \frac{d\Phi_2}{d\eta} + \frac{N}{4}\left[\left(\frac{d\Phi_0}{d\eta}\right)^2 + (\eta - \Phi_0)\frac{d^2\Phi_0}{d\eta^2}\right] = c_2. \tag{23}$$

The corresponding boundary conditions for Φ_i are given from (20) as

$$\Phi_i(0) = 0, \quad \Phi_i'(0) = 0, \qquad (i = 0, 1, 2). \tag{24}$$

The solutions of (22) satisfying (24) are

$$\Phi_i = c_i(1 - \eta - e^{-\eta}), \qquad (i = 0, 1) \tag{25}$$

after removing the exponentially growing terms. Using (25), equations (18) and (21) give the following three-term inner expansion for f as follows:

$$f^{(in)} = 1 - \varepsilon c_0(1 - \eta - e^{-\eta}) - \varepsilon^2 c_1(1 - \eta - e^{-\eta}) + \ldots \tag{26}$$

On the other hand, the three-term outer expansion for f is obtained from (13) and (17) as

$$f^{(ou)} = -c_0\xi - \varepsilon c_1\xi + \varepsilon^2\left(\frac{N}{4} - c_2\right)\xi + \ldots \tag{27}$$

To determine the constants c_0, c_1 and c_2 we use the method of the asymptotic matching principle of Van Dyke [11] given by:

The p-term inner expansion of (the q-term outer expansion) = the q-term outer expansion of (the p-term inner expansion), (28) where p and q are any two integers such that p is either q or $q + 1$.

Then

$$\begin{aligned} f^{(in)}_{1\text{-}term} &= 1 \\ &= 1 \text{ (rewritten in outer variable)} \\ &= 1 \text{ (expanded in powers of } \varepsilon\text{)}, \end{aligned} \tag{29}$$

$$\begin{aligned} f^{(ou)}_{1\text{-}term} &= -c_0\xi \\ &= -c_0(1-\varepsilon\eta) \quad \text{(rewritten in inner variable)}, \\ &= -c_0 \quad \text{(expanding in powers of } \varepsilon \text{ and retaining one term)}. \end{aligned} \tag{30}$$

Matching (29) and (30) we find

$$c_0 = -1. \tag{31}$$

Similarly, we have

$$\begin{aligned} f^{(in)}_{2\text{-}term} &= 1 + \varepsilon(1-\eta-e^{-\eta}) \\ &\approx 1 + \varepsilon\left(1 - \frac{1-\xi}{\varepsilon}\right) \quad \text{(rewriting in outer variable and removing the transcendentally small term)} \\ &= \xi + \varepsilon \quad \text{(expanding in powers of } \varepsilon \text{ and retaining two terms)}, \end{aligned} \tag{32}$$

$$\begin{aligned} f^{(ou)}_{2\text{-}term} &= \xi - \varepsilon c_1\xi \\ &= 1 - \varepsilon\eta - \varepsilon c_1(1-\varepsilon\eta) \quad \text{(rewriting in inner variable)} \\ &\approx 1 - \varepsilon\eta - \varepsilon c_1 \quad \text{(expanding in powers of } \varepsilon \text{ and retaining two terms)} \end{aligned} \tag{33}$$

Matching (32) and (33) we get

$$c_1 = -1. \tag{34}$$

And we have

$$\begin{aligned} f^{(in)}_{3\text{-}term} &= 1 + \varepsilon(1-\eta-e^{-\eta}) + \varepsilon^2(1-\eta-e^{-\eta}) \\ &\approx 1 + \varepsilon\left(1-\frac{1-\xi}{\varepsilon}\right) + \varepsilon^2\left(1-\frac{1-\xi}{\varepsilon}\right) \quad \text{(rewriting in outer variable and removing transcendentally small terms)}, \\ &= \xi + \varepsilon\xi + \varepsilon^2 \quad \text{(expanding in powers of } \varepsilon\text{)} \end{aligned} \tag{35}$$

$$\begin{aligned} f^{(ou)}_{3\text{-}term} &= \xi + \varepsilon\xi + \varepsilon^2\left(\frac{N}{4} - c_2\right)\xi, \\ &\approx \xi + \varepsilon\xi + \varepsilon^2\left(\frac{N}{4} - c_2\right). \end{aligned} \tag{36}$$

Matching (35) and (36) we find

$$c_2 = \frac{N}{4} - 1. \tag{37}$$

Now from (12), (13) and (14) we get the equation for f_3 as

$$f_1^{(ou)'''} - f_3^{(ou)'} + \frac{N}{4}[2f_0^{(ou)'}f_1^{(ou)'} - \{f_0^{(ou)} - \xi\}f_1^{(ou)''} - f_1^{(ou)}f_0^{(ou)''}] = c_3. \tag{38}$$

The substituting from (17), (31), (34) and (37) into (38) and subsequent integration leads to

$$f_3^{(ou)} = \left(\frac{N}{2} - c_3\right)\xi, \tag{39}$$

which satisfies (15). Again, solving (23) with Φ_0 given by (25) and (31), we have

$$\Phi_2(\eta) = \eta + e^{-\eta} + \frac{N}{8}(\eta+1)e^{-\eta} - \left(\frac{N}{8}+1\right), \tag{40}$$

which satisfies the boundary conditions (24). Thus the 4-term outer and inner expansions for f are

$$f^{(ou)} = \xi + \varepsilon\xi + \varepsilon^2\xi + \varepsilon^2\left(\frac{N}{2} - c_3\right)\xi \tag{41}$$

and

$$\begin{aligned} f^{(in)} &= 1 + \varepsilon(1-\eta-e^{-\eta}) + \varepsilon^2(1-\eta-e^{-\eta}) - \\ &- \varepsilon^3\left[\eta + e^{-\eta} + \frac{N}{8}(\eta+1)e^{-\eta} - \left(\frac{N}{8}+1\right)\right]. \end{aligned} \tag{42}$$

Matching (41) and (42) as before we get

$$c_3 = \frac{3N}{8} - 1.$$

Hence, for large n, i.e. for small K, the pressure gradient P given by (14) becomes

$$P = -n^2 - n + \left(\frac{N}{4} - 1\right) + \frac{1}{n}\left(\frac{3N}{8} - 1\right) + 0\left(\frac{1}{n^2}\right) \tag{44}$$

and the 4-term outer expansion for f is

$$f^{(ou)} = \xi + \frac{1}{n}\xi + \frac{1}{n^2}\xi + \left(\frac{N}{8}+1\right)\frac{\xi}{n^3}. \tag{45}$$

The 4-term inner expansion (42) with $\varepsilon = n^{-1}$ shows the boundary layer behaviour near the channel wall for large n, i.e. for small K, the boundary layer thickness being of order $\sqrt{K}$. It is evident from (44) that the rate of pressure drop decreases with the increase in N when $N > 0$ (i.e. volume sinks of mass) and the pressure drop increases with increasing $|N|$ when $N < 0$ (i.e. volume sources of mass). Thus for fixed K (small), the volume sources of mass increase and sinks of mass decrease the overall hydraulic resistance of the channel. Also for fixed N, a decrease in K results in an increase in the pressure gradient along the channel.

III. SOLUTION FOR SMALL N AND ANY VALUE OF K

It is seen from (7) that for small N the solution of the equation (7) satisfying the boundary conditions (11) gives rise to a regular perturbation problem unlike the one given in Section II. Thus we expand f and P as follows:

$$f = f_0 + Nf_1 + N^2 f_2 + \dots \tag{46}$$

$$P = P_0 + NP_1 + N^2 P_2 + \dots \tag{47}$$

The boundary conditions for f_i are given from (11) as

$$f_0(0) = f_0''(0), \quad f_0(1) = 1, \quad f_0'(1) = 0 \tag{48}$$

and

$$f_i(0) = f_i''(0) = 0, \quad f_i(1) = f_i'(1) = 0, \qquad (i = 1, 2). \tag{49}$$

Substituting (46) and (47) in (7) and equating the coefficients of N^0, N, N^2, ... we get

$$f_0''' - n^2 f_0' = P_0, \tag{50}$$

$$f_1''' - n^2 f_1' + \frac{1}{4}[f_0'^2 - (f_0 - \xi) f_0''] = P_1, \tag{51}$$

$$f_2''' - n^2 f_2' + \frac{1}{4}[2 f_0' f_1' - \{f_1''(f_0 - \xi) + f_0'' f_1\}] = P_2. \tag{52}$$

The solutions of the equations (50) and (51) satisfying the boundary conditions (48) and (49) are

$$f_0(\xi) = \frac{\sinh n\xi - n\xi \cosh n}{\sinh n - n \cosh n}, \tag{53}$$

$$f_1(\xi) = \frac{n \sinh n\xi}{4(\sinh n - n\cosh n)^3}\left[\frac{1}{2n}(\sinh^2 n - 2\cosh^2 n) + \right.$$
$$\left. + \frac{1}{4n^2}\sinh n . \cosh n + \frac{3}{4}\sinh n . \cosh n + \frac{3}{4n^3}\sinh^2 n\right] +$$
$$+ \xi\left[\frac{1 + \cosh^2 n}{4(\sinh n - n\cosh n)^2} - \frac{P_1}{n^2}\right] + \frac{n \cosh n}{4(\sinh n - n - n\cosh n)^2} \cdot$$
$$\cdot \left[\frac{\xi}{n}\cosh n\xi - \frac{1}{n^2}\sinh n\xi\right] - \frac{\sinh n}{16(\sinh n - n\cosh n)^2} \cdot$$
$$\cdot \left[\xi^2 \sinh n\xi - \frac{3\xi}{n}\cosh n\xi + \frac{3}{n^2}\sinh n\xi\right] \tag{54}$$

with

$$P_0 = \frac{n^3 \cosh n}{\sinh n - n \cosh n}, \tag{55}$$

$$P_1 = \frac{4n^2(2\cosh n + \cosh^3 n) - 3n \sinh^2 n . \cosh n - 3n^2(3\sinh n + 2\sinh^3 n)}{16(n\cosh n - \sinh n)^3} \tag{56}$$

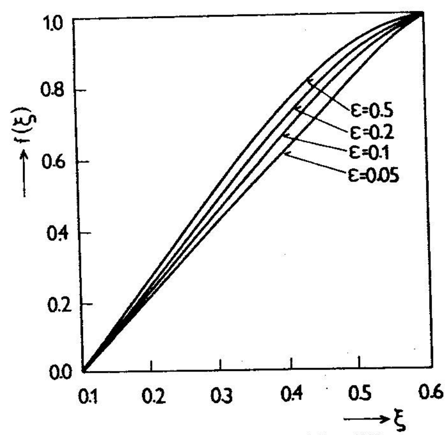

Figure 1: Variation of $f(\xi)$ with ξ for different ε.

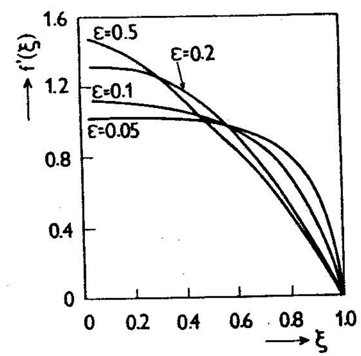

Figure 2: Variation of $f'(\xi)$ with ξ for different ε.

Using (53) and (54), the values of $f(\xi)$ and $f'(\xi)$ against ξ for several values of the permeability constant $\varepsilon = \frac{1}{n}$ with $N = 0.02$ are plotted in Figures 1 and 2, respectively. It is seen that $f(\xi)$ decreases steadily with decreasing ε (Figure

1). This reduction in the normal component of the velocity with decreasing ε can be attributed to the decelerating influence of the porous material. On the other hand figure 2 shows that a decrease in ε results in a progressive flattening of $f'(\xi)$, which is the profile corresponding to the velocity component parallel to channel walls. it may be noted in this connection that $f'(\xi)$ remains almost uniform for a given value of ε over the central portion of the channel and its value decreases with an increase in ε. However, near the channel walls, there is a steep gradient in $f'(\xi)$ (exhibiting the boundary layer behaviour) and the magnitude of $f(\xi)$ increses with decreasing ε unlike the behaviour of $f'(\xi$ near the central portion of the channel.

REFERENCES

[1] Berman, A. S.: J. Appl. Phys. *24* (1953), 1232.
[2] Berman, A. S.: J. Appl. Phys. *27* (1956), 1557.
[3] Sellers, J. R.: J. Appl. Phys. *26* (1955), 489.
[4] Yuan, S. W.: J. Appl. Phys. *27* (1956), 267.
[5] Yuan, S. W., Finkelstein, A. B.: Trans. Amer. Soc. Mech. Engns. *78* (1956), 719.
[6] Donoughe, P. L.: NACA T N 3759 August (1956).
[7] Morduchow, M.: Quart. Appl. Math. *14* (1957), 361.
[8] Aladiev, S. I., Zaichik, L. I.: J. Appl. Math. *14* (1957), 361.
[9] Na, T. Y., Gupta, A. S., Nanda, A.: J. Nat. Acad. Math. *1* (1983), 1.
[10] Na, T. Y. *Computational Methods in Engineering Boundary Value Problems*, Academic Press 1979.
[11] Van Dyke, M.: *Perturbation Methods in Fluid Mechanics*, Academic Press 1964.

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ГИДРОДИНАМИЧЕСКИЙ ПОТОК В ПОРИСТОМ КАНАЛЕ С ОБЬЕМНЫМИ ИСТОЧНИКАМИ ИЛИ СТОКАМИ МАСС

Приведены решения подобия для стационарного тока вязкой несжимаеиой жидкости через пористую среду в канале с объемными источниками или стоками масс. Показано, что проблема может быть решена методом помеченных ассимптотических разложений для малых значений проницаемости среды (k), соответствующих значениям параметра N источника или слоя. Падение давления требует уменьшения скорости тока в зависимости от k. Для фиксированного k падение давления уменьшается с увеличением N для $N > 0$ (объемные стоки масс) и увеличивается с увеличением $|N|$ для $N < 0$ (объемные источники масс).