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ON THE EVOLUTION OF THE SQUEEZED VACUUM STATE OF AN ANHARMONIC OSCILLATOR

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We find that if an anharmonic oscillator is initially in the squeezed vacuum state, then the initial squeezing of the variantces of the quadrature operators can exhibit periodical revivals.

I. INTRODUCTION

Today, when it is possible to generate squeezed states (for reviews on the subject see [1—3]) of the electromagnetic field in the laboratory [4—7], new perspectives are opened in quantum optics — theoretical as well as experimental ones. In this situation it is worth-while to analyse the influence of the material media on the squeezing properties of the light field.

Previously Tanas [8] has studied the interaction of the coherent light field with a nonabsorbing linear medium modelled as an anharmonic oscillator with the Hamiltonian

$$H = \hbar\omega a^+ a + \frac{\hbar\lambda}{2}(a^+)^2 a^2. \tag{1.1}$$

He has found that if a large number of photons ($\sim 10^6$) is present initially in the system, then the light becomes squeezed significantly over the appropriate time scale of $\lambda t \sim 10^{-6}$ (i.e. the variance in one quadrature is sufficiently less than 1/4).

Later the model described above has been studied by several authors [9—13]. In particular, Peřinová and Lukš have studied in great detail the statistical properties of the radiation passing through the nonlinear medium modelled as a third order dissipative oscillator interacting with squeezed light.

The recent paper by Gerry [15] has also been devoted to the problem of the interaction of matter with squeezed light. He has studied the solvable model of nonabsorbing nonlinear medium modelled as an anharmonic oscillator (1.1) interacting with squeezed light described as an SU(1,1) coherent state. He has shown that the anharmonic medium tends to revoke the squeezing of an initially squeezed SU(1,1) coherent state. Moreover, he has found that the greater the initial squeezing, the more rapidly it is revoked and that at longer times the variances of the quadrature operators tend to oscillate.

We will analyse the model of the anharmonic oscillator very similar to that considered by Gerry, the typical feature of which are the periodical revivals of the squeezing of the variances of the quadrature operators.

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II. THE MODEL

Gerry in his paper [15] has studied the dynamics of the nonlinear oscillator with the Hamiltonian (1.1) rewritten in terms of the generators K_0 and $K_\pm$ of the Lie algebra of the SU(1,1) group given in terms of the bosonic operators a and a^+ ($[a, a^+] = 1$):

$$K_0 = \frac{1}{4}(a^+ a + aa^+); \quad K_+ = \frac{1}{2}(a^+)^2; \quad K_- = \frac{1}{2}(a)^2. \tag{2.1}$$

The generators $K_\pm$ and K_0 satisfy the commutation relations:

$$[K_0, K_\pm] = \pm K_\pm; \quad [K_-, K_+] = 2K_0. \tag{2.2}$$

Using (2.1) the Hamiltonian (1.1) can be rewritten (up to constant terms) as follows:

$$H = \hbar\omega K_0 + \lambda\hbar K_+ K_-. \tag{2.3}$$

In our analysis we will also consider the Hamiltonian in the form (2.3), but with a different realization of the SU(1,1) Lie algebra. Namely, we will use the Holstein-Primakoff realization of the SU(1,1) Lie algebra [16—18] with the Bargmann index [19] $k = 1/2$:

$$K_0 = \frac{1}{2}(a^+ a + aa^+), \quad K_+ = \sqrt{a^+ a}\, a^+; \quad K_- = a\sqrt{a^+ a}. \tag{2.4}$$

using the generators (2.4) we can find the solution of the timedependent Schrödinger equation for the state vector $|\Psi(t)\rangle$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{2.5}$$

with the Hamiltonian given by the relation (2.3). If we assume the initial state (at $t = 0$) to be an SU(1,1) generalized coherent state [15, 20, 21], which for the squeezed vacuum is

$$|\Psi(t=0)\rangle \equiv |\xi\rangle = (1-|\xi|^2)^{1/4} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+1/2)}{n!\Gamma(1/2)}\right]^{1/2} \xi^n |2n\rangle \equiv \sum_{n=0}^{\infty} Q_n |2n\rangle, \quad (2.6)$$

then the state vector $|\Psi(t)\rangle$ for $t > 0$ can be obtained:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} Q_n e^{-i(2\omega n + \lambda(2n)^2)t} |2n\rangle. \quad (2.7)$$

The parameter ξ can be written as:

$$\xi = -|\xi| e^{2i\Phi} = -\tanh \Theta/2\, e^{2i\Phi}, \quad (2.8)$$

where Θ and Φ are the SU(1,1) group parameters (Θ has the range $(-\infty, +\infty)$ and Φ is from the interval $(0, \pi)$). The parameters $|\xi|$ and Θ are related to the average photon number $\bar{n} = \langle a^+ a\rangle$ in the following way:

$$\bar{n} = \frac{|\xi|^2}{1-|\xi|^2} = \sinh^2 \Theta/2. \quad (2.9)$$

For the purposes of the following calculations we will write down the mean values of the photon number operator $\langle a^+ a\rangle$, the photon amplitude $\langle a\rangle$ and the squared photon amplitude $\langle a^2\rangle$:

$$A_0 \equiv \langle a^+ a\rangle = \sum_{n=0}^{\infty} 2nP_n = \bar{n}, \quad (2.10.a)$$

$$A_1 \equiv \langle a\rangle e^{i(\omega t - \Phi)} = 0, \quad (2.10.b)$$

$$A_2 \equiv \langle a^2\rangle e^{2i(\omega t - \Phi)} = -|\xi| \sum_{n=0}^{\infty} P_n(2n+1) e^{-4i\lambda(2n+1)t}, \quad (2.10.c)$$

where P_n is the distribution of the squeze vacuum state $|\xi\rangle$:

$$P_n \equiv |Q_n|^2 = (1-|\xi|^2)^{1/2} \left(\frac{|\xi|}{2}\right)^{2n} \frac{(2n)!}{(n!)^2}. \quad (2.11)$$

With the distributions (2.11) the expression (2.10.c) for A_2 can be calculated in a closed form [22]

$$A_2 = \frac{-|\xi|(1-|\xi|^2)^{1/2} e^{-4i\lambda t}}{(1-|\xi|^2 e^{-8i\lambda t})^{3/2}}. \quad (2.10.d)$$

From (2.10) it follows that the average photon number is the integral of motion (this is the consequence of the commutation relation $[K_0, H] = 0$) and the mean photon amplitude is equal to zero.

III. LIGHT SQUEEZING

To analyse the squeezing we introduce two Hermitian timedependent quadrature operators

$$a_1(t) = \frac{1}{2}(a e^{i(\omega t - \delta)} + a^+ e^{-i(\omega t - \delta)}), \quad (3.1)$$

$$a_2(t) = \frac{1}{2i}(a e^{i(\omega t - \delta)} - a^+ e^{-i(\omega t - \delta)}), \quad (3.2)$$

where δ is an arbitrary phase chosen to be equal to Φ. The commutation relation for the operators $a_i(t)$ is:

$$[a_1(t), a_2(t)] = \frac{i}{2}. \quad (3.3)$$

One of the consequences of this commutator is the uncertainty relation

$$V_1(t) . V_2(t) \geqslant \frac{1}{16}, \quad (3.4)$$

where $V_i(t)$ are the variances of the quadrature opeators $a_i(t)$:

$$V_i(t) = \langle a_i^2\rangle - \langle a_i\rangle^2. \quad (3.5)$$

Since the squeezed states are defined as the states with a smaller uncertainty in one quadrature of the field than that associated with the coherent field (see, for instance, [1—3]) the squeezing condition can be written as:

$$V_i(t) < \frac{1}{4} \quad \text{for} \quad i = 1 \quad \text{or} \quad i = 2. \quad (3.6)$$

The variances of the quadrature operators a_i can be expressed through the mean values of the photon operators (2.10):

$$V_1(t) = \frac{1}{2}\left[A_0 + \frac{1}{2} + \mathrm{Re}A_2 - 2(\mathrm{Re}A_1)^2\right] \quad (3.7.a)$$

$$V_2(t) = \frac{1}{2}\left[A_0 + \frac{1}{2} - \mathrm{Re}A_2 - 2(\mathrm{Im}A_1)^2\right] \quad (3.7.b)$$

or, using the relations (2.10) for the functions $A_i(t)$, we obtain:

$$V_{\substack{1\\2}}(t) = \frac{1}{4}\left[\frac{1+|\xi|^2}{1-|\xi|^2} \mp \frac{2|\xi|(1-|\xi|^2)^{1/2}}{(1-2|\xi|^2\cos 8\lambda t + |\xi|^4)^{3/4}} \cos\left(4\lambda t + \frac{3}{2}\varphi\right)\right], \quad (3.8)$$

where

$$\varphi = \text{arctg}\,\frac{|\xi|^2 \sin 8\lambda t}{1 - |\xi|^2 \cos 8\lambda t}. \tag{3.9}$$

From the explicit expressions for $V_i(t)$ several properties of the variances of the quadrature operators follow:

1) The functions $V_i(t)$ are strictly periodical with the period $T = \dfrac{\pi}{2\lambda}$

$$V_i(t) = V_i(t + \pi/2\lambda). \tag{3.10}$$

This means that if at $t = 0$ one quadrature is squeezed, then this squeezing will appear periodically at later times.

2) There exists a relation between the variances of the quadrature operators:

$$V_1(t) = V_2(t + \pi/4\lambda). \tag{3.11}$$

This means that the variances $V_1(t)$ and $V_2(t)$ are shifted mutually in time by the half-period of the squeezing revivals.

3) The sum of the variances $V_i(t)$ is the integral of motion in the discussed model:

$$V_1(t) + V_2 t) = \bar{n} + \frac{1}{2} = \frac{1}{2}\cosh\Theta. \tag{3.12}$$

4) To determine how repadly the squeezing in revoked in the initial moments of the evolution ($t \ll \pi/\lambda$) we evaluate the time derivative of the function $V_1(t)$ (this variance is initially squeezed) at time $t = 0$ and examine its dependence on the initial conditions, particularly on the initial photon number. For the first derivative we obtain:

$$\left.\frac{\partial V_1(t)}{\partial t}\right|_{t=0} = 0. \tag{3.13}$$

Since the first derivative vanishes, we have to calculate the second:

$$\left.\frac{\partial^2 V_1(t)}{\partial^2 t}\right|_{t=0} = 8\lambda^2\sqrt{\bar{n}(\bar{n}+1)}\,[15\bar{n}^2 + 12\bar{n} + 1]. \tag{3.14}$$

This derivative is strictly positive and it increases by the increasing initial photon number. From this it follows that the more photons in the initial state the more rapidly the squeezing is revoked (this is the situation identical to that in Gerry's case). Nevertheless, it should be stressed once more that the initial squeezing is periodically restored at a long-time scale (see relation (3.10)).

5) Finally it is also worth-wile to mention that the variances $V_i(t)$ have a very simple form for the times $t = k\pi/8\lambda$ where $k = 0, 1, 2 \ldots$ (see Table 1).

The time evolution of the variance $V_i(t)$ for various values of the initial squeezing (or, which is the same, for various values of the initial photon numbers) are plotted in Figure 1.

Table 1

The values of the variances $V_i(t)$ expressed through the parameters $|\xi|$ and Θ for the times $t = \dfrac{k\pi}{8\lambda}$; $k = 0, 1, 2 \ldots$

	$V_1(t)$	$V_2(t)$
$t = 0$	$\frac{1}{4}\frac{1-\lvert\xi\rvert}{1+\lvert\xi\rvert} = \frac{1}{4}e^{-\Theta}$	$\frac{1}{4}\frac{1+\lvert\xi\rvert}{1-\lvert\xi\rvert} = \frac{1}{4}e^{+\Theta}$
$t = \frac{\pi}{8\lambda}$	$\frac{1}{4}\frac{1+\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2} = \frac{1}{4}\cosh\Theta$	$\frac{1}{4}\frac{1+\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2} = \frac{1}{4}\cosh\Theta$
$t = \frac{\pi}{4\lambda}$	$\frac{1}{4}\frac{1+\lvert\xi\rvert}{1-\lvert\xi\rvert} = \frac{1}{4}e^{+\Theta}$	$\frac{1}{4}\frac{1-\lvert\xi\rvert}{1+\lvert\xi\rvert} = \frac{1}{4}e^{-\Theta}$

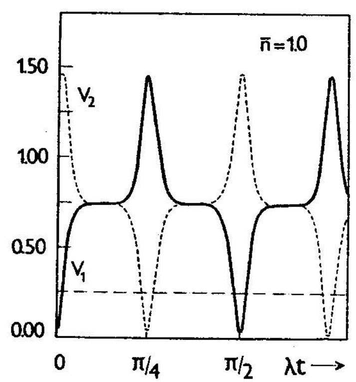

Figure 1.a Time evolution of the variances $V_1(t)$ and $V_2(t)$ for $\bar{n} = 1.0$.

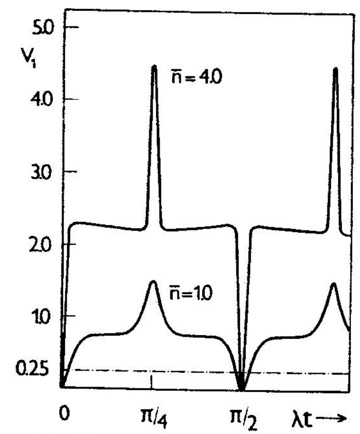

Figure 1.b Time evolution of the variance $V_1(t)$ for $\bar{n} = 1.0$ and $\bar{n} = 4.0$.

IV. CONCLUSIONS AND DISCUSSION

Thus we can conclude that in the present model of the anharmonic oscillator the squeezing of the variances exhibits periodical revivals for *any* value of the initial squeezing. This periodicity is preserved not only for an SU(1,1) GCS, but for any initial state of the oscillator. Particularly, if at $t = 0$ the system is in the Glauber conherent state, then the variances of the quadrature operators oscillate with period $T = \pi/\lambda$ (see [13]). Moreover, it can be shown that the variances can be squeezed in the evolution.

In the present paper we have analysed the dynamics of the anharmonic oscillator without dissipations. To make the problem more realistic, the dissipations should be taken into account as it has been proposed by Peřinová and Lukš [14]. This problem will be discussed elsewhere.

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ЭБОЛЮЦИЯ СЖАТОГО СОСТОЯНИЯ АНГАРМОНИЧЕСКОГО ОСЦИЛЛЯТОРА

В работе показано, что если ангармонической осциллятор первоначально находится в сжатом состоянии, то начальное сжатие флюктуаций квадратурных операторов периодически восстанавливается.