acta phys. slov. 39 (1989), No. 5

NEW APROACH TO THE SOLUTION OF PROBLEMS OF INHOMOGENEOUS THIN PLATES WITH A HOLE PLANAR

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In the work a new approach is presented to problems of inhomogeneous isotropic thin planar plates with a circular hole. In the case of weakly inhomogeneous materials the solutions are expressed in the form of expansions for given boundary conditions. This approach allows not only to determine directly the nth term in the expansion (using the Sherman-Lauricello integral equation) but also to find the radius of convergence of the expansion and even to sum it up.

Finally, we present numerical calculations of some special cases with the following boundary conditions: a) the contour of the circular hole is effected by a uniform normal stress; b) the infinite thin planar plate with a circular hole uniformly expands in all directions. We have found the qualitative and quantitative effects of inhomogeneities on the stress distribution around the hole.

I. INTRODUCTION

The use of composite materials (inhomogeneous, anisotropic etc.) in various regions of science and technology brings a variety of unsolved practical and theoretical problems. In this connection, a new branch of science was created, with the main emphasis on the formulation and solution of important technical problems.

Various problems of the theory of elasticity of inhomogeneous and anisotropic bodies were investigated in the works of Muschelishvili N. I. [9], Savin G. N. [12], Lechnickij S. G. [5], Lomakin V. A. [6], Michlin S. G. [7], Ambarcumyan S. A. [1], Kosmodamyanskij A. S. [4], Sarkisyan V. S. [13—17], Brilla J. [20], Kolchin G. B. [3], Mishiku M. and Teodoseu K. [8], and many other authors.

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II.

It is well known that the plane problem of inhomogeneous bodies leads to the solution of the third order differential equation with variable coefficients [9, 12]

$$\frac{\partial^3 G}{\partial z^2 \partial \bar{z}} + 2\,\mathrm{Re}\left\{A_1(z,\bar{z})\frac{\partial^2 G}{\partial z \partial \bar{z}} + A_2(z,\bar{z})\frac{\partial G}{\partial \bar{z}} + A_3(z,\bar{z})\frac{\partial G}{\partial z}\right\} = f(z,\bar{z}) \qquad (1)$$

with the boundary condition

$$G(\sigma) = H(\sigma), \qquad (2)$$

where

$$A_1(z,\bar{z}) = E(z,\bar{z})\frac{\partial}{\partial z}\left[\frac{1}{E(z,\bar{z})}\right]$$

$$A_2(z,\bar{z}) = \frac{E(z,\bar{z})}{4}\frac{\partial^2}{\partial z^2}\left[\frac{1+\nu(z,\bar{z})}{E(z,\bar{z})}\right]$$

$$A_3(z,\bar{z}) = \frac{E(z,\bar{z})}{2}\frac{\partial^2}{\partial z \partial \bar{z}}\left[\frac{1-\nu(z,\bar{z})}{E(z,\bar{z})}\right].$$

Here $f(z,\bar{z})$ is a function depending on external forces, $H(\sigma)$ is a function defined on a contour (boundary), $E(z,\bar{z})$ and $\nu(z,\bar{z})$ are real functions defining inhomogeneities of the thin planar plate.

In further investigations it is assumed that the body is characterized by the weak inhomogeneity of the type [10—18, 20, 21]

$$E(z,\bar{z}) = E_0[1 + \delta e(z,\bar{z})], \qquad (3)$$

where $E_0 = \text{const}$, δ is the given physical parameter, $e(z,\bar{z})$ is the inhomogeneity function.

The solution of the equation given above, with respect to condition (3), is given in the form [13—18, 20]

$$G(z,\bar{z}) = G_0(z,\bar{z}) + \sum_{n=1}^{\infty} \delta^n G_n(z,\bar{z}), \qquad (4)$$

where the ccomplex functions $G_n(z,\bar{z})$ of the stress are defined by the following sequence of the boundary problems

$$\begin{aligned} L[G_0(z,\bar{z})] &= 0 \\ G_0(\sigma) &= H(\sigma) \end{aligned} \qquad (5)$$

$$L[G_n(z,\bar z)] = V[G_{n-1}(z,\bar z)]$$
$$G_n(\sigma) = 0, \tag{6}$$

where $L[\ ]$ and $V[\ ]$ are third order differential operators (see [14, 18]).

The solution of the boundary problem (5) under the fixed boundary condition, i.e. by the fixed function $H(\sigma)$, is the following (9, 12)

$$G_0(z,\bar z) = \varphi_0(z) + z\overline{\varphi_0'(z)} + \overline{\psi_0(z)}, \tag{7}$$

where

$$\varphi_0(z) = -\frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)\,d\sigma}{\sigma - z}$$
$$\psi_0(z) = -\frac{1}{2\pi i}\int_\gamma \frac{\overline{H(\sigma)}\,d\sigma}{\sigma - z} - \frac{R}{z}\varphi_0'(z). \tag{8}$$

One can show that for the arbitraty inhomogeneity function $e(z,\bar z)$ the operator $V[G_{n-1}(z,\bar z)]$ can be written in the form

$$V[G_{n-1}(z,\bar z)] = \frac{(n+1)^2(A_n(z,\bar z)}{(z\bar z)^{n+2}}, \tag{9}$$

where

$$A_n(z,\bar z) = \frac{1}{(n+1)^2}\,2\,\mathrm{Re}\left\{(z\bar z)^{n+2}\left[\frac{\partial e}{\partial z}\frac{\partial^2 G_{n-1}}{\partial z\,\partial \bar z} + \frac{1+\nu}{4}\frac{\partial^2 e}{\partial z^2}\frac{\partial G_{n-1}}{\partial z} + \right.\right.$$
$$+\frac{1-\nu}{4}\frac{\partial^2 e}{\partial z\,\partial\bar z} - \frac{1+\nu}{4}\left(\frac{\partial e}{\partial z}\right)^2\frac{\partial G_{n-2}}{\partial \bar z} - \frac{1-\nu}{4}\frac{\partial G_{n-2}}{\partial z}\frac{\partial e}{\partial z}\frac{\partial e}{\partial \bar z} =$$
$$= 2\,\mathrm{Re}\{f_1(z,\bar z) + i f_2(z,\bar z)\} \tag{10}$$

and $f_1(z,\bar z)$ and $f_2(z,\bar z)$ are arbitraty real functions, $f_1(z,\bar z)$ should be non-vanishing.

From equation (10) it follows that

$$A_n(z,\bar z) = 2f_1(z,\bar z) \tag{11}$$

and that the inhomogeneity function $e(z,\bar z)$ satisfies the following equation

$$\frac{\partial e}{\partial z}\frac{\partial^2 G_{n-1}}{\partial z\,\partial\bar z} + \frac{1+\nu}{4}\frac{\partial^2 e}{\partial z^2}\frac{\partial G_{n-1}}{\partial \bar z} + \frac{1-\nu}{4}\frac{\partial^2 e}{\partial z\,\partial \bar z}\frac{\partial G_{n-1}}{\partial z} - \frac{1+\nu}{4}\left(\frac{\partial e}{\partial z}\right)^2\frac{\partial G_{n-2}}{\partial \bar z} -$$
$$-\frac{1-\nu}{4}\frac{\partial e}{\partial z}\frac{\partial e}{\partial \bar z}\frac{\partial G_{n-2}}{\partial z} = \frac{f_1(z,\bar z) + i f_2(z,\bar z)}{(z\bar z)^{n+2}}. \tag{12}$$

Since n is an arbitrary number, one can put $n = 1$. In this case equation (12) takes the very simple form

$$\frac{\partial e}{\partial z}\frac{\partial^2 G_0}{\partial z\,\partial\bar z} + \frac{1+\nu}{4}\frac{\partial^2 e}{\partial z^2}\frac{\partial G_0}{\partial \bar z} + \frac{1-\nu}{4}\frac{\partial^2 e}{\partial z\,\partial\bar z}\frac{\partial G_0}{\partial z} = \frac{f_1(z,\bar z) + i f_2(z,\bar z)}{(z\bar z)^3}. \tag{13}$$

From equation (13) one can obtain an arbitrary number of inhomogeneity functions $e(z,\bar z)$ depending of $f_1(z,\bar z)$, $f_2(z,\bar z)$ and $G_0(z,\bar z)$.

Since the functions $f_1(z,\bar z)$ and $f_2(z,\bar z)$ are arbitrary, we shall suppose that

1. $f_1(z,\bar z) = A_0 \quad (A_0 = \text{const} \neq 0)$.

Then from equation (11) we obtain

$$A_n(z,\bar z) = 2A_0. \tag{14}$$

According to equations (9) and (14) the boundary problem (6) takes the following form

$$L[G_n(z,\bar z)] = \frac{2(n+1)^2 A_0}{(z\bar z)^{n+2}} \tag{15}$$
$$G_n(\sigma) = 0, \quad (n \geqslant 1).$$

The solution of this equation is given by the formulas [8, 14—18]

$$G_n(z,\bar z) = \varphi_n(z) + z\overline{\varphi_n'(z)} + \overline{\psi_n(z)} + IG_n(z,\bar z), \tag{16}$$

where $IG_n(z,\bar z)$ is a particular solution, which has in the assumed case the form

$$IG_n(z,\bar z) = -\frac{2A_0}{n z^n \bar z^{n+1}}. \tag{17}$$

According to equation (17), the integral Sherman-Lauricell equation has the form

$$\frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\sigma)\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_\gamma \frac{\sigma\overline{\varphi_n'(\sigma)}\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi_n(\sigma)}\,d\sigma}{\sigma - z} = \frac{2A_0}{2\pi i n}\int_\gamma \frac{\sigma\,d\sigma}{\sigma - z}.$$

From this equation we obtain

$$\varphi_n(z) = 0, \quad \psi_n(z) = \frac{2A_0}{nz},$$

and consequently

$$G_n(z,\bar z) = \frac{2A_0}{n}\left(\frac{1}{\bar z} - \frac{1}{z^n \bar z^{n+1}}\right). \tag{18}$$

Inserting equation (18) into equation (4) we obtain

$$G_n(z, \bar{z}) = G_0(z, \bar{z}) + 2A_0 \sum_{n=1}^{\infty} \frac{\delta^n}{n} \left(\frac{1}{\bar{z}} - \frac{1}{z^n \bar{z}^{n+1}} \right). \tag{19}$$

The components of the stress are determined from the relations given in [14] and the coefficient of the concentration is

$$K = K_0 + 4A_0 \sum_{n=1}^{\infty} \frac{\delta^n}{r^{2n+2}}. \tag{20}$$

The power series in (20) converges to

$$\sum_{n=1}^{\infty} \left(\frac{\delta}{r^2} \right)^n = \frac{\delta}{r^2 - \delta} \tag{21}$$

if the condition

$$\delta < r^2 \tag{22}$$

is satisfied.

According to this equation and equation (20) we obtain

$$K_1 = K_0 + 4A_0 \frac{\delta}{r^2(r^2 - \delta)}. \tag{23}$$

2. Let

$$f_1(z, \bar{z}) = \frac{B_0(n - m + 1)}{n + 1} (z^m + \bar{z}^m) \tag{24}$$

$$B_0 = \text{const} \neq 0, \; n > m.$$

Under the assumption (24) the boundary problem (6) can be rewritten in the form

$$L[G_n(z, \bar{z})] = \frac{B_0(n - m + 1)(n + 1)}{(z\bar{z})^{n+2}} (z^m + \bar{z}^m) \tag{25}$$

$$G_n(\sigma) = 0.$$

Solving (25) like (15) we obtain

$$G_n(z, \bar{z}) = B_0 \left\{ \frac{1}{n - m} \left(\frac{1}{\bar{z}^{m+1}} - \frac{1}{z^{n-m} \bar{z}^{n+1}} \right) + \frac{1}{n} \left(\frac{1}{z^{m-1}} - \frac{1}{z^n \bar{z}^{n-m+1}} \right) + \right.$$

$$\left. + \frac{m-1}{n} \left(- \frac{z}{\bar{z}^m} + \frac{1}{\bar{z}^{m+1}} \right) \right\}.$$

In this case, using equation (19), we obtain

$$K_2 = K_0 + \frac{4B_0 \cos m\Theta}{r^2} \left\{ \frac{\delta}{r^2 - \delta} - \frac{m-1}{r^{2m-2}} \lg(1 - \delta) \right\}, \; (0 < \delta < 1). \tag{26}$$

Here it is assumed that $\sum_{n=1}^{\infty} \frac{\delta^n}{n} = -\lg(1 - \delta)$.

3. Let

$$f_1(z, z) = \frac{C_0(n - m + 1)}{n + 1} \mathrm{i}(z^m - \bar{z}^m) \tag{27}$$

$$C_0 = \text{const} \neq 0, \; (n > m).$$

According to (27), the solution of the problem (6) is given as

$$G_n(z, \bar{z}) = C_0 \mathrm{i} \left\{ \frac{1}{n - m} \left(\frac{1}{\bar{z}^{m+1}} - \frac{1}{z^{n-m} \bar{z}^{n+1}} \right) + \frac{1}{n} \left(- \frac{1}{z^{m-1}} + \frac{1}{z^n \bar{z}^{n-m+1}} \right) + \right.$$

$$\left. + \frac{m-1}{n} \left(- \frac{z}{\bar{z}^m} + \frac{1}{\bar{z}^{m+1}} \right) \right\}. \tag{28}$$

In this case the concentration coefficient, with respect to equation (28), has the following form

$$K_3 = K_0 + \frac{4C_0 \sin m\Theta}{r^2} \left\{ \frac{\delta}{\delta - r^2} - \frac{m-1}{r^{2m-2}} \lg(1 - \delta) \right\} (0 < \delta < 1). \tag{29}$$

III. SOME EXAMPLES

a) The contour of the hole is influenced by a uniform normal pressure [9]. In this case

$$H(\sigma) = -PR\sigma.$$

According to equations (7), (8), (23), (26), (29) the concentration coefficient in the limit $r \to 1$ (i.e. on the contour of the hole) takes the form

$$K_{11} = 1 + \frac{4(1 + \nu)\delta}{1 - \delta}$$

$$K_{12} = 1 + \frac{4(1 + \nu) \cos m\Theta}{1 - \delta} \{\delta - (m - 1)(1 - \delta) \cdot \lg(1 - \delta)\} \quad (m \geqslant 1) \tag{30}$$

$$K_{13} = 1 - \frac{4(1+\nu)\sin m\Theta}{1-\delta}\{\delta + (m-1)(1-\delta)\cdot \lg(1-\delta)\} \quad (m \geqslant 1).$$

b) The case of a uniformly expanded infinite thin planar plate with a circular hole [9]. In this case

$$K_{21} = 2 + \frac{4(1+2\nu)\delta}{1-\delta}$$

$$K_{22} = 2 + \frac{4(1+2\nu)\cos m\Theta}{1-\delta}\{\delta - (m-1)(1-\delta)\cdot \lg(1-\delta)\} \qquad (31)$$

$$K_{23} = 2 - \frac{4(1+2\nu)\sin m\Theta}{1-\delta}\{\delta + (m-1)(1-\delta)\cdot \lg(1-\delta)\}.$$

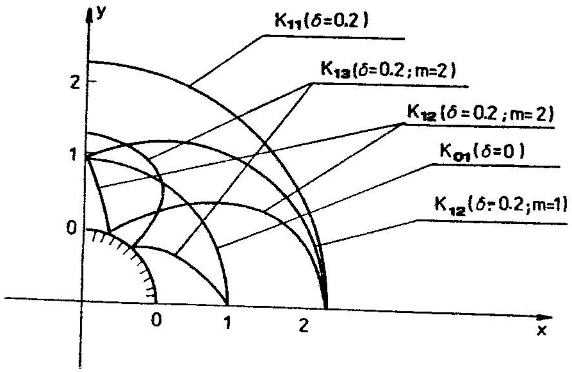

Fig. 1.

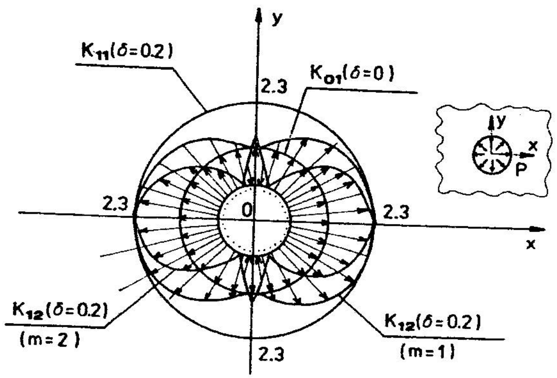

Fig. 2.

In Tables 1 and 2*) are given the values of the coefficients of the concentration on the contour of the hole determined from equations (30) and (31).

In Figures 1—3 there are given the diagrams and the distribution of the stress around the holes under the boundary conditions a) and b)

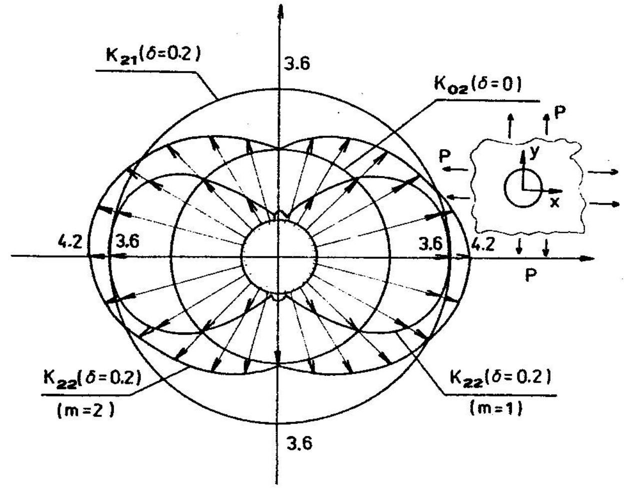

Fig. 3.

IV. CONCLUDING REMARKS

From the results given above we draw the following conclusions:

1. We have found that for an arbitrary inhomogeneity function $e(z,\bar{z})$ (is calculated from equation (12) or (13)), one can determine from equation (19) not only the $G_n(z,\bar{z})$, but also the sum of the series (4) (when $0 < \delta < 1$).

2. We have shown that the function $f_1(z,\bar{z})$ considerably impacts (not only quantitative, but qualitative, too) on the distribution of the stress around the hole (see Fig. 1—3).

3. For the effect of inhomogeneity in the problems solved in the previous section under the boundary conditions a) and b) we obtain an estimate 20—25 %.

4. From Tables 1 and 2 it follows that changing the small physical parameter δ one can enlarge or diminish the zone of the compression of the material.

Finally we point out that the use of inhomogeneous materials could be advantageous in science and technology.

Table 1

δ	0	0.1			0.2			0.3		
Θ K	K_{01}	K_{11}	K_{12}	K_{13}	K_{11}	K_{12}	K_{13}	K_{11}	K_{12}	K_{13}
0°	1	1.58	1.58	1	2.3	2.30	1	3.22	3.22	1
15°	—	—	1.56 1.50	0.85 0.71	—	2.25 2.13	0.66 0.35	—	3.15 2.93	0.42 −0.11
30°	—	—	1.50 1.29	0.71 0.50	—	2.13 1.65	0.35 −1.13	—	2.93 2.11	−0.11 −0.93
45°	—	—	1.41 1.00	0.59 0.42	—	1.92 1.00	0.08 −1.30	—	2.58 1.00	−0.58 −1.22
60°	—	—	1.29 0.71	0.50 0.50	—	1.65 0.35	−1.13 −1.13	—	2.11 −0.11	−0.93 −0.93
75°	—	—	1.15 0.50	0.44 0.71	—	1.34 −0.13	−1.25 0.35	—	1.58 −0.93	−1.15 −0.11
90°	—	—	1.00 0.42	0.42 1.00	—	1.00 −1.30	−1.30 1.00	—	1.00 −2.22	−1.22 1.00

Table 2

δ	0	0.1			0.2			0.3		
Θ K	K_{02}	K_{21}	K_{22}	K_{23}	K_{21}	K_{22}	K_{23}	K_{21}	K_{22}	K_{23}
0°	2	2.71	2.71 3.00	2 2	3.6	3.60 4.22	2 2	4.74	4.74 5.73	2 2
15°	—	—	2.69 2.87	1.82 1.79	—	3.54 3.92	1.59 1.51	—	4.65 5.23	1.29 1.12
30°	—	—	2.62 2.50	1.65 1.64	—	3.39 3.11	1.20 1.15	—	4.37 3.87	0.63 0.48
45°	—	—	2.50 2	1.50 1.58	—	3.13 2	0.87 1.02	—	3.94 2	0.06 0.24
60°	—	—	2.35 1.50	1.38 1.64	—	2.80 0.89	0.61 1.15	—	3.37 0.13	−0.37 0.48
75°	—	—	2.18 1.13	1.31 1.79	—	2.41 0.08	0.46 1.51	—	2.71 −1.23	−0.65 1.12
90°	—	—	2 1	1.29 2	—	2 −0.22	0.4 2	—	2 −1.73	−0.74 2

*) In Tables 1 and 2 the upper number corresponds to $m = 1$ and the lower to $m = 2$.

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Received August 16th, 1988
Accepted for publication February 8th, 1989