acta phys. slov. 39 (1989), No. 3

DIFFUSIONS AND NOISE ASSOCIATED TO GENERALIZED ORNSTEIN-UHLENBECK PROCESSES

BEZÁK, V.[1]), Bratislava

The autor discusses one-dimensional diffusions (Brownian motions) $x(t)$ described by the Langevin equations $\dot{x} - \mu F(x, t) = \mu f$ with general stationary zero-centred Gaussian random forces $f(t)$. In particular, he deals with the Langevin equation corresponding to linear driving forces: $F = \alpha - \beta x$. In such a case, he calls $x(t)$ the "generalized Ornstein-Uhlenbeck processes" (since they are, in contrast with the original Ornstein-Uhlenbeck process, non-Markovian) and explains how to construct their probability densities $p(x, t; x_0)$ directly, without any use of the Fokker-Planck-type equations or without representing p by means of path integrals. (The construction is possible owing to the gaussianity of the processes $x(t)$ when $F = \alpha - \beta x$). For $\mu = \text{const}$, $\alpha = 0$, $\beta = \text{const}$, the author calculates the spectral functions $S_x(\omega)$ to $x(t)$ for two autocorrelation functions $W_f(|u|)$ of $f(t)$: a. $W_f(|u|) \sim \exp(-|u|/\tau_f)$ and b. $W_f(|u|) \sim \exp(-\pi u^2/(2\tau_f)^2)$. (In both cases $\tau_f > 0$.) The spectral function $S_x(\omega)$ is shown to equal a Lorentzian (centred at $\omega = 0$) in case a., and to exhibit a remarkable departure from the lorentzianity in case b. All the main results of the author's calculations are presented graphically. Attention is also paid to possible multidimensional generalizations.

I. INTRODUCTION

According to the modern theory of stochastic processes, diffusions can essentially be studied in two ways which are — at least in principle, without taking into account a computational (technical) aspect — tantamount. To recall them, let us consider a one-dimensional diffusion process $x(t)$, interpreting $x(t)$ as instantaneous positions of an inertialess Brownian particle. (This is the same as considering the Brownian motion in the so-called overdamped regime.) The first way of quantifying the Brownian motion is to formulate and slove the Fokker-Planck equation for the density $p(x, t; x_0)$ of the probability that the particle, after leaving the point x_0 at a time $t_0 = 0$, will achieve the point x at a time $t > 0$ [1], [3]. The second way is to analyse the Itô equation for $x(t)$ [2], [3] (one of the simplest stochastic differential equations). (We might equally speak of the Stratonovich equation. As we will not consider a state-dependent stochastic term in the Itô equation, the concurrent Stratonovich equation will not look differently). When dividing to Itô equation formally by the time differential dt, we obtain what has been called the Langevin equation. If we admit the presence of some drift due to a deterministic force $F(x, t)$, we may write the Langevin equation in its simplest form

$$\gamma\dot{x}(t) - F(x(t), t) = f(t) \tag{1.1}$$

with a positive constant γ (meaning a friction coefficient). Yet we prefer to use the mobility $\mu = 1/\gamma$ (a non-negative constant) of the Brownian particle, so we rewrite the Langevin equation into the (trivially modified) from

$$\dot{x}(t) - \mu F(x(t), t) = \mu f(t). \tag{1.2}$$

Following Langevin, one has usually defined $f(t)$ as a zero-centred Gaussian random function of the white-noise type:

$$\langle f(t)\rangle = 0, \tag{2}$$

$$\langle f(t_1)f(t_2)\rangle = \Lambda\delta(t_1 - t_2) \tag{3}$$

(with a constant $\Lambda > 0$). Then we can represent p as the path integral

$$p(x, t; x_0) = \int_{x_0, 0}^{x, t} \mathscr{D}x(\tau) \exp\left\{-\frac{1}{4D}\,\text{M.p.}\int_0^t d\tau[\dot{x}(\tau) - \mu F(x(\tau), \tau)]^2\right\} \tag{4}$$

with the diffusion coefficient

$$D = \frac{1}{2}\Lambda\gamma^{-2} = \frac{1}{2}\Lambda\mu^2 > 0 \tag{5}$$

(cf. [4], [5]). When writing the "differential" $\mathscr{D}x(\tau)$ of the path integration (functional integration), one must pay attention to the proper normalization since p, being the probability density, has to satisfy the condition

$$\int_{-\infty}^{\infty} dx p(x, t; x_0) = 1 \tag{6}$$

for all values x_0 and for all times $t > 0$. We have introduced the symbol M.p.$\int$ (see Appendix III in [5]) in order to emphasize that it is not the classical (Riemann-Lebesgue) integral. Indeed, our "Markov-process integral" M.p.$\int$ is much (if not entirely) synonymous with the stochastic integral of the Itô type. (Here it must be pointed out that the integral M.p.$\int$ must not be interpreted in the Stratonovich sense in expression (4); the Stratonovich integrals are good for Feynman's quantum-mechanical path integrals, but if they are to be used here, formula (4) would have to be modified!)

One can easily prove (cf. again [5]) that p, when defined by the path integral

[1]) Matematicko-fyzikálna fakulta UK, Mlynská dolina F-2, 842 15 BRATISLAVA, Czechoslovakia

(4), satisfies the diffusion equation with a drift term (i.e. the Fokker-Planck equation) for $t > 0$,

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial}{\partial x}[F(x, t)p], \tag{7}$$

and the initial condition

$$p(x, 0; x_0) = \delta(x - x_0). \tag{8}$$

(No boundaries will be taken into account in the present paper.)

Clearly, the white-noise property of the stationary zero-centred Gaussian force $f(t)$ is a *condicio sine qua non* if the path integral (4) is to be used. Without the white-noise property of $f(t)$, the function $p(x, t; x_0)$ would not satisfy the Chapman-Kolmogorov (Smoluchowski) equation

$$\int_{-\infty}^{\infty} dx_1\, p(x, t - t_1; x_1)\, p(x_1, t_1; x_0) = p(x, t; x_0) \tag{9}$$

for each intermediate time $t_1 (0 < t_1 < t)$. But equation (9) is indispensable if equation (7) and condition (8) are validated (equation (9) follows as their corollary). Equation (9) defines the markovianity of the process $x(t)$. This means that no memory effect is germane to the process $x(t)$. To put it in another way, we may take equation (9) as a definition of a semigroup and the path-integral representation (4) as its consequence. So the circle of reasoning pertaining to the white-noise case of the Langevin force $f(t)$ has essentially become closed.

In the present paper, however, we pose another question: what if the Gaussian random force $f(t)$ is *not* endowed with the white-noise property? Is it then possible to derive a differential equation (similar to equ. (7) though not necessarily identical with it) for the probability density p? Our answer is positive for the case of the free diffusion (when $F = 0$). Moreover, we can derive the probability density $p(x, t; x_0)$ for a whole class of the forces $F(x, t)$: for the linear forces

$$F(x, t) = \alpha(t) - \beta(t)x, \tag{10}$$

where α, β are independent of x.

What is certain a priori is the loss of the markovianity of (i.e. the occurrence, may be implicit, of the memory effects in) $x(t)$ if the stationary zero-centred Gaussian random function $f(t)$ is defined by an autocorrelation function

$$W_f(t_1, t_2) \equiv W_f(|t_1 - t_2|) = \langle f(t_1)f(t_2)\rangle \tag{11}$$

with a *non-zero width*. The width of the function $W_f(|u|)$ (with the definition region $(-\infty, \infty)$ for u) determines a correlation time $\tau_f > 0$ of the process $f(t)$.

In practice one employs, as a rule, some standard autocorrelation function for W_f. We shall consider two such functions and so we define two cases:

Case a:

$$W_f^a(|u|) = \frac{\Lambda}{2\tau_f}\exp\left(-\frac{|u|}{\tau_f}\right); \tag{12.a}$$

Case b:

$$W_f^b(|u|) = \frac{\Lambda}{2\tau_f}\exp\left(-\frac{\pi u^2}{4\tau_f^2}\right). \tag{12.b}$$

They both have been defined with the same variance

$$\sigma_f^2 = \langle [f(t)]^2\rangle = \frac{\Lambda}{2\tau_f} > 0. \tag{13}$$

Clearly, the white-noise case (formula (3)) corresponds to the limit when $\tau_f \to +0$. (Λ is fixed.)

There are not many examples of simply defined non-Markovian processes $x(t)$ yet. For this very reason, we believe that it is interesting — despite the restrictive choice of the linearity of F (formula (10)) — to analyse the processes $x(t)$ defined by the autocorrelation functions (12.a), (12.b) explicitly and to show to what extent they differ from their white-noise analogue. With this intention we shall discuss first, in Section II, free diffusions. Afterwards, in Section III, we shall devote attention to the generalized Ornstein-Uhlenbeck processes (taking the linear driving forces time-independent). Emphasis will be put on the spectral theory of such processes. Finally, in Section IV, we will suggest possibilities of further generalizations. In particular, we shall take the coefficients α, β of F time-dependent and shall also mention of the multidimensional version of the problem.

II. FREE DIFFUSIONS WITH NON-WHITE-NOISE LANGEVIN FORCES

The simplest case of our concern is the free diffusion. Then $\alpha = \beta = 0$. If t is fixed, we may consider

$$x(t) - x_0 = \mu\int_0^t dt' f(t') \tag{14}$$

as a single zero-centred Gaussian random variable. Its variance is

$$\langle [x(t) - x_0]^2\rangle = \mu^2\int_0^t\int_0^t dt'\,dt''\, W_f(|t' - t''|). \tag{15}$$

It is natural to define the diffusion coefficient D by the integral

$$D = \frac{1}{2}\mu^2 \int_{-\infty}^{\infty} du\, W_f(|u|). \tag{16}$$

We regard formula (16) as convenient for any autocorrelation function $W_f(|u|)$ of the stationary Gaussian random functions $f(t)$. Quite generally (i.e. not only in the white-noise case) formula (16) implies formula (5).

We find it useful to introduce a new "chronometry" selectively for each autocorrelation function $W_f(|u|)$: to implement this idea, we introduce a chronometric variable χ, requiring it to be an increasing function of t (for $t > 0$, such that $\chi = \chi_0 = 0$ at $t_0 = 0$), by the relation

$$\langle [x(t) - x_0]^2 \rangle = 2D\chi. \tag{17}$$

If $\tau_f \to +0$, then $\chi \to t$, since relation (17) must not contradict the white-noise result (when the r.h. side equals $2Dt$). To have a dimensionless time variable we define the "chronometric ratio" r:

$$\chi = rt \tag{18}$$

(for each autocorrelation function $W_f(|u|)$ independently; r is an increasing function — such that $0 < r < 1$ for $t > 0$ — of the dimensionless argument t/τ_f). Formulae (15)—(18) give us the following expression demonstrating the functional dependence of r on W_f:

$$r = \frac{1}{t}\int_0^t \int_0^{t'} dt'\,dt''\, W_f(|t' - t''|) \Big/ \int_{-\infty}^{\infty} du\, W_f(|u|). \tag{19}$$

If we insert the autocorrelation functions (12.a), (12.b) here, we obtain, respectively, the functions

$$r \equiv r_a(\xi) = 1 - \frac{1}{\xi}(1 - e^{-\xi}), \tag{20.a}$$

$$r \equiv r_b(\xi) = \frac{1}{\xi}\int_0^{\xi} d\xi'\, \operatorname{erf} \xi' = \operatorname{erf} \xi - \frac{1}{\sqrt{\pi}} \frac{1 - \exp(-\xi^2)}{\xi}, \tag{20.b}$$

where

$$\xi = \frac{t}{\tau_f} > 0. \tag{21}$$

($\operatorname{erf} \xi$ denotes the error function,

$$\operatorname{erf} \xi = (2/\sqrt{\pi}) \int_0^{\xi} du \exp(-u^2).)$$

The values of r are indeed bounded between zero and unity; $r = 0$ for $\xi = 0$ and $r \to 1$ for $\xi \to \infty$ (Fig. 1).

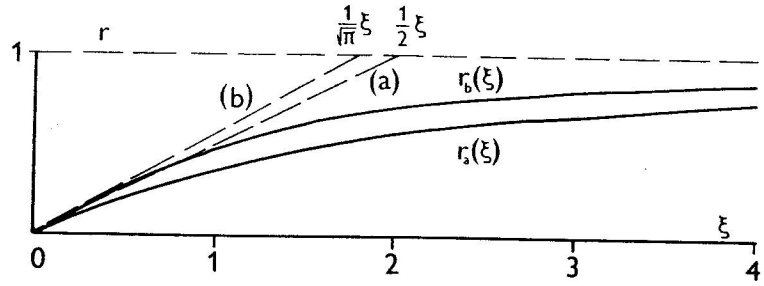

Fig. 1. Dependence of the chronometric ratio r on the reduced time variable $\xi = t/\tau_f$ for the free diffusion in two cases (a. and b.) distinguished by the autocorrelation function W_f (W_f^a and W_f^b). The dashed straight lines correspond to the tangents to the curves $r = r_a(\xi)$ and $r = r_b(\xi)$ at $\xi \to +0$.

The probability density of the random variable $x(t) - x_0$ is equal to the conditional probability density $p(x, t; x_0)$ of the process $x(t)$:

$$p(x, t; x_0) = \frac{1}{(4\pi D\chi)^{1/2}} \exp\left[-\frac{(x - x_0)^2}{4D\chi}\right] \tag{22}$$

($\chi > 0$).

Evidently, p fulfils — for each autocorrelation function W_f of the process $f(t)$ defined above — the equation

$$\frac{\partial p}{\partial \chi} = D \frac{\partial^2 p}{\partial x^2}. \tag{23}$$

Thus, χ plays a role of a "proper diffusin time" of the non-Markovian diffusion corresponding to W_f. The function $p(x, t; x_0)$ is, of course, the fundamental solution (i.e. Green's function) to equation (23).

Equ. (23) indicates that we may speak of a Markovian process $x(\chi)$, regardless of the non-markovianity of the original process $x(t)$. This possibility is, however, exceptional and does only concern the free-diffusion case. As we shall soon show, if we take $\beta \neq 0$, we can neither use the concept of the "proper diffusion time" as something replacing effectively the normal time t (as for $\beta = 0$) nor formulate a simple equation for p with coefficients independent of the initial point x_0.

III. GENERALIZED ORNSTEIN-UHLENBECK PROCESSES (WITH NON-WHITE-NOISE LANGEVIN FORCES)

Now we intend to analyse the case when α, β are arbitrary constants. However, the constant α can always be easily removed (from consideration): by making the substitution $x(t) = \varphi(t) + \mu\alpha t$, we obtain the Langevin equation for $\varphi(t)$ in which α is no more present. Thus, without loss of generality, we may ignore the constant α and focus attention to the equation

$$\dot{x}(t) + \beta\mu x(t) = \mu f(t). \tag{24}$$

Note that $x(t)$ need not correspond to a position of a *particle* at all (observable, say, with the aid of a microscope): it is usual to take the notion "Brownian motion" in an abstract sence. (In our preceding paper [6], we have analysed a stochastic integrating circuit, or an RC-chain, with a stationary Gaussian random source voltage $V_S(t)$ in the role of $f(t)$; we have used $Q(t)$ — the instantaneous electric charge on one of the capacitor electrodes — for $x(t)$, the conductance $1/R$ for μ and the reciprocal value $1/C$ of the capacitance for β.)

If the zero-centred Gaussian random function $f(t)$ is defined with the white-noise autocorrelation function (3), then $x(t)$ is nothing but the classical Ornstein-Uhlenbeck process [7]. Now we will show what happens with the Ornstein-Uhlenbeck process if $f(t)$ is not of the white-noise type. In such a case, we speak of the generalized Ornstein-Uhlenbeck processes.

III.1. Probability densities

III.1.1. The single probability density $p(x, t; x_0)$

Owing to the linearity of equation (1.2) with $F = -\beta x$, the expression

$$x(t) - x_0 \exp(-\beta\mu t) = \mu \int_0^t \mathrm{d}t' \exp[\beta\mu(t' - t)] f(t') \tag{25}$$

may be considered, for any fixed value $t > 0$, as a single zero-centred Gausian random variable. Its variance is

$$\langle [x(t) - x_0 \exp(-\beta\mu t)]^2 \rangle = \mu^2 \int_0^t \int_0^t \mathrm{d}t' \mathrm{d}t'' \exp[-\beta\mu(t' + t'')]\, W_f(|t' - t''|). \tag{26}$$

On a parallel with formula (19) we define the function

$$\psi(t) = \int_0^t \int_0^t \mathrm{d}t' \mathrm{d}t'' \exp[-\beta\mu(t' + t'')]\, W_f(|t' - t''|) \Big/ \int_{-\infty}^{\infty} \mathrm{d}u\, W_f(|u|). \tag{27}$$

Clearly, for $\beta \to +0$ we obtain that $\psi \to \chi$ (cf. definitions (18), (19)). Keeping in mind the definition of the diffusion coefficient D by formula (16), we can write at once the probability density

$$p(x, t; x_0) = \frac{1}{(4\pi D\psi)^{1/2}} \exp\left\{ -\frac{[x - x_0 \exp(-\beta\mu t)]^2}{4D\psi} \right\}. \tag{28}$$

(It would be apposite to call ψ — on the analogy of the variable χ of Section II — the "proper time", were there not one reason against it: viz., the numerator in the exponent of expression (28) involves very logically the normal time t, not the variable ψ.)

For the correlation time τ_f equal to zero, we obtain the original Ornstein-Uhlenbeck result:

$$\psi_{OU}(t) = \lim_{\tau_f \to +0} \psi(t) = \frac{1 - \exp(-2\beta\mu t)}{2\beta\mu}. \tag{29}$$

Thus, our distribution function (28) is the most straightforward generalization of the classical Ornstein-Uhlenbeck distribution function. (A brief summary of the original Ornstein-Uhlenbeck theory is in § 3.8.4 of the manual [3].)

Now we will enter upon a discussion about whether there is a possibility to derive a differential equation for the general probability density

$$P(x, t) = \int \mathrm{d}x_0 p(x, t; x_0) P_0(x_0) \tag{30}$$

such that $P(x, 0) = P_0(x)$ (a function given in advance). If $\psi(t)$ is different from $\psi_{OU}(t)$, we do not expect that there may exist a purely differential equation for p (given by expression (28)). In regard to the appearance of the memory in the process $x(t)$, we do rather expect that $P(x, t)$ should obey an integro-differential equation — an equation that, as far as we know, has not been derived yet. Of course, the equation must be linear, i.e. valid for any prescribed initial function $P_0(x)$ in integral (30). (Only the normalization to unity, $\int \mathrm{d}x\, P_0(x) = 1$, must be required.) For the original Ornstein-Uhlenbeck case, we write:

$$P_{OU}(x, t) = \int \mathrm{d}x_0 p_{OU}(x, t; x_0) P_0(x_0), \tag{31}$$

$$\frac{\partial P_{OU}}{\partial t} = D \frac{\partial^2 P_{OU}}{\partial x^2} - \beta\mu \frac{\partial}{\partial x}(x P_{OU}), \tag{32}$$

$$\frac{\partial p_{OU}}{\partial t} = D \frac{\partial^2 p_{OU}}{\partial x^2} - \beta\mu \frac{\partial}{\partial x}(x p_{OU}). \tag{33}$$

It is not at all clear how to write a memory-involving equation which would have its counterpart in equation (32) in the Ornstein-Uhlenbeck limit: we mean an equation valid for *every* function (30), not only for the function $p(x, t; x_0)$.

Namely, if the function $p(x, t; x_0)$ is taken alone in case of the linear driving force $F = -\beta x$, we can easily write a differential equation for it (although not of the Fokker-Planck type). For this purpose, it is suitable to use the following new variables:

$$X = x - x_0 \exp(-\beta\mu t), \tag{34}$$

$$T = \psi(t), \tag{35}$$

where $T_0 = \psi(t_0) = 0$ for $t_0 = 0$. The function $p(x, t; x_0) \equiv p(X, T; x_0)$ (cf. formula (28)) satisfies simply the equation

$$\frac{\partial p}{\partial T} = D\frac{\partial^2 p}{\partial x^2}. \tag{36}$$

Hence, as

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \frac{1}{\psi'}\left(\frac{\partial}{\partial t} - \beta\mu x_0 \, e^{-\beta\mu t}\frac{\partial}{\partial x}\right),$$

we obtain the equation

$$\frac{\partial p}{\partial t} = D\psi'\frac{\partial^2 p}{\partial x^2} - \beta\mu x_0 \, e^{-\beta\mu t}\frac{\partial p}{\partial x}, \tag{37}$$

where $\psi' = d\psi/dt$. Unfortunately, one of the coefficients at the differential operators in equ. (37) involves x_0 and so the differential operator of the r.h. side of equ. (37) is not applicable to the general function $P(x, t)$ given by integral (30); that is why we must exclude equation (37) from further consideration.

Nevertheless, even though lacking a universal equation for $P(x, t)$ in a form of an evolution equation (a generalized Fokker-Planck equation), we may assert that integral (30) with the kernel $p(x, t; x_0)$ given by expression (28) does represent the probability density $P(x, t)$ (for $F = -\beta x$ and any $P_0(x)$) quite *exactly*, whichever stationary Gaussian (zero-centred) random function $f(t)$ is used.

III.1.2. The joint (multisite and multitime) probability density $p_n(x_1, t_1, \ldots, x_n, t_n; x_0)$

Consider n events: event 1, ..., event n. We define the event i by stipulating that $x_i = x(t_i)$ $(t_i > 0; i = 1, \ldots, n)$. On an "$x$ versus t" chart, we identify the event i with the point t_i, x_i. In regard to the mapping (25), every two events, i and j, are correlated. Besides, the joint probability density p_n for all the events 1, ..., n depends on the initial position x_0 of the process $x(t)$. As the Gaussian random variables $x_i - x_0 \exp(-\beta\mu t_i)$ are zero-centred, we define the covariance matrix $\boldsymbol{W}_x$ with the elements

$$W_x(t_i, t_j) = \langle [x_i - x_0 \exp(-\beta\mu t_i)][x_j - x_0 \exp(-\beta\mu t_j)] \rangle. \tag{38}$$

The matrix $\boldsymbol{W}_x$ is symmetric and independent of x_0 — cf. the r.h. side of relation (25). Let $\boldsymbol{X}$ be the n-dimensional vector (written as a column) with the elements $x_i - x_0 \exp(-\mu t t_i)$; $i = 1, \ldots, n$. According to the general theory of the Gaussian multivariate probability densities (cf. e.g. [3]) we may write

$$p_n(1, \ldots, n; x_0) = (2\pi)^{-n/2} |\det \boldsymbol{W}|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{X}^T . \boldsymbol{W}^{-1} . \boldsymbol{X}\right), \tag{39}$$

where $\boldsymbol{W}^{-1}$ is the matrix inverse to $\boldsymbol{W}$, i.e. $\boldsymbol{W}^{-1} . \boldsymbol{W} = \boldsymbol{1}$. det $\boldsymbol{W}$ is the determinant to $\boldsymbol{W}$ and $\boldsymbol{X}^T$ is the transposed vector (written as a row) to $\boldsymbol{X}$.

When $n = 2$, the execution of the multiplications and summations required by formula (39) is not awkward yet. So we obtain the probability density

$$p_2(1, 2; x_0) = \frac{1}{2\pi}(\det \boldsymbol{W})^{-1/2} \exp\left(-\frac{W_{22}x_1^2 - 2W_{12}x_1x_2 + W_{11}x_2^2}{2 \det \boldsymbol{W}}\right), \tag{40}$$

where

$$\det \boldsymbol{W} = W_{11}W_{22} - W_{12}^2 > 0. \tag{41}$$

(We have written for brevity $W_{ij} \equiv W_x(t_i, t_j)$.)

III.1.3. The autocorrelation function $W_x(t_1, t_2)$

According to formula (39), the multivariate probability density is determined uniquely by the matrix elements W_{ij}. Thus, the total information about the statistical properties of the process $x(t)$ can essentially be determined from the sole function $W_x(t_1, t_2)$ $(t_1 > 0, t_2 > 0)$. The function $W_x(t_1, t_2)$ is the autocorrelation function of the process $x(t)$. With respect to relation (25) we can write explicity the relationship between W_x and W_f:

$$W_x(t_1, t_2) = \mu^2 \exp[-\beta\mu(t_1 + t_2)] \int_0^{t_1} dt' \int_0^{t_2} dt'' \exp[\beta\mu(t' + t'')] \, W_f(|t' - t''|). \tag{42}$$

(Check the symmetry: $W_x(t_1, t_2) = W_x(t_2, t_1)$.) The process $x(t)$ is not stationary; it becomes, however, stationary when the time t is much longer than the correlation time τ_f of the process $f(t)$. Therefore, we expect that

$$W_x(t_1, t_2) \to W_{x,as}(|t_1 - t_2|) \qquad \text{if } t_1 \to \infty, \; t_2 \to \infty \tag{43}$$

whilst $|t_1 - t_2| < \infty$ (asymptonic stationarity).

We intend to calculate the function $W_x(t_1, t_2)$ explicitly for case *a.* and case

b. Then, in paragraph III.1.4, we shall calculate, utilizing the asymptotic function $W_{x,as}$, the spectral density of the process $x(t)$ (for both cases).

In general, we write

$$W_x(t_1, t_2) = W_{x,as}(|t_1 - t_2|) + \Delta W_x(t_1, t_2). \tag{44}$$

For our further calculations, it is formally advantageous to use the parameters

$$a = \beta\mu \geqslant 0, \tag{45}$$

$$b = 1/\tau_f > 0. \tag{46}$$

As it is seen from formula (42), we have to calculate the integral

$$I = \frac{2}{\Lambda}\int_0^{t_1} dt' \int_0^{t_2} dt'' \exp[a(t' + t'')]\, W_f(|t' - t''|). \tag{47}$$

This integral is invariant under the exchange $t_1 \to t_2$, $t_2 \to t_1$, so it is sufficient to perform the integration under the condition that $t_2 > t_1$. Then we have to integrate over the three regions on the (t', t'')-plane shown in Fig. 2: two triangle

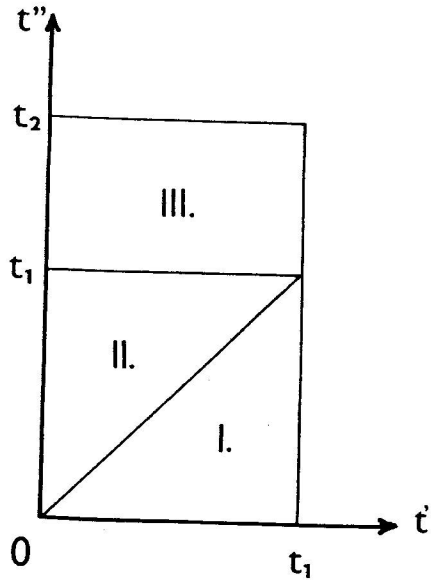

Fig. 2. The integration region in the calculation of the functions $W_x(t_1, t_2)$.

regions (I and II) and one oblong region (III). When integrating over the triangle regions, we may make use of the new variables

$$u = t'' - t', \quad v = t' + t''.$$

The region I and region II contribute to the integral (47) equally; for their contribution $I_I + I_{II} = 2I_I$ we obtain the expression

$$I_I + I_{II} = \frac{2}{\Lambda}\int_0^{t_1} du\left(\int_0^{t_1} dv + \int_0^{2t_1 - u} dv\right)\exp(av)\, W_t(|u|) =$$

$$= \frac{2}{\Lambda a}\int_0^{t_1} du[\exp(2at_1)\exp(-au) - 1]\, W_t(|u|). \tag{48}$$

The contribution of the region III to the integral I is

$$I_{III} = \frac{2}{\Lambda}\int_0^{t_1} dt' \int_0^{t_2} dt'' \exp[a(t' + t'')]\, W_f(t'' - t'). \tag{49}$$

The autocorrelation function $W_x(t_1, t_2)$ is given by the expression

$$W_x(t_1, t_2) = D\exp[-a(t_1 + t_2)](I_I + I_{II} + I_{III}), \tag{50}$$

where D is given by formula (5).

Case a.:

When using the function W_f^a (definition (12.a)) in formulae (48), (49), we obtain the results:

$$I_I^a + I_{II}^a = \frac{1}{a}\{\exp(2at_1) - 1\} - \frac{1}{a - b}\{1 - \exp[-(b - a)t_1]\} -$$

$$- \frac{1}{b + a}\{\exp(2at_1) - \exp[-(b - a)t_1]\},$$

$$I_{III}^a = \frac{b}{b^2 - a^2}\{\exp[(b + a)t_1] - 1\}\{\exp[-(b - a)t_1] - \exp[-(b - a)t_2]\}.$$

The sum of these expressions, when multiplied by $D\exp[-a(t_1 + t_2)]$, gives the function $W_x^a = W_{x,as}^a + \Delta W_x^a$ for $t_2 > t_1$. Yet the same result must be obtained for $t_2 < t_1$. Thus, after some regrouping of the terms, we obtain, respectively, the stationary and nonstationary part of the autocorrelation function W_x^a:

$$W_{x,as}^a(|t_1 - t_2|) = \frac{D}{a}\frac{b}{b + a}\left\{\exp(-a|t_1 - t_2|) + \right.$$

$$\left. + \frac{a}{b - a}[\exp(-a|t_1 - t_2|) - \exp(-b|t_1 - t_2|)]\right\}, \tag{51.a}$$

$$\Delta W_x^a(t_1, t_2) = \frac{D}{a}\left\{\frac{a}{b-a}\left(\frac{b}{b+a}\right)[\exp(-bt_1 + at_2) + \exp(-at_1 - bt_2)] - \right.$$

$$\left. - \exp[-a(t_1 + t_2)] - \exp[-a(t_1 + t_2)]\right\}. \tag{52.a}$$

Case b.:

Using the function W_f^b (definition (12.b)) we obtain the results:

$$I_I^b + I_{II}^b = \frac{1}{a}\left\{\exp\left(\frac{a^2}{\pi b^2}\right)\exp(2at_1)\left[\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\, b\left(t_1 + \frac{2a}{\pi b^2}\right)\right) - \right.\right.$$

$$\left.\left. - \operatorname{erf}\left(\frac{a}{\sqrt{\pi}b}\right)\right] - \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\, bt_1\right)\right\},$$

$$I_{III}^b = \frac{1}{2a}\exp\left(\frac{a^2}{\pi b^2}\right)[\Phi_{a,b}(t_1, t_2) - \Phi_{a,b}(t_1, t_1)],$$

where

$$\Phi_{a,b}(t_1, t_i) = \exp(2at_i)\left[\operatorname{erf}\left(\frac{\sqrt{\pi}b}{2}\left(t_i - \frac{6a}{\pi b^2}\right)\right) - \operatorname{erf}\left(\frac{\sqrt{\pi}b}{2}t_i - t_1 - \frac{6a}{\pi b^2}\right)\right] -$$

$$- \operatorname{erf}\left(\frac{\sqrt{\pi}}{b}\left(t_i - \frac{2a}{\pi b^2}\right)\right) + \exp(2at_1)\operatorname{erf}\left(\frac{\sqrt{\pi}b}{2}t_i - t_1 - \frac{2a}{\pi b^2}\right)\right),$$

$i = 1, 2$.

To obtain the fuction $W_{x,as}^b$ we must assemble those terms of the sum $I_I^b + I_{II}^b + I_{III}^b$ which, when multiplied by the factor $D\exp[-a(t_1 + t_2)]$, remain potentially non-zero for $t_1 \to \infty$, $t_2 \to \infty$. The remaining terms will give the function ΔW_x^b. So we arrive at the result:

$$W_{x,as}^b(|t_1 - t_2|) = \frac{D}{2}\exp\left(\frac{a^2}{\pi b^2}\right)\operatorname{erfc}\left(\frac{a}{\sqrt{\pi}b}\right)\exp(-a|t_1 - t_2|), \tag{51.b}$$

where $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. (Note that $\operatorname{erf}(x) \to 1$ for $x \to \infty$.) To express the function ΔW_x^b concisely, let us introduce the denotations

$$\bar{t} = \frac{1}{2}(t_1 + t_2 - |t_1 - t_2|), \qquad \bar{T} = \frac{1}{2}(t_1 + t_2 + |t_1 - t_2|),$$

and use the function $\Phi_{a,b}$ defined above. Then we can write:

$$\Delta W_x^b(t_1, t_2) = \frac{D}{a}\left\{\exp\left(\frac{a^2}{\pi b^2}\right)\left[\exp(-a|t_1 - t_2|)\operatorname{erfc}\left(\frac{\sqrt{\pi}b}{2}\left(\bar{t} + \frac{2a}{\pi b^2}\right)\right) + \right.\right.$$

$$\left.\left. + \frac{1}{2}(\Phi_{a,b}(\bar{t}, \bar{T}) - \Phi_{a,b}(\bar{t}, \bar{t}))\right] - \exp(-a(t_1 + t_2))\operatorname{erf}\left(\frac{\sqrt{\pi}b}{2}\bar{t}\right)\right\}. \tag{52.b}$$

The white-noise (Ornstein-Uhlenbeck) limiting case:

If we let the parameter b tend to infinity in formulae (51), (52) (no matter whether we take into account case a. or case b.), we reproduce the Ornstein-Uhlenbeck classical results:

$$W_{x,as}^{OU}(|t_1 - t_2|) = \frac{D}{a}\exp(-a|t_1 - t_2|), \tag{53}$$

$$\Delta W_x^{OU}(t_1, t_2) = -\frac{D}{a}\exp[-a(t_1 + t_2)]. \tag{54}$$

(Note that the function $\Phi_{a,b}$, in case b., falls to zero if $b \to \infty$. Hence, the expressions in [ ] in formula (52.b) vanish for $b \to \infty$.)

III.1.4. The spectral density $S_x(\omega)$

The non-stationary part $\Delta W_x(t_1, t_2)$ (cf. formulae (52.a), (52.b), (54)) is only then of interest to experimentalists when there is a means of measuring some ensemble behaviour of the processes $x(t)$ (we mean an ensemble $\{x(t)\}$ of the stochastic functions $x(t)$ described by the same stochastic equation — equ. (1.1) or (1.2)) during some initial period whose length of duration T is less than, say, ten times the values $a^{-1}(a = \beta\mu)$. If T is much longer than a^{-1}, then we may neglect the initial non-stationarity of $x(t)$; then we may identify the ensemble average of any functional $\mathcal{F}(\{x(t)\})$ with the value

$$\langle\mathcal{F}(\{x(t)\})\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\,\mathcal{F}(\{x(t)\}).$$

Moreover, then we can define the spectral density $S_x(\omega)$ for the process $x(t)$ by the well-known (Wiener-Khinchin) formula

$$S_x(\omega) = \int_{-\infty}^{\infty}\mathrm{d}u\exp(-\mathrm{i}\omega u)\, W_{x,as}(|u|) \tag{55}$$

(cf. [3]).

Our interest is to calculate $S_x(\omega)$ for both cases, *a.* and *b.*, with the aim to show whether the respective spectral densities $S_x^a(\omega)$, $S_x^b(\omega)$ do, or do not resemble, the Ornstein-Uhlenbeck spectral density $S_x^{OU}(\omega)$.

Case a.:

After substituting expression (51.a) for $W_{x,as}$ in formula (55) and carrying out the integration, we obtain the function

$$S_x^a(\omega) = 2D \frac{b^2}{(a^2 + \omega^2)(b^2 + \omega^2)}. \tag{56.a}$$

Case b.:

As the t-dependence is expression (51.b) is simpler than in expression (51.a), the calculation of the spectral density $S_x^b(\omega)$ is quite easy and gives the result:

$$S_x^b(\omega) = 2D \exp\left(\frac{a^2}{\pi b^2}\right) \operatorname{erfc}\left(\frac{a}{\sqrt{\pi} b}\right) \frac{1}{a^2 + \omega^2}. \tag{56.b}$$

Comparison with the white-noise (Ornstein-Uhlenbeck) case:

In the limit of the zero correlation time τ_f (i.e. for $b = 1/\tau_f \to \infty$) we obtain the Ornstein-Uhlenbeck function

$$S_x^{OU}(\omega) = 2D \frac{1}{a^2 + \omega^2}. \tag{57}$$

To compare the spectral densities $S_x^a(\omega)$, $S_x^b(\omega)$ and $S_x^{OU}(\omega)$ we take a as the frequency unit, i.e. introduce the dimensionless frequency variable

$$\zeta = \frac{\omega}{a}. \tag{58}$$

As it is seen from formulae (56.a), (56.b), the spectral densities $S_x^a(\omega)$, $S_x^b(\omega)$ depend on one dimensionless parameter, namely

$$\eta = \frac{b}{a}, \tag{59}$$

which grows to infinity in the Ornstein-Uhlenbeck limiting case.

In Fig. 3, we have plotted the reduced spectral densities

$$\sigma_\eta^a(\zeta) \equiv \frac{a^2}{2D} S_x^a(\omega) = \frac{1}{(1 + \zeta^2)(1 + \zeta^2/\eta^2)}, \tag{60.a}$$

$$\sigma_\eta^b(\zeta) \equiv \frac{a^2}{2D} S_x^b(\omega) = \exp\left(\frac{1}{\pi\eta^2}\right) \operatorname{erfc}\left(\frac{1}{\sqrt{\pi}\eta}\right) \frac{1}{1 + \zeta^2} \tag{60.b}$$

as functions of the variable $\zeta \geqslant 0$ with some fixed values of the parameter η. Clearly, all the curves corresponding to $S_x^a(\omega)$ with different fixed values of η

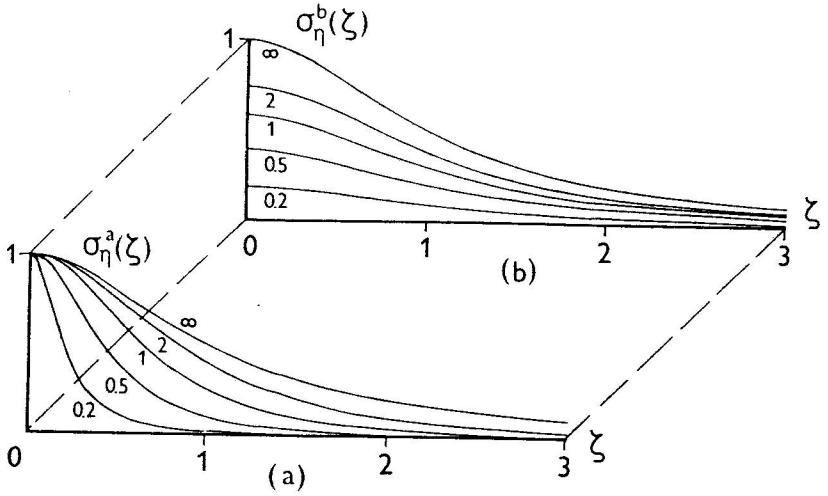

Fig. 3. The reduced spectral densities $\sigma_\eta(\zeta)$ for case a. and case b.; ζ is the reduced frequency, $\omega = \beta\mu\zeta$. The numerals at the curves mean the values of the parameter $\eta = (\beta\mu\tau_f)^{-1}$. In both cases, the upper curves (for $\eta = \infty$) are identical, corresponding to the Ornstein-Uhlenbeck limiting case.

meet at one point due to the common maximum value $S_x^a(0)$. Not so do behave the curves corresponding to $S_x^b(\omega)$: they do not start (at $\zeta = 0$) from one point; $S_x^b(0)$ is an increasing function of the parameter η with some asymptotic (maximum) value for $\eta \to \infty$. (Note, besides, that $S_x^b(0) \to 0$ for $\eta \to +0$.)

Forget, for the moment, the restriction on the non-negative values of ζ. (Really, it may formally be advantageous to use $\exp(-i\omega t)$ in the Fourier analysis and thus the negative values of ω will also necessarily be considered.) Then formulae (60.a), (60.b) correspond to "spectral lines" centred at $\zeta_0 = 0$. With this interpretation, we may say that the case b. gives "Lorentzian lines". (The same is true for the Ornstein-Uhlenbeck limiting case.)

To compare the shapes of the "lines" for various values of η, we normalize all the spectral densities to some common constant by defining the functions

$$s_\eta^a(\zeta) = \frac{1 + \eta}{\eta} \frac{1}{(1 + \zeta^2)(1 + \zeta^2/\eta^2)}, \tag{61.a}$$

$$s_\eta^b(\zeta) = \frac{1}{1 + \zeta^2}. \tag{61.b}$$

These are drawn Fig. 4, in obvious correspondence with the curves of Fig. 3. All the curves (b) of Fig. 3 have shrunk in Fig. 4 into one single curve (half of the

Lorentzian with the unit "halfwidth"). The area below each of the curves in Fig. 4 above the semi-axis $\zeta > 0$ is equal to $\pi/2$.

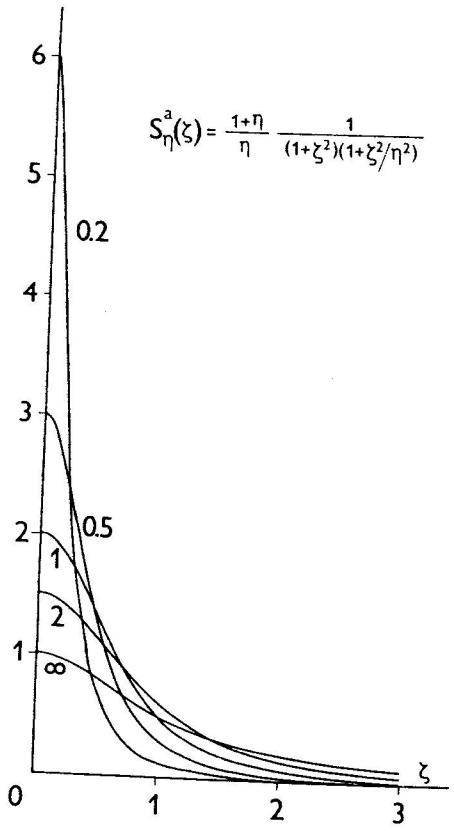

Fig. 4. The reduced spectral "half-lines" $s^a_\eta(\zeta) = [(1+\eta)/\eta]\,\sigma^a_\eta(\zeta)$ for case a. The numerals at the curves have the same meaning as in Fig. 3. The curve with $\eta = \infty$ corresponds to the Lorentzian. This, being validated for the Ornstein-Uhlenbeck limiting case, does also correspond to all the curves $s^b_\eta(\zeta)$ of case b.

IV. CONCLUDING REMARKS

The basic idea of the present paper can be summarized as follows. If we confine ourselves to the Langevin equations with driving forces $F(x, t)$ which are linear in x and with stochastic forces $f(t)$ which are Gaussian (but not necessarily white-noise) random functions, then we can directly construct the corresponding conditional probability densities $p(x, t; x_0)$ (with $t_0 = 0$) without any need to use an equation of the Fokker-Planck type. (We must say quite plainly that it is not so easy to formulate an equation analogous to the Fokker-Planck equation if the white-noise property of $f(t)$ is abandoned.)

So far we have dealt with the simplest one-dimensional case defined by the time-independent mobility μ and the time-independent coefficients α, β (cf. formula (10)). (It has been possible to take $\alpha = 0$.) Now we will derive the function $p(x, t; x_0)$ in the case when α, β, μ are arbitrary functions of the time variable t. We write, for $t > 0$,

$$x(t) = \bar{x}(t) + \xi(t) \tag{62}$$

and assume that

$$\bar{x}(0) = x_0, \qquad \xi(0) = 0. \tag{63}$$

The stochastic function $x(t)$ obeys the equation

$$\dot{x} - \mu(\alpha - \beta x) = \mu f \tag{64}$$

($x(0) = x_0$). Owing to the linearity of relation (64), we may consider $x(t)$ as a single Gaussian random variable for any fixed value $t > 0$. The function $\bar{x}(t)$ is deterministic; we define it as

$$\bar{x}(t) = \langle x(t) \rangle. \tag{65}$$

Since $f(t)$ is zero-centred, the function $\bar{x}(t)$ obeys the equation

$$\dot{\bar{x}} + \mu\beta\bar{x} = \mu\alpha. \tag{66}$$

The solution satisfying the first of conditions (63) is

$$\bar{x}(t) = x_0 \exp\left[-\int_0^t \mathrm{d}t'\,\mu(t')\,\beta(t')\right] + \int_0^t \mathrm{d}t''\,\mu(t'')\,\alpha(t'') \exp\left[-\int_{t''}^t \mathrm{d}t'\,\mu(t')\,\beta(t')\right]. \tag{67}$$

The stochastic function $\xi(t)$ obeys the equation $\dot{\xi} + \mu\beta\xi = \mu f$. (68)

The solution satisfying the second of the conditions (63) is

$$\xi(t) = \int_0^t \mathrm{d}t'')\,\mu(t'')f(t'') \exp\left[-\int_{t''}^t \mathrm{d}t'\,\mu(t')\,\beta(t')\right]. \tag{69}$$

If t is fixed, $\xi(t)$ is a zero-centred Gaussian random variable whose variance

$$\sigma_x^2(t) = \langle \xi^2(t) \rangle \tag{70}$$

is a linear functional of the autocorrelation function $W_f(|t_1 - t_2|)$ (formula (11)), namely:

$$\sigma_x^2(t) = \int_0^t\int_0^t \mathrm{d}t_1\mathrm{d}t_2\,\mu(t_1)\,\mu(t_2)\,W_f(|t_1 - t_2|) \exp\left[-\left(\int_{t_1}^t + \int_{t_2}^t\right)\mathrm{d}t'\,\mu(t')\,\beta(t')\right]. \tag{71}$$

The probability density p is then given by the expression

$$p(x, t; x_0) = \frac{1}{\sqrt{2\pi}\sigma_x(t)} \exp\left\{-\frac{[x - \bar{x}(t)]^2}{2\sigma_x^2(t)}\right\}. \tag{72}$$

New problems may arise when two or more Langevin equations are considered simultaneolusly. For instance, if we include into consideration the inertia of the Brownian particle, i.e. a mass $m > 0$, we obtain — still for a one-dimensional motion — the system of two equations:

$$\dot{x} - v = 0, \qquad m\dot{v} + \gamma v - F(x, t) = f(t). \tag{73}$$

This is a linear system if $F = \alpha - \beta x$. We refrain from writing the corresponding probability density $p(x, v, t; x_0, v_0)$ here. (For $\alpha = 0$, $\beta = \text{const}$, $\gamma = \text{const}$, cf. e.g. [8].)

In general, we may consider $\boldsymbol{X}(t)$ as a stochastic vector function of the time variable $t > 0$ in an n-dimensional Euclidean space. We represent $\boldsymbol{X}(t)$ as a column with elements $x_i(t)$; $i = 1, \ldots, n$. Similarly, we define $\boldsymbol{f}(t)$ as a column with elements $f_i(t)$, all taken as zero-centred Gaussian (possibly mutually correlated) random functions. The autocorrelation function (in the stationary case) of $\boldsymbol{f}(t)$,

$$\boldsymbol{W}_f(|t_1 - t_2|) = \langle \boldsymbol{f}(t_1)\, \boldsymbol{f}^T(t_2)\rangle, \tag{74}$$

is a symmetric $n \times n$ matrix function. (Recall that $\boldsymbol{f}\boldsymbol{f}^T$ is a dyadic.) Instead of equation (64), we write now the vector equation

$$\dot{\boldsymbol{X}} + \boldsymbol{\beta}\boldsymbol{X} = \boldsymbol{f} + \boldsymbol{\alpha}, \tag{75}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are two $n \times n$ matrices (which may be time-dependent). (We have incorporated $\boldsymbol{\mu}$ in the "forces" $\boldsymbol{F} = \boldsymbol{\alpha} - \boldsymbol{\beta}\boldsymbol{X}$ and $\boldsymbol{f}$.) Writing

$$\boldsymbol{X}(t) = \bar{\boldsymbol{X}}(t) + \boldsymbol{\Xi}(t) \tag{76}$$

and assuming that

$$\bar{\boldsymbol{X}}(0) = \boldsymbol{X}_0, \qquad \boldsymbol{\Xi}(0) = \boldsymbol{0}, \tag{77}$$

we obtain, taking for simplicity $\boldsymbol{\beta}$ as a time-independent matrix, the fuctions

$$\bar{\boldsymbol{X}}(t) = \exp(-\boldsymbol{\beta}t) \,.\, \boldsymbol{X}_0 + \int_0^t \mathrm{d}t' \exp[\boldsymbol{\beta}(t' - t)] \,.\, \boldsymbol{\alpha}(t'), \tag{78}$$

$$\boldsymbol{\Xi}(t) = \int_0^t \mathrm{d}t' \exp[\boldsymbol{\beta}(t' - t)] \,.\, \boldsymbol{f}(t'). \tag{79}$$

(Note that now the location of $\boldsymbol{X}_0$, $\boldsymbol{\alpha}(t')$ and $\boldsymbol{f}(t')$ in expressions (78), (79) may not be changed since we must respect the matrix multiplication which need not be commutative.)

Generally, for $n > 1$, we have to use the $n \times n$ matrix function

$$\boldsymbol{W}_X(t, t) = \langle \boldsymbol{\Xi}(t)\, \boldsymbol{\Xi}^T(t)\rangle \tag{80}$$

(instead of te function $\sigma_x^2(t)$ for $n = 1$). Explicitly:

$$\boldsymbol{W}_X(t, t) = \int_0^t \int_0^t \mathrm{d}t_1 \mathrm{d}t_2 \exp[\boldsymbol{\beta}(t_1 - t)] \,.\, \boldsymbol{W}_f(|t_1 - t_2|) \,.\, \exp[\boldsymbol{\beta}^T(t_2 - t)]. \tag{81}$$

The probability density p is given by the expression

$$p(\boldsymbol{X}; t; \boldsymbol{X}_0) = \frac{1}{(2\pi)^{n/2}|\det \boldsymbol{W}_X(t, t)|^{1/2}} \exp\left\{-\frac{1}{2}[\boldsymbol{X}^T - \bar{\boldsymbol{X}}^T(t)] \,.\right.$$

$$\left. \,.\, \boldsymbol{W}_X^{-1}(t, t) \,.\, [\boldsymbol{X} - \bar{\boldsymbol{X}}(t)]\right\}, \tag{82}$$

where $\boldsymbol{W}_X^{-1}(t, t)$ is the matrix inverse to $\boldsymbol{W}_X(t, t)$ (with fixed t).

If β were time-dependent, we could still derive $\boldsymbol{X}(t)$, $\boldsymbol{\Xi}(t)$ as functionals of $\alpha(t)$, $\beta(t)$ (in analogy to expressions (67), (68) valid for $n = 1$), but only by employing (for $n > 1$) one rather discomforting element of the theory: the time ordering operation. (We mean the time ordering as it was introduced by Dyson in the quantum field theory.) We desist from commenting further on this issue.

Our final remark concerns generalized Langevin equations which are stochastic integro-differential equations. To exemplify such an equation we take $n = 1$ and write

$$\dot{x} - I(\{x\}) = f. \tag{83}$$

We assume that $I(\{x\})$ is a linear functional of $x(t)$. For example,

$$I(\{x\}) = \int_0^t \mathrm{d}t' \, K(t - t') \, x(t'). \tag{84}$$

(The kernel $K(u)$ must be some reasonable function such that $K(u) \to 0$ if $u \to \infty$. Naturally, the linear functional $I(\{x\})$ could eventually be defined in a more general form, too.) If the kernel K is distinct from the Dirac delta function, the presence of the functional $I(\{x\})$ itself manifests the non-markovianity of the process $x(t)$, regardless of whether $f(t)$ has, or has not, the white-noise property. (As above, we do not require the white-noise property of $f(t)$.) Particularly, when we insert integral (84) into equation (83) and define $f(t)$ as stationary (random function), we may calculate the spectral density $S_x(\omega)$ of the process $x(t)$. (The process becomes stationary when $t \to \infty$.) Then we may inquire, say, about the influence of various memory kernels K upon the spectral density $S_x(\omega)$; or we may question how much the shape of $S_x(\omega)$ may deform from the

Lorentzian when a delay time (characterizing the kernel K) varies from zero to some large enough value — naturally, we should obtain a class of curves on the "S_x versus ω" chart resembling those shown in Fig. 4 although parametrized otherwise.

Of course, we could equally discuss the generalized Langevin equation problem for $n > 1$, i.e. for vector processes $\boldsymbol{X}(t)$.

Problems with generalized Langevin equations emerge (and have already been analysed to a large extent in literature) in the theory of superionic conductors or, more generally, in the theory of the self-diffusion in crystals respecting atomic (or molecular) interactions. Here, however, one new complicating point must be embodied in the theory: the presence of some periodic force $\boldsymbol{F}_{per}$ (due to the crystalline arrangement of the atoms), in addition to the functional $I(\{x\})$, in the generallized Langevin equations (cf., e.g., [9], [10]).

We are aware that the confinement to the linearity of the driving forces F — or, in general, the linearity of the functionals $I(\{x\})$ — is a drawback because of which various interesting applications are excluded from our range of view. Yet we do believe that the exact solvability of the problems tied to the linear stochastic equations, such as equation (83), is a doubtless advantage which should be utilized as far as possible. When one solves analytically linear problems in an exact way, one prepares, in fact, conditions under which eventually more demanding theories of cognate non-linear problems can be formed.

REFERENCES

[1] Kvasnikov, I. A.: *Termodinamika i statističeskaja fizika. Teorija neravnovesnych sistem.* Izd. MGU, Moskva 1987.

[2] Gichman, I. I., Skorochod, A. V.: *Stochastičeskie differencialnye uravnenija i ich priloženija.* Naukova dumka, Kiev 1982.

[3] Gardiner, C. W.: *Handbook of Stochastic Methods,* 2nd Ed. Springer, Berlin 1985. (Russian translation: Mir, Moskva 1986.)

[4] Bezák, V.: Acta Phys. Slov. *28* (1978), 12

[5] Bezák, V.: Acta Phys. Slov. *28* (1978), 24

[6] Bezák, V.: Acta Phys. Slov. *39* (1989), 22

[7] Uhlenbeck, G. E., Ornstein, L. S.: Phys. Rev. *36,* (1930), 823. (Copied in: *Selected Papers on Noise and Stoch. Processes,* ed. Wax, N. Dover, New York 1954.)

[8] Chandrasekhar, S.: Revs. Mod. Phys. *15,* (1943), 1. (Copied in: *Selected Papers on Noise and Stoch. Processes,* ed. Wax, N. Dover, New York 1954.)

[9] Munakata, T.: Phys. Rev. B *33* (1986), 8016.

[10] Coffey, W., Evans, M., Grigolini, P.: *Molecular Diffusion and Spectra.* Wiley, New York 1984. (Russian translation: Mir, Moskva 1987.)

Received November 11th, 1988.
Accepted for publication December 12th, 1988

ДИФФУЗИИ И ШУМ СВЯЗАННЫЕ С ОБОБЩЕННЫМИ ПРОЦЕССАМИ ОРНШТЕЙНА-УЛЕНБЕКА

Изучены одномерные диффузии (броуновские движения) описываемые уравнениями Ланжевена $\dot{x} - \mu F(x, t) = \mu f$ с общими гауссовскими случайными силами $f(t)$ (центрированными в нуле) в случае линейных действующих сил $F = \alpha - \beta x$. Автор называет такие процессы «обобщенными процессами Орнштейна-Уленбека» (которые — в отличие к подлинному ОУ-процессу — немарковские). Автор объясняет как конструировать плотности вероятности $p(x, t; x_0)$ для обобщенных ОУ-процессов прямо без применения таких концепций как уравнение Фоккера-Планка или функциональные интегралы по траекториям. (Конструкция возможна из-за гауссового характера процессов $x(t)$, если $F = \alpha - \beta x$.) В случае, когда $\mu = \text{const}$, $\alpha = 0$, $\beta = \text{const}$., автор исчисляет спектральную плотность $S_x(\omega)$ или $x(t)$ избирая две автокорреляционные функции $W_f(|u|)$ для $f(t)$: *a.* $W_f(|u|) \sim \exp(-|u|/\tau_f)$, *b.* $W_f(|u|) \sim \exp(-\pi u^2/(2\tau_f)^2)$. (В обоих случаях $\tau_f > 0$.) Автор показывает, что спектральная плотность $S_x(\omega)$ типа лоренцевой линии (центрированной в $\omega = 0$) в случае *a.*, но что в случае *b.* имеются значительные отклонения от лоренцевой формы. Все свои главные результаты представляет автор графически. Внимание предоставляется также многомерным обобщениям.