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ON STOCHASTIC *RC*-CHAINS

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A stochastic theory is formulated for the electrical *RC*-circuit with a source voltage $V_S(t)$ taken as a stationary Gaussian process of the Ornstein–Uhlenbeck type. The conductance $1/R(t)$ and the reciprocal capacitance $1/C(t)$ are considered either as stochastic functions (stationary Gaussian functions of the Ornstein–Uhlenbeck type) or as constants. It is shown how the mathematical analysis of the *RC*-circuit alters in three cases: 1. when R is stochastic and C constant, 2. when R is constant and C stochastic and 3. when both quantities, R and C, are stochastic. It is elucidated that the first two cases, but not the third, allow to derive results (e.g. cumulants of the capacitor voltage or charge) without any use of a perturbational calculation.

I. INTRODUCTION

Recently a mathematical monograph on stochastic systems has appeared [1] which seems to be appropriate to mathematically educated non-mathematicians. (I call so those people, including myself, who are experienced in a lot of the basic methods of mathematics — say "engineering mathematics" — and who wish, when reading a new mathematical manual, to learn something practicable without being forced to devote too much time to study a heavy "overconceptualized" treatise.) Ref. [1] is divided into twelve chapters and only the last deals with applications. One of the short paragraphs, which exemplify ten applications, has been entitled "*RC*-chain with a random capacitance". Being aware of problems in the noise theory, I was curious to learn what was, in fact, hidden by such a title. I was surprised to find a striking disparity between the title and the contents of the paragraph. Fortunately, a remedy was easy — if the mathematical contents of the paragraph were to stay unchanged, the title ought to read contrariwise, i.e. like this: "*RC*-chain with a constant capacitance and a random resistance".

My intent in the present paper is to show how to solve the problem as it was *declared*, but not solved, in Ref. [1]. To do this, I will draw a parallel with that problem which the author of Ref. [1] has actually solved, but misconceived physically. In addition, I will also pay attention to a further (i.e. third) problem which may be considered as a generalization of the former two. I hope that the juxtaposition of all these problems is interesting enough since similar problems (and, naturally, even much more complicated ones) occur in the noise theory of electronic devices and circuits.

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II. THE *RC*-CHAIN WITH RANDOM ELEMENTS

In the present paper I will analyse the circuit according to Fig. 1.

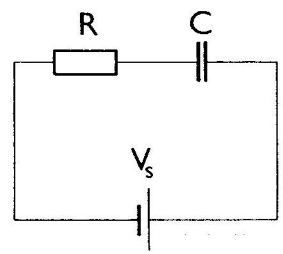

Fig. 1. The *RC*-circuit.

II.1. The *RC*-chain with R = const., C = const.

Let us suppose first, following the logic of introductory textbooks on electricity, that both the resistance R and the capacitance C are constant in time. Evidently, the source voltage V_S equals the sum of the Ohmic voltage RI and the capacitor voltage V_C,

$$V_S = RI + V_C. \tag{1}$$

If Q is the electric charge on one of the capacitor electrodes, we may write

$$V_C = Q/C \tag{2}$$

and

$$I = dQ/dt. \tag{3}$$

After differentiating equation (2) with respect to the time variable t we obtain the relation

$$I = C\frac{dV_C}{dt} \tag{4}$$

for $dC/dt = 0$. When substituting expression (4) into equation (1) we obtain the differential equation

$$\frac{dV_C}{dt} + \Gamma V_C = \Gamma V_S \tag{5}$$

with the damping constant

$$\Gamma = \frac{1}{RC} > 0. \tag{6}$$

If V_S is also a constant, we obtain the general solution

$$V_C(t) = V_C(0)\,e^{-t\Gamma} + V_S(1 - e^{-t\Gamma}). \tag{7}$$

Equation (1) then gives the expression for the electric current

$$I(t) = I(0)\,e^{-t\Gamma} \tag{8}$$

where

$$I(0) = \frac{V_S - V_C(0)}{R}. \tag{9}$$

Now let us make the ansatz (and from now on I shall hold by it throughout this paper) that the source voltage V_S is a stationary Gaussian process. Its mean value keeps constant in time:

$$\mu_S = \langle V_S(t) \rangle, \quad d\mu_S/dt = 0. \tag{10.1}$$

For simplicity, I shall take $V_S(t)$ as the so-called Ornstein – Uhlenbeck process. This has been defined as the stationary Gaussian process with the autocorrelation function

$$W_S(t_1, t_2) = \langle [V_S(t_1) - \mu_S][V_S(t_2) - \mu_S] \rangle \tag{10.2}$$

chosen as the simple exponential:

$$W_S(t_1, t_2) = \sigma_S^2 \exp(-|t_1 - t_2|\,\varkappa_S). \tag{11}$$

Here σ_S^2 is the dispersion of V_S

$$\sigma_S^2 = \langle [V_S(t) - \mu_S]^2 \rangle, \tag{12}$$

and $\varkappa_S > 0$ is the reciprocal value of what may be called the source correlation time τ_S, i.e. $\varkappa_S = 1/\tau_S$.

Owing to the linear relation between V_C and V_S (cf. equ. (5)), the voltage $V_C(t)$ — and the same statement is true, in accord with eq. (4), for the current $I(t)$ — represents again a Gaussian process. My first aim is to calculate the mean

$$\mu_C(t) = \langle V_C(t) \rangle \tag{13}$$

and the autocorrelation function

$$W_C(t_1, t_2) = \langle [V_C(t_1) - \mu_C(t_1)][V_C(t_2) - \mu_C(t_2)] \rangle. \tag{14}$$

As $d\Gamma/dt = 0$, the general stochastic solution to equ. (5) is

$$V_C(t) = \Gamma \int_0^t dt'\, V_S(t')\,e^{-(t-t')\Gamma} + V_C(0)\,e^{-t\Gamma}. \tag{15}$$

After averaging it, we obtain the relation

$$\mu_C(t) = \mu_S + [\mu_C(0) - \mu_S]\,e^{-t\Gamma}. \tag{16}$$

In my further analysis, I shall distinguish between the following two cases:

Case a. defined by the assumption that $V_C(0)$ is a deterministic value given in advance; then obviously $\mu_C(0) = V_C(0)$; and

Case b. defined by the assumption that $V_C(t)$ is a stationary Gaussian process; then $d\mu_C/dt = 0$ and hence $\mu_C(t) = \mu_C(0) = \mu_S = \text{const.}$ (Of course, the cases a. and b. have to be understood as two typical situations but not as all possibilities.)

Using the definition of W_C by formula (14) we obtain the result

$$W_C^a(t_1, t_2) = \Gamma^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'\, W_S(t_1', t_2') \exp[-(t_1 + t_2 - t_1' - t_2')\,\Gamma] \tag{17.a}$$

for the case a. In the case a. the dispersion

$$[\sigma_C^a(t)]^2 = W_C^a(t, t) \tag{18.a}$$

is an increasing function of the variable t, with some asymptotic value $W_C^a(\infty, \infty)$.

To calculate $W_C(t_1, t_2)$ for the case b. we may rewrite relation (15) into the form

$$V_C(t) - \mu_C - [V_C(0) - \mu_C]\,e^{-t\Gamma} = \Gamma \int_0^t dt'\,[V_S(t') - \mu_S]\,e^{-(t-t')\Gamma}. \tag{15.b}$$

(Recall that $\mu_C = \mu_S$ in case b.) By raising the equality (15.b) to the second power we obtain — after performing the averaging and some elementary calculation — the expression

$$W_C^b(0, t) = [\sigma_C^b]^2 \cosh(t\Gamma) - \frac{1}{2}\,\Gamma^2 e^{t\Gamma} \int_0^t \int_0^t dt_1'\,dt_2' \cdot W_S(t_1', t_2') \exp[-(t_1' + t_2')\,\Gamma] \tag{17.b}$$

for the case b. (Note that

$$[\sigma_C^b]^2 = W_C^b(0, 0) = W_C^b(t, t), \quad \text{i.e.} \quad d\sigma_C^b/dt = 0$$

in case b.) The function $W_C(t_1, t_2)$ is always symmetric but now, owing to the stationarity of $V_C(t)$, it depends only on the difference $t_1 - t_2$; therefore, $W_C^b(0, t) = W_C^b(t, 0) = W_C^b(|t|)$ in case b. If we take the limit $t \to \infty$ in formula (17.b), we obtain the formula

$$[\sigma_C^b]^2 = \Gamma^2 \int_0^\infty \int_0^\infty dt_1' dt_2' W_S(t_1', t_2') \exp[-(t_1' + t_2')\Gamma] \qquad (18.b)$$

for the case b.

Formulae (17), (18) hold for an arbitrary autocorrelation function $W_S(t_1, t_2)$. In particular, if we take into account the Ornstein – Uhlenbeck autocorrelation function (11), we obtain the results as follows:

Case a:

$$W_C^a(t_1, t_2) = \frac{\Gamma^2 \sigma_S^2 e^{-(t_1+t_2)\Gamma}}{\Gamma + \varkappa_S} \left\{ \frac{1}{\Gamma}(e^{2t_1\Gamma} - 1) + \right.$$

$$\left. + \frac{1}{\Gamma - \varkappa_S}[e^{2t_1\Gamma}(e^{(t_2-t_1)(\Gamma-\varkappa_S)} - 1) - e^{t_1(\Gamma-\varkappa_S)} - e^{t_2(\Gamma-\varkappa_S)} + 2] \right\} \qquad (19.a)$$

for $t_2 \geq t_1$. If $t_1 \leq t_2$, we must exchange t_1 and t_2.

$$[\sigma_C^a(t)]^2 = W_C^a(t, t) = [\sigma_C^a(\infty)]^2$$

$$\left[1 - e^{-2t\Gamma} - \frac{2\Gamma}{\Gamma - \varkappa_S}(e^{-t(\Gamma+\varkappa_S)} - e^{-2t\Gamma})\right] \qquad (20.a.1)$$

where

$$[\sigma_C^a(\infty)]^2 = \frac{\Gamma\sigma_S^2}{\Gamma + \varkappa_S}. \qquad (20.a.2)$$

In the special case when $\varkappa_S \to 0$ (i.e. in the approximation of the infinite source correlation time, $\tau_S \to \infty$), the autocorrelation function $W_C^a(t_1, t_2)$ is reduced to the form

$$\lim_{\varkappa_S \to 0} W_C^a(t_1, t_2) = \sigma_S^2(1 - e^{-t_1\Gamma})(1 - e^{-t_2\Gamma}). \qquad (21.a)$$

Case b:

$$W_C^b(0, t) = [\sigma_C^b]^2 \cosh(t\Gamma) - \frac{1}{2} e^{t\Gamma}[\sigma_C^a(t)]^2, \qquad (19.b)$$

$$[\sigma_C^b]^2 = [\sigma_C^a(\infty)]^2. \qquad (20.b)$$

Even if $[\sigma_C^a(t)]^2$ were not specified by formula (19.a), i.e. if we were not considering $W_S(t_1, t_2)$ in the Ornstein – Uhlenbeck form, relations (19.b), (20.b) would still be formally exact. The function $W_C^b(0, t)$ tends to zero for $t \to \infty$. This can

be seen directly in the case of the Ornstein – Uhlenbeck process $V_S(t)$; namely, if expressions (20.a) are inserted into formula (19.b)

$$W_C^b(0, t) = \frac{\Gamma\sigma_S^2}{\Gamma^2 - \varkappa_S^2}(\Gamma e^{-t\varkappa_S} - \varkappa_S e^{-t\Gamma}). \qquad (21.b.1)$$

On assuming simultaneously with $t \to \infty$ ($t\Gamma \gg 1$) that $\varkappa_S \to 0$ ($0 < \varkappa_S \ll \Gamma$), so that product $t\varkappa_S$ need not be either too small or too large, we obtain the simple exponential

$$\lim_{\substack{t \to \infty \\ \varkappa_S \to 0}} W_C^b(0, t) = \sigma_S^2 e^{-t\varkappa_S}. \qquad (21.b.2)$$

II.2. The *RC*-chain with random *R* and *C* = const.

Assume that Γ in equation (5) is fluctuating randomly. For simplicity, I take $\Gamma(t)$ as a stationary Gaussian process non-correlated with the process $V_S(t)$. Moreover, let us reduce a little the problem by accepting the assumption that $V_C(0)$ may be taken as a variable independent (statistically) of the process $\Gamma(t)$. So we have already completely defined the *mathematical* problem whose formal (perturbational) solution has been suggested in § 12.8 of Ref. [1]. From the viewpoint of *physics* we must, if we want to save the validity of equation (5), consider temporal fluctuations of the resistance R, or rather of the conductance $\Sigma = 1/R$, but not of the capacitance C (cf. relation (4)).

The formal stochastic solution to equ. (5) is

$$V_C(t) = V_C(0) \exp\left[-\int_0^t dt' \Gamma(t')\right] +$$

$$+ \int_0^t dt'' V_S(t'') \frac{\partial}{\partial t''} \exp\left[-\int_{t''}^t dt' \Gamma(t')\right]. \qquad (22)$$

This can be developed in Γ. Then, if we confine ourselves to the third order in Γ, we obtain all that Adomian has written about this problem in [1]. Let us go, however, further: Owing to the gaussianity of Γ, we can carry out the averaging of $V_C(t)$, $V_C(t_1) V_C(t_2)$, etc., with respect to Γ without the necessity to rely upon the development of $V_C(t)$ in Γ at all.

I shall denote by $\langle\rangle_S$ the averaging with respect to V_S when the averaging with respect to Γ is not considered yet. Similarly, $\langle\rangle_\Gamma$ will mean the averaging with respect to Γ alone. The non-indexed brackets $\langle\rangle$ will denote the averaging with respect to all random variables under consideration (now it means with respect to both V_S and Γ). Clearly, in our case

$$\langle\rangle = \langle\langle\rangle_S\rangle_\Gamma = \langle\langle\rangle_\Gamma\rangle_S.$$

When subjecting equ. (22) to the averaging $\langle\,\rangle_\Gamma$, one obtains $\langle V_C(t)\rangle_\Gamma$ as a stochastic functional of V_S:

$$\langle V_C(t)\rangle_\Gamma = V_C(0)\exp\left[-t\mu_\Gamma + \frac{1}{2}\int_0^t\int_0^t dt'd\bar{t}'\,W_\Gamma(t',\bar{t}')\right] + \int_0^t dt''\,V_S(t'')\frac{\partial}{\partial t''}\exp\left[-(t-t'')\mu_\Gamma + \frac{1}{2}\int_{t''}^t\int_{t''}^t dt'd\bar{t}'\,W_\Gamma(t',\bar{t}')\right] \tag{23}$$

where

$$\mu_\Gamma = \langle\Gamma(t)\rangle,\quad d\mu_\Gamma/dt = 0, \tag{24.1}$$

$$W_\Gamma(t_1,t_2) = \langle[\Gamma(t_1)-\mu_\Gamma][\Gamma(t_2)-\mu_\Gamma]\rangle_\Gamma. \tag{24.2}$$

Because of the stationarity of $\Gamma(t)$, the autocorrelation function $W_\Gamma(t_1,t_2)$ is, in fact, a function of the variable $|t_1-t_2|$. For instance, we may take

$$W_\Gamma(t_1,t_2) = \sigma_\Gamma^2\exp[-|t_1-t_2|\,\varkappa_\Gamma] \tag{25}$$

where σ_Γ^2, $\varkappa_\Gamma > 0$ are two constants. Then

$$\int_{t''}^t\int_{t''}^t dt_1'dt_2'\,W_\Gamma(t_1',t_2') = \frac{2\sigma_\Gamma^2}{\varkappa_\Gamma}\left[t-t''-\frac{1}{\varkappa_\Gamma}\left(1-e^{-(t-t'')\varkappa_\Gamma}\right)\right] \tag{26}$$

(for $t > t''$).

When putting expression (26) into formula (23), one observes that the condition

$$\sigma_\Gamma^2 < \mu_\Gamma\varkappa_\Gamma \tag{27}$$

must be fulfilled. Namely, the term in the mean $\langle V_C(t)\rangle_\Gamma$ proportional to $V_C(0)$ must vanish for $t\to\infty$. (Cf. formula (7).) Moreover, we must also presume the fulfilment of the condition

$$\sigma_\Gamma \ll \mu_\Gamma, \tag{28}$$

otherwise the gaussianity of $\Gamma(t)$ would not be a sound concept. (We know from physics that both the resistance R and the capacitance C are non-negative. Each Gaussian process $\Gamma(t)$ allows both positive and negative values of Γ at any distance from zero. However, we do commit no faux pas if we admit the gaussianity of the proces $\Gamma(t)$ provided that condition (28) is satisfied: then, viz., the statistical weight of the negative values of Γ occurring in $\Gamma(t)$ is negligible.)

If we express $V_C(t_1)$ and $V_C(t_2)$ according to formula (22) and take their product, we obtain a sum of four integral terms; these can be averaged with respect to Γ. So we obtain the expression for the second statistical moment of $V_C(t)$:

$$\langle V_C(t_1)\,V_C(t_2)\rangle_\Gamma = \sum_{i=1}^4 M_{i\Gamma}(t_1,t_2), \tag{29}$$

where

$$M_{1\Gamma}(t_1,t_2) = [V_C(0)]^2\exp[-(t_1+t_2)\mu_\Gamma]\exp\left\{\frac{1}{2}\left[\int_0^{t_1}\int_0^{t_1} dt_1'd\bar{t}_1'\,W_\Gamma(t_1',\bar{t}_1') + 2\int_0^{t_1}dt_1'\int_0^{t_2}dt_2'\,W_\Gamma(t_1',t_2') + \int_0^{t_2}\int_0^{t_2}dt_2'd\bar{t}_2'\,W_\Gamma(t_2',\bar{t}_2')\right]\right\}, \tag{29.1}$$

$$M_{2\Gamma}(t_1,t_2) = V_C(0)\exp(-t_1\mu_\Gamma)\int_0^{t_2}dt_2''\,V_S(t_2'')\frac{\partial}{\partial t_2''}\Big\{\exp[-(t_2-t_2'')\mu_\Gamma]\cdot \exp\left\{\frac{1}{2}\left[\int_0^{t_1}\int_0^{t_1}dt_1'd\bar{t}_1'\,W_\Gamma(t_1',\bar{t}_1') + 2\int_0^{t_1}dt_1'\int_{t_2''}^{t_2}dt_2'\,W_\Gamma(t_1',t_2') + \int_{t_2''}^{t_2}\int_{t_2''}^{t_2}dt_2'd\bar{t}_2'\,W_\Gamma(t_2',t_2')\right]\right\}\Big\}, \tag{29.2}$$

$$M_{3\Gamma}(t_1,t_2) = M_{2\Gamma}(t_2,t_1), \tag{29.3}$$

$$M_{4\Gamma}(t_1,t_2) = \int_0^{t_1}dt_1''\int_0^{t_2}dt_2''\,V_S(t_1'')\,V_S(t_2'')\frac{\partial^2}{\partial t_1''\partial t_2''}\cdot \Big\{\exp[-(t_1-t_1''+t_2-t_2'')\mu_\Gamma]\cdot\exp\left\{\frac{1}{2}\left[\int_{t_1''}^{t_1}\int_{t_1''}^{t_1}dt_1'd\bar{t}_1'\,W_\Gamma(t_1',\bar{t}_1') + 2\int_{t_1''}^{t_1}dt_1'\int_{t_2''}^{t_2}dt_2'\,W_\Gamma(t_1',t_2') + \int_{t_2''}^{t_2}\int_{t_2''}^{t_2}dt_2'd\bar{t}_2'\,W_\Gamma(t_2',\bar{t}_2')\right]\right\}\Big\}. \tag{29.4}$$

The derivation of these expressions has been based on the use of the formula

$$\left\langle\exp\left[\int dt\alpha(t)\,\Gamma(t)\right]\right\rangle_\Gamma = \exp\left[\mu_\Gamma\int dt\alpha(t)\right]\cdot\exp\left[\frac{1}{2}\int\int du\,dv\alpha(u)\,W_\Gamma(u,v)\,\alpha(v)\right] \tag{30}$$

valid for any Gaussian random function $\Gamma(t)$. In the role of the function $\alpha(t)$ (which may be more or less arbitrary but independent of $\Gamma(t)$), "indicators" have been used. (The "indicator" for an interval (a, b) is a function defined as unity for $a < t < b$ and as zero outside the interval (a, b).) Thus the calculation of the expressions $M_{i\Gamma}(t_1, t_2)$ was quite simple and straightforward, only the results look formidable at first sight.

I do not want more to exhibit such lengthy calculations — neither the author of Ref. [1] did. Instead, my aim will only be the calculation of the mean $\langle V_C(t)\rangle$.

Therefore, let us return to expression (23). We must average it with respect to V_S. When doing so, we need not distinguish between the case a. and case b. (defined in Section II.1). (Anyway, the distinction would become relevant if we were calculating higher moments of V_C, including the second $\langle V_C(t_1)\, V_C(t_2)\rangle$.) The result is

$$\mu_C(t) = \langle V_C(t)\rangle = \mu_S + [\mu_C(0) - \mu_S] \exp\left[-t\mu_\Gamma + \frac{1}{2}\int_0^t\int_0^t dt'\, d\bar{t}'\, W_\Gamma(t', \bar{t}')\right]. \quad (31)$$

In particular, if we take the autocorrelation function W_Γ in the Ornstein–Uhlenbeck form (24), we may employ expression (25) in the exponent:

$$\mu_C(t) = \mu_S + [\mu_C(0) - \mu_S] \exp\left\{-t\mu_\Gamma + \frac{\sigma_\Gamma^2}{\varkappa_\Gamma}\left[t - \frac{1}{\varkappa_\Gamma}(1 - e^{-t\varkappa_\Gamma})\right]\right\}. \quad (31.1)$$

In the approximation of large conductance correlation times, when $\tau_\Sigma = \tau_\Gamma = 1/\varkappa_\Gamma \to \infty$, we obtain the Gaussian temporal behaviour

$$\mu_C(t) \approx \mu_S + [\mu_C(0) - \mu_S] \exp\left(-t\mu_\Gamma - \frac{1}{2}t^2\sigma_\Gamma^2\varkappa_\Gamma\right) \quad (31.2)$$

if $0 < t\varkappa_\Gamma \ll 1$.

Generally, i.e. not only in the approximation of the large correlation times τ_Σ, the asymptotic temporal behaviour of $\mu_C(t)$ is

$$\mu_C(t) \approx \mu_S + [\mu_C(0) - \mu_S] \exp\left(-\frac{\sigma_\Gamma^2}{\varkappa_\Gamma^2}\right) \exp\left[-\left(\mu_\Gamma - \frac{\sigma_\Gamma^2}{\varkappa_\Gamma}\right)t\right] \quad (31.3)$$

if $1 \ll t\varkappa_\Gamma$. (Recall condition (27).)

This method, contrary to the method of the Γ-development proposed in Ref. [1], has allowed us to consider relatively high values of σ_Γ^2 (provided, however, that conditions (27), (28) are satisfied). Formula (31.1) for the Ornstein–Uhlenbeck process $\Gamma(t)$ is exact. Rewrite it in the form

$$\mu_C(t) = \mu_S + [\mu_C(0) - \mu_S] \exp(-t\mu_\Gamma)\cdot \exp\left[\frac{\sigma_\Gamma^2}{\varkappa_\Gamma^2}\varphi(t\varkappa_\Gamma)\right], \quad (32)$$

where

$$\varphi(x) = x + e^{-x} - 1. \quad (33)$$

The parameter $y = \sigma_\Gamma^2/\varkappa_\Gamma^2$ is a dimensionless degree of the intensity of the fluctuations of the damping parameter $\Gamma = 1/RC$. The variable $x = t\varkappa_\Gamma$ is a dimensionless time. If $y \to 0$, then $\sigma_\Gamma \to 0$ and this means that fluctuations of Γ

are absent; then $\mu_\Gamma = \Gamma =$ const. (Formula (29) is reduced to formula (15).) The function $\varphi(x)$ is shown in Fig. 2.

There are many reasons why the conductance may fluctuate. They have been described in modern books on noise (e.g. in [2], [3]). One of the most comprehensible models explaining the fluctuations of Σ, and hence also of $\Gamma = \Sigma/C = 1/RC$ (if C is constant), in semiconductors is related to electron-hole recombination processes. Several other noise mechanisms for semiconductors and metals are also well known. To decide which noise mechanism prevails (for which resistor and under which physical conditions) belongs to the competence of the solid-state physics and depends on many factors. I refrain from discussing these problems in this paper.

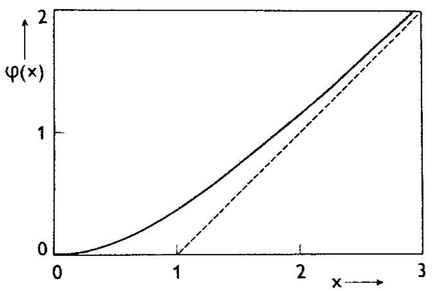

Fig. 2. The function $\varphi(x)$ (full line). The dashed line corresponds to the asymptote $x - 1$. The function $\varphi(x)$ determines (cf. formula (32)) the dependence of the average capacitor voltage μ_C on the conductance fluctuations.

II.3. The RC-chain with random C and $R =$ const.

This problem seems to me more academic than the problem of Section II.2, but I will suggest its solution. The academicity is connected with the fact that usually capacitance fluctuations are negligible against the conductance fluctuations. Perhaps we may consider two planar electrodes and a turbulent dielectric gas or liquid between them, so that the permittivity will fluctuate temporally around some constant value; then we may consider $C(t)$, or $1/C(t)$ if need be, as a stationary stochastic process. Other problems where the capacitance may fluctuate remarkably enough will occur, I believe, in the theory of electrets [4].

For $dC/dt \neq 0$, we may **not** start from equation (5). The best thing to do is first to write, using relations (1), (2) and (3), the equation for the charge Q on one of the capacitor electrodes:

$$\frac{dQ}{dt} + \Gamma Q = \frac{1}{R}V_S. \quad (34)$$

If we assume that $A(t) = 1/C(t)$ is a stationary Gaussian process such that

$$\mu_A = \langle A(t) \rangle, \quad d\mu_A/dt = 0, \tag{35.1}$$

$$W_A(t_1, t_2) = [A(t_1) - \mu_A][A(t_2) - \mu_A], \tag{35.2}$$

we obtain the relations

$$\mu_\Gamma = \frac{1}{R}\,\mu_A, \tag{36.1}$$

$$W_\Gamma(t_1, t_2) = \frac{1}{R^2}\,W_A(t_1, t_2). \tag{36.2}$$

(The reciprocal value $1/C$ might perhaps be called "anticapacitance".) Equation (34) is simpler than equation (5) since now the right-hand side does not involve a product of two stochastic functions (because R has been fixed).

The analysis of equ. (34) (when $\Gamma(t)$ is the process defined in Section II.2) is much the same as the analysis that has already been explained for equ. (5). Therefore, it need not be repeated here in all details.

For instance, if the initial charge $Q(0)$ is independent (statistically) of the process $\Gamma(t)$, we may write $\langle Q(t) \rangle_\Gamma$ as the following (stochastic) functional of V_S:

$$\langle Q(t) \rangle_\Gamma = Q(0) \exp\left[-t\mu_\Gamma + \frac{1}{2}\int_0^t \int_0^t dt'\, d\bar{t}'\, W_\Gamma(t', \bar{t}')\right] +$$
$$+ \frac{1}{R}\int_0^t dt''\, V_S(t'') \exp\left[-(t - t'')\mu_\Gamma + \frac{1}{2}\int_{t''}^t \int_{t''}^t dt'\, d\bar{t}'\, W_\Gamma(t', \bar{t}')\right]. \tag{37}$$

By averaging it with respect to V_S, we obtain the temporal behavior of the mean:

$$\mu_Q(t) = \langle Q(t) \rangle = \mu_Q(0) \exp\left[-t\mu_\Gamma + \frac{1}{2}\int_0^t \int_0^t dt'\, d\bar{t}'\, W_\Gamma(t', \bar{t}')\right] +$$
$$+ \frac{\mu_S}{R}\int_0^t dt'' \exp\left[-(t - t'')\mu_\Gamma + \frac{1}{2}\int_{t''}^t \int_{t''}^t dt'\, d\bar{t}'\, W_\Gamma(t', \bar{t}')\right] \tag{38}$$

($\mu_Q(0) = \langle Q(0) \rangle_S$). In the special case, when $\varkappa_\Gamma \to 0$ (i.e. in the approximation of the infinite capacitance correlation time $\tau_C = \tau_A = 1/\varkappa_C = 1/\varkappa_\Gamma \to \infty$), we see that $W_\Gamma \to \sigma_\Gamma^2 = \text{const}$. Then

$$\mu_Q(t) \approx \mu_Q(0) \exp\left(-t\mu_\Gamma + \frac{1}{2}t^2\sigma_\Gamma^2\right) -$$
$$- \frac{\mu_S}{R} \exp\left(-\frac{\mu_\Gamma^2}{2\sigma_\Gamma^2}\right)\left\{\text{daw}\left(\frac{1}{\sqrt{2}}\frac{\mu_\Gamma}{\sigma_\Gamma}\right) - \text{daw}\left[\frac{\sigma_\Gamma}{\sqrt{2}}\left(\frac{\mu_\Gamma}{\sigma_\Gamma^2} - t\right)\right]\right\} \tag{39.1}$$

where

$$\text{daw}(x) = \int_0^x du \exp(u^2 - x^2) = -\mathrm{i}\frac{\sqrt{\pi}}{2}\exp(-x^2)\,\text{erf}(\mathrm{i}x).$$

(This is Dawson's function; it has been tabulated in [5]. Its mathematical properties can be found, e.g., in [6]. Note that some authors used to prefer the Kramp function $\Phi(x)$ (cf. e.g. [7], p. 125),

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x du \exp(u^2) = -\mathrm{i}\,\text{erf}(\mathrm{i}x) = \frac{2}{\sqrt{\pi}}\exp(x^2)\,\text{daw}(x),$$

rather than the Dawson function daw (x).)

Formula (39.1) is acceptable for the times t satisfying the condition

$$0 < t < \frac{\mu_\Gamma}{\sigma_\Gamma^2}, \tag{39.2}$$

which is, owing to condition (28), consistent with the condition

$$t\sigma_\Gamma \ll 1. \tag{39.3}$$

II.4. The *RC*-chain with random *R* and random *C*

This problem is much more difficult than the preceding two problems. Let $\Sigma(t)$, $A(t)$ be two non-correlated stationary Gaussian processes ($\Sigma = 1/R$, $A = 1/C$). Then $\Gamma(t) = \Sigma(t)\,A(t)$ (cf. definition (6)) is also a stationary process. The third stochastic process of the problem in question, non-correlated with $\Sigma(t)$ and $A(t)$, is $V_S(t)$. If μ_Σ, μ_A are, respectively, the means of $\Sigma(t)$, $A(t)$, the mean of $\Gamma(t)$ is

$$\mu_\Gamma = \mu_\Sigma \mu_A \tag{40}$$

and the autocorrelation function of $\Gamma(t)$ is

$$W_\Gamma(t_1, t_2) = [W_\Sigma(t_1, t_2) + \mu_\Sigma^2][W_A(t_1, t_2) + \mu_A^2] - \mu_\Gamma^2. \tag{41}$$

Now our basic stochastic differential equation is this:

$$\frac{dQ}{dt} + \Sigma\cdot A\cdot Q = \Sigma\cdot V_S. \tag{42}$$

Its formal (stochastic) solution is

$$Q(t) = Q(0) \exp\left[-\int_0^t dt'\,\Gamma(t')\right] + \int_0^t dt''\,\Sigma(t'')\,V_S(t'') \exp\left[-\int_{t''}^t dt'\,\Gamma(t')\right]. \tag{43}$$

If we take $Q(0)$ as statistically independent of both the processes $\Sigma(t)$, $A(t)$, the averaging with respect to A is simple:

$$\langle Q(t)\rangle_A = Q(0)\exp\left[-\mu_A\int_0^t dt'\Sigma(t')\right]\cdot$$
$$\cdot\exp\left[\frac{1}{2}\int_0^t\int_0^t dt'\,d\bar{t}'\,\Sigma(t')\,\Sigma(\bar{t}')\,W_A(t',\bar{t}')\right]+$$
$$+\int_0^t dt''\Sigma(t'')\,V_S(t'')\exp\left[-\mu_A\int_{t''}^t dt'\Sigma(t')\right]\cdot$$
$$\cdot\exp\left[\frac{1}{2}\int_{t''}^t\int_{t''}^t dt'\,d\bar{t}'\,\Sigma(t')\,\Sigma(\bar{t}')\,W_A(t',\bar{t}')\right]. \qquad (44)$$

Here we have arrived at a point when it is impossible to proceed without applying a perturbational calculation. Namely, the next step should be to subject expression (44) to the averaging with respect to Σ but the presence of the $(\Sigma\cdot\Sigma)$-terms in the exponents prevents us from the possibility to do it *exactly* and at ease. On the other hand, in regard to the assumptions that

$$\sigma_\Sigma \ll \mu_\Sigma, \qquad (45)$$
$$\sigma_A \ll \mu_A \qquad (46)$$

(which have necessarily to be taken into consideration since otherwise the gaussianity of $\Sigma(t)$ and $A(t)$ would not be physically acceptable), we may expect that a certain perturbational calculation will be pretty effective. First of all, we may develop the $(\Sigma\cdot\Sigma)$-exponentials into series with regard to W_A. If we content ourselves with the approximation linear in W_A, we obtain the result

$$\langle Q(t)\rangle_A \approx Q(0)\left[1+\frac{1}{2}\int_0^t\int_0^t dt'\,d\bar{t}'\,\Sigma(t')\,\Sigma(\bar{t}')\,W_A(t',\bar{t}')\right]\cdot$$
$$\cdot\exp\left[-\mu_A\int_0^t dt_1'\Sigma(t_1')\right]+\int_0^t dt''\Sigma(t'')\,V_S(t'')\cdot$$
$$\cdot\left[1+\frac{1}{2}\int_{t''}^t\int_{t''}^t dt'\,d\bar{t}'\,\Sigma(t')\,\Sigma(\bar{t}')\,W_A(t',\bar{t}')\right]\exp\left[-\mu_A\int_{t''}^t dt_1'\Sigma(t_1')\right]. \qquad (47)$$

Similarly, writing $\Sigma = \mu_\Sigma + (\Sigma - \mu_\Sigma)$ in the exponents of expression (47), we may perform the development of the exponentials with respect to $\Sigma - \mu_\Sigma$. The majority of values of $\Sigma - \mu_\Sigma$ is of the same order of magnitude as σ_Σ. I shall assume, for simplicity, that σ_A (and then also W_A) is of the same order of magnitude as σ_Σ. (I regard this assumption as realistic but not as indispensable;

for example, it may hold if the fluctuations of $\Sigma(t)$ and $A(t)$ derive from the thermodynamic fluctuations in the resistor and in the capacitor which both are kept at the same temperature.)

Setting aside the W_A-term and putting, in the development of expression (47) in $\Sigma - \mu_\Sigma$, together the terms independent of $\Sigma - \mu_\Sigma$, we obtain the zero-order approximation:

$$\mu_Q^{(0)}(t) = \langle Q(t)\rangle^{(0)} = \mu_Q(0)\exp(-t\mu_\Sigma\mu_A) - \frac{\mu_S}{\mu_A}\left[1-\exp(-t\mu_\Sigma\mu_A)\right]. \qquad (48)$$

(Naturally, we could have derived this result from formula (38) for $R = 1/\mu_\Sigma$, if we had neglected W_Γ.)

We need not write the first-order expression $\langle Q(t)\rangle_A^{(1)}$ at all, since this, when averaged with respect to Σ, will vanish. The second-order term to $\langle Q(t)\rangle$ is

$$\langle Q(t)\rangle^{(2)} = \frac{1}{2}\mu_Q(0)\,\mu_\Sigma^2\int_0^t\int_0^t dt'\,d\bar{t}'\,W_A(t',\bar{t}')\exp(-t\mu_\Gamma) -$$
$$-\mu_S\mu_A\int_0^t dt''\exp[-(t-t'')\mu_\Gamma]\int_{t''}^t dt'\,W_\Sigma(t',t''). \qquad (49)$$

Following this method, we can derive $\langle Q(t)\rangle$ in the form of a series

$$\langle Q(t)\rangle = \langle Q(t)\rangle^{(0)} + \langle Q(t)\rangle^{(2)} + \langle Q(t)\rangle^{(4)} + \ldots. \qquad (50)$$

Of course, the calculations of high-order terms require more and more effort.

In particular, if we take W_Σ, W_A in the Ornstein–Uhlenbeck form, i.e. if

$$W_\Sigma(t_1,t_2) = \sigma_\Sigma^2\exp(-|t_1-t_2|\,\varkappa_\Sigma), \qquad (51)$$
$$W_A(t_1,t_2) = \sigma_A^2\exp(-|t_1-t_2|\,\varkappa_A), \qquad (52)$$

we can easily accomplish the integrations (according to formula (49)) in quadratures:

$$\int_0^t\int_0^t dt'\,d\bar{t}'\,W_A(t',\bar{t}') = \frac{2\sigma_A^2}{\varkappa_A}\left[t-\frac{1}{\varkappa_A}(1-e^{-t\varkappa_A})\right], \qquad (53)$$

$$\int_0^t dt''\exp[-(t-t'')\mu_\Gamma]\int_{t''}^t dt'\,W_\Sigma(t',t'') =$$
$$= \frac{\sigma_\Sigma^2}{\varkappa_\Sigma}\left\{\frac{1}{\mu_\Gamma}[1-\exp(-t\mu_\Gamma)] - \frac{1}{\varkappa_\Sigma+\mu_\Gamma}[1-\exp(-t(\varkappa_\Sigma+\mu_\Gamma))]\right\}. \qquad (54)$$

III. CONCLUDING REMARKS

The simple *RC*-chain (Fig. 1) represents a rudimentary example of a kind of problems which was and continues to be of considerable interest to electronics. In the present paper, I have considered, having followed the author of Ref. [1], the *RC*-chain under the assumption that the source voltage V_S has been defined as a stationary Gaussian process. Frankly speaking, however, the gaussianity of V_S need not be required at all: owing to the linearity of eqs. (5), (34) and (42), our calculations would not have changed essentially if we had admitted some non-gaussianity, but still stationarity, of the process $V_S(t)$. (Complications would only arise in calculations of third-order and higher-order correlation functions of $V_C(t)$ or $Q(t)$.) On the other hand, if we had not taken $1/R$ and $1/C$ as the (stationary) *Gaussian* processes (or as constants), we should have got serious formal troubles.

I have shown, in constrast to Ref. [1], that if one of the parameters $1/R$, $1/C$ is kept constant whilst the second is taken as a stationary Gaussian process, the the *RC*-chain can be analysed in a non-perturbative way. I have also suggested that if both $1/R$ and $1/C$ represent (though independent) stationary Gaussian processes with non-zero dispersions, the possibility of a non-perturbative calculation ceases to exist.

Stochastic problems due to electric circuits were well elucidated in many books (e.g. [8], [9]). Cognate stochastic problems were solved, e.g. in monographs [10], [11]. When confronted with these, Adomian's monograph [1] (complementing them to a certain extent) may be rated quite favourably (despite the interpretative slip in one of its paragraphs which had provoked me into writing the present paper).

Nevertheless, a moral of the story described in the present paper should be drawn (and I would like to address it to some of our younger colleagues) as follows. The scientific wisdom does not consist purely in our ability to manifest some technical skill in using sophisticated mathematical formalisms. It is equally (if not more) important also to know what our equations and our results actually mean.

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О СТОХАСТИЧЕСКИХ *RC*-ЦЕПЯХ

Сформулирована стохастическая теориая для электрических *RC*-цепей с источником напряжения $V_S(t)$, состояние которого можно описать как стационарный гауссовский процесс типа Орнштайн—Уленбека. Проводимость $1/R(t)$ и обратная емкость $1/C(t)$ рассматриваются либо как стохастические функции (стационарная гауссовская функция Орнштайн—Уленбековского типа), либо как постоянная. Показано, как математический анализ *RC*-контуров меняется в трех случаях: 1. когда *R* стохастично и *C* постоянно; 2. когда *R* постоянно и *C* стохастично; 3. когда *R* и *C* имеют стохастическое поведение. Показано, что в первых двух случаях (это означает либо нетривиальные моменты напряжения в конденсаторе, либо его заряды) результаты можно получить, не применяя теорию возмущений.