SSQM AND NONLINEAR EQUATIONS[1)]

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The method of obtaining the superpartner potential in supersymmetric quantum mechanics (SSQM) is discussed in connection with nonlinear equations and reflectionless potentials. The application of the SSQM in Plasma Physics is discussed.

I. INTRODUCTION

The study of the particle-like behaviour of nonlinear fields originally initiated by Einstein to systematically derive the motion equations of a particle in an external field, took a new turn with the discovery of the soliton solutions [1].

The soliton-type properties have been found by now in a great variety of nonlinear physical systems such as the Korteveg de Vries (KdV), the sine-Gordon, the nonlinear Schrödinger (NSL) and others.

In the seventies theoretical physics has developed a new fruitful conception of supersymmetry, whose main idea is to treat bosons and fermions equally [2].

The interesting advantage of supersymmetry is the unambiguous way of incorporating the fermions into the soliton system; it was first done for nonlinear equations via direct supersymetrization in ref. [3].

From this supersoliton theory, which is given by the supersoliton Lagrangian in the (1 + 1) space-time dimension

$$L = \frac{1}{2}[(\partial_\mu \varphi)^2 - V^2(\varphi) + \psi(\mathrm{i}\not{\partial} + V'(\varphi))\psi], \qquad (1.1)$$

where φ is a Bose field and ψ is a Fermi field, we can obtain SSQM as a restriction to the (0 + 1) space-time dimension.

Indeed, if we substitute into (1.1) the following restriction:

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$$\partial_\mu \to \partial_t,$$
$$\varphi \to x(t),$$
$$\mathrm{i}\not{\partial} \to \mathrm{i}\partial_t \sigma_2,$$
$$\psi \to \psi^T \sigma_2^-,$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the components of which are to be interpreted as anticommuting c-numbers and σ_k denotes the usual Pauli matrices, then $L \to L_{SSQM}$, where:

$$L_{SSQM} = \frac{1}{2}[(\partial_t x)^2 - V^2(x) + \psi^T(\mathrm{i}\partial_t + \sigma_2 V'(x))\psi]. \tag{1.2}$$

The corresponding Hamiltonian has the known form:

$$H_{SSQM} = \frac{1}{2}p^2 + \frac{1}{2}V^2(x) + \frac{1}{2}i[\psi_1, \psi_2]V'(x); \tag{1.3}$$

proposed by Witten and also by Salamonson and Van Holten [4].

In the present work we show the role of SSQM for nonlinear equations like the NLS eq. and the KdV eq. and also the application of SSQM in Plasma Physics.

2. SUPERSYMMETRIC QUANTUM MECHANICS

We shall start with the Schrödinger factorization in QM: Let us assume *a* one-dimensional Schrödinger eq.

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x)\right)\psi(x) = E\psi(x) \tag{2.1}$$

and the factorization in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + v\right)\left(\frac{-\mathrm{d}}{\mathrm{d}x} + v\right)\psi = E\psi. \tag{2.2}$$

If we denote

$$A^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + v, \tag{2.3}$$

we can write

$$A^+ A\psi = E\psi = H^+\psi,$$

but it gives

$$A^+A^- = H^+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v^2 + v_x = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+.$$

Let us choose the zero of energy so that the groundstate ψ_0^+ in H_+ has zero energy

$$H_+\psi_0^+ = A^+A^-\psi_0^+ = 0,$$

implying from

$$A^-\psi_0^+ = 0 \tag{2.4}$$

This is a first-order differential eq.

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + v(x)\right)\psi_0^+ = 0,$$

constraining

$$v = \frac{\psi_{0x}^+}{\psi_0^+}. \tag{2.5}$$

If we consider the factorization in the form

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + v\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + v\right)\psi = E\psi,$$

we get

$$A^-A^+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v^2 - v_x = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_- = H_-. \tag{2.6}$$

Now suppose ψ^+ is any eigenfunction of H_+

$$H_+\psi^+ = E_+\psi^+; \tag{2.7}$$

then $A^-H_+\psi_+ = E_+(A^-\psi_+) = A^-A^+A^-\psi_+$.

Either $A^-\psi_+ = 0$ (so that $E_+ = 0$ and ψ_+ is the ground state) or $H_-(A^-\psi_+) = E_+(A^-\psi_+)$.

Thus, every eigenstate of H_+ except the ground state gives rise (via A^-) to an eigenstate of H_- with the same eigenvalue.

The ground state in H_+, with zero energy, does not correspond to any eigenstate of H_-. It means that the Hamiltonian H_+ has the same spectrum as H_- plus one ground state more.

If we denote the solution ψ_0^- of the zero-energy Schrödinger eq. with H_-

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-\right)\psi_0^- = -\psi_{0xx}^- + (v^2 - v_x)\psi_0^- = 0, \tag{2.8}$$

we get

$$v = -\frac{\psi_{0x}^-}{\psi_0^-} \tag{2.9}$$

and by comparing with (2.3) we have:

$$\psi_0^+ \sim \frac{1}{\psi_0^-}. \tag{2.10}$$

The factorization presented here can be written in a supersymmetric way. In the matrix formulation H_{SSQM} (1.3) becomes a 2×2 matrix as well:

$$H_{SSQM} = \frac{1}{2}\begin{pmatrix} -\frac{d^2}{dx^2} + v^2(x) + v_x(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + v^2(x) - v_x(x) \end{pmatrix}. \tag{2.11}$$

Then

$$H_{SUSY} = 2H_{SSQM} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \{Q, Q^+\}, \tag{2.12}$$

where the "supercharges" are defined

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$$

The other relations are

$$Q^2 = (Q^+)^2 = 0, \quad [H_{SUSY}, Q] = [H_{SUSY}, Q^+] = 0. \tag{2.13}$$

The eigenfunctions of H_{SUSY} are:

$$\psi_{SUSY} = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$$

and they have the properties:

$$Q\psi_{SUSY} = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \quad \text{unles} \quad A^-\psi^+ = 0,$$

$$Q^+\psi_{SUSY} = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$$

We can call the levels $\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$ "bosonic" and the levels $\begin{pmatrix} 0 \\ \psi^- \end{pmatrix}$ "fermionic" in view of the "fermionic" nature of the "super algebra" in relations (2.12), (2.13).

In the theory of the spectral transforms and solitons [5] the Schrödinger factorization (2.2) is equivalent to the Miura transformation between V_+, v

$$V_+ = v^2 + v_x,$$

which connects the Kdv eq. with the modified KdV (mKdV) eq. The same is valid for V_-, because the mKdV eq. is invariant under the transformation of $v \to -v$.

In this sense the Miura trasformation represents the supersymmetric "square root".

3. THE SSQM AND REFLECTIONLESS POTENTIALS IN NONLINEAR EQUATIONS

First we show how reflectionless potentials appear in a toy NLS model with a Grassmann one-component real variable θ.

The NLS eq. has the form

$$(\mathrm{i}\,\partial_t + \partial_x^2 + |\mathscr{E}(x,t)|^2)\mathscr{E}(x,t) = 0. \tag{3.1}$$

This eq. 3.1 has the known particular solution

$$\mathscr{E}(x,t) = e^{\frac{it}{2L^2}}\frac{1}{L}\,\mathrm{sech}\,\frac{x}{L\sqrt{2}}, \quad L > 0. \tag{3.2}$$

Using the variable θ we shall define the superfield $E(x_a, \theta)$ in the following way:

$$E(x_a, \theta) = \mathscr{E}(x_a) + \mathrm{i}\theta\psi(x_a), \tag{3.3}$$

where x_a is the usual two-dimensional space-time commuting coordinate, $\mathscr{E}(x_a)$ is the solution of the NLS eq. and $\psi(x_a)$ is a new field.

Now, we postulate the NLS eq. for the superfield $E(x_a, \theta)$ as follows

$$(\mathrm{i}\,\partial_t + \partial_x^2 + |E(x_a,\theta)|^2 E(x_a,\theta) = 0.$$

Using the anticommutativy of θ, we obtain two independent equations of motion coupling $\mathscr{E}(x_a)$ with $\psi(x_a)$

$$\mathrm{i}\,\partial_t + \partial_x^2 + |\mathscr{E}(x_a|^2)\mathscr{E}(x_a) = 0, \tag{3.4a}$$

$$(\mathrm{i}\,\partial_t + \partial_x^2 + 2|\mathscr{E}(x_a)|^2\psi(x_a) - \mathscr{E}^2(x_a)\psi^*(x_a) = 0. \tag{3.4b}$$

The first eq. (3.4a) is the NLS eq. with the particular solution (3.2). We can see immediately that the system of the coupled eqs. (3.4a, b) has the solution $\mathscr{E}(x,t)$ in the form (3.2) and the solution $\psi(x,t)$, which has the form

$$\psi(x,t) = e^{\frac{it}{2L^2}}\frac{L^{-1}}{\mathrm{ch}\frac{x}{L\sqrt{2}}}. \tag{3.5}$$

From (3.5) and (3.2) it follows directly

$$\mathscr{E}(x,t) = \psi(x,t). \tag{3.6}$$

It means that eq. (3.4b) can be rewritten as

$$(\mathrm{i}\,\partial_t + \partial_x^2 + |\psi(x,t)|^2)\psi(x,t) = 0,$$

which is the nonlinear eq. of the same type as that for $\mathscr{E}(x,t)$.

Now, if we take $\mathscr{E}(x, t)$ in the form (3.2) and put it into the eq. (4.9), we obtain

$$\left(\mathrm{i}\partial_t + \partial_x^2 + \frac{L^{-2}}{\mathrm{ch}^2 \frac{x}{L\sqrt{2}}}\right)\psi(x, t) = 0. \tag{3.7}$$

If we make the following substitution

$$U_0 = L^{-2}, \qquad \alpha = \frac{1}{\sqrt{2}} L^{-1}$$

in eq. (4.7), we can rewrite it in the following form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi + \left(E + \frac{U_0}{\mathrm{ch}^2 \alpha x}\right) = 0, \tag{4.8}$$

which is the one-dimensional Schrödinger eq. with the known potential $\sim U_0/\mathrm{ch}^2 \alpha x$, the solution of which is given in ref. [6].

It is well known that the transmission coefficient in the potential $\sim U_0/\mathrm{ch}^2 \alpha X$-well has the form

$$D = \frac{\mathrm{sh}^2 \frac{\pi\sqrt{E}}{\alpha}}{\mathrm{sh}^2 \frac{\pi\sqrt{E}}{\alpha} + \cos^2 \frac{\pi}{2}\sqrt{1 - \frac{4U_0}{\alpha^2}}}. \tag{4.9}$$

An interesting phenomenon arises, when the following condition is valid:

$$|U_0| = N(N+1)\,\alpha^2. \tag{4.10}$$

Then the transmission coefficient is trivial, i.e. $D = 1$.

With our substitution $\alpha = \frac{1}{\sqrt{2}} L^{-1}$ we have:

$$|U_0| = \frac{1}{2} N(N+1)\,L^{-2}; \tag{4.11}$$

thus, it means that the potential will not reflect, when the corresponding values $|U_0|$ are follows L^{-2}, $3L^{-2}$, $6L^{-2}$...

Now we show the relation of solitons in the non-linear KdVeq. and the SSQM.

It is well known [5] that a potential of the form

$$U(x) = -\frac{1}{L^2 \mathrm{ch}^2 \frac{x}{L\sqrt{2}}}$$

can be regarded as one-soliton solution of the KdV eq. for $t = 0$, i.e. of the equation:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{4.12}$$

The KdV one-soliton solution for all t is

$$u(x, t) = -\frac{1}{L^2 \mathrm{ch}^2 \left(\frac{x - \frac{2}{L^2}t}{L\sqrt{2}}\right)}.$$

Let us consider now a function $v(x, t)$ satisfying the mKdV eq.:

$$v_t + 6\left(\frac{1}{2L^2} - v^2\right)v_x + v_{xxx} = 0. \tag{4.13}$$

Then, if we define

$$V_- = v^2 - v_x - \frac{1}{2L^2}, \tag{4.14}$$

as usual in SSQM, it can be easily shonw that V_- satisfies the KdV eq. The same is valid for V_+

$$V_+ = v^2 + v_x - \frac{1}{2L^2}. \tag{4.15}$$

If we take (4.14) as the one-soliton solution of the KdV eq., then with a suitable boundary condition on v one can interpret V_+ in (4.15) as a two-soliton solution of the KdV eq. and it corresponds to $N = 2$, the above mentioned potential.

This is shown by Sukumar in ref. [7], where the general connection between the N-soliton solutions of the KdV, SSQM, the inverse scattering method and the construction of the reflectionless potentials is discussed.

We shall now concern ourselves with the correspodence between the SSQM and the super NLS (SNLS) eqs.

4. THE SUPER-GENERALIZATION OF THE NLS EQUATION

A super extension of the Ablowitz-Kaub-Newell-Segur scheme [8] was presented by Gürses and Oğuz in ref. [9] to obtain the SNLS model.

Their extension gives the following SNLS eqs.:

$$\mathrm{i}\mathscr{E}_t + \mathscr{E}_{xx} - 2k_1|\mathscr{E}|^2 - 4\psi_x\psi - 4k_2\mathscr{E}\psi\psi^* = 0\,, \tag{4.1}$$

$$\mathrm{i}\psi_t + 2\psi_{xx} - k_1|\mathscr{E}|^2\psi - 2k_2\mathscr{E}\psi_x^* - k_2\mathscr{E}_x\psi^* = 0\,, \tag{4.2}$$

where k_i is a constant.

Here we shall present another supper extension of the NLS. eq., which is in correspodence with SSQM.

We can immediately see that the eq. (4.1) for $k_1 = -\frac{1}{2}$ has the solution $\mathscr{E}(x, t)$ in the form (3.2) when

$$\frac{\psi_x}{\psi^*} = -k_2\mathscr{E}\,. \tag{4.3}$$

Eq. (4.3) is the equation for ψ, where $\mathscr{E}$ is the one-soliton.solution (3.2). We search for the solution of eq. (4.3) in the form

$$\psi(x, t) = \mathrm{e}^{if(t)}\psi(x)$$

and we get

$$\psi(x, t) = \mathrm{e}^{\frac{it}{4L^2}} \exp\left(-\sqrt{2}k_2 \operatorname{arctg} e^{\frac{x}{\sqrt{2}L}}\right). \tag{4.4}$$

It can be directly shown that the functions (3.2), (4.4) satisfy the following nonlinear eq.

$$\mathrm{i}\psi_t + \psi_{xx} + \left(\frac{1}{4L^2} + \frac{1}{L}|\mathscr{E}|^2\right)\psi + 2k_2\mathscr{E}\psi_x^* + k_2\mathscr{E}_x\psi^* = 0\,, \tag{4.5}$$

which is a modification of eq. (4.2).

Thus eqs. (4.1) and (4.5) are in correspodence with SSQM. It can be shown in the following way: Let us put

$$A_\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + k_2\mathscr{E}(x)\,. \tag{4.6}$$

Then

$$H_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k_2^2\mathscr{E}^2 \pm k_2\mathscr{E}_x\,. \tag{4.7}$$

From eq. (4.3) we get

$$\psi_x = -k_2\mathscr{E}\psi\,,$$

$$\psi_{xx} = -k_2\mathscr{E}_x\psi - k_2\mathscr{E}\psi_x = -k_2\mathscr{E}_x\psi + k_2^2\mathscr{E}^2\psi\,, \tag{4.8}$$

$$\frac{\psi_{xx}}{\psi} = k_2^2\mathscr{E}^2 - k_2\mathscr{E}_x\,,$$

but it is the Schrödinger eq. for V_- in SSQM.

5. THE APPLICATION OF SSQM TO NONLINEAR EQUATIONS IN PLASMA PHYSICS

First we pay attention to the Zakharov Z eqs.

$$\mathrm{i}\mathscr{E}_t + \mathscr{E}_{xx} - n\mathscr{E} = 0\,, \tag{5.1}$$

$$n_{tt} - (n + |\mathscr{E}|^2)_{xx} = 0\,, \tag{5.2}$$

which describe the propagation of the Langmuir waves in plasmas.

Here, $\mathscr{E}$ denotes t he slowly varying envelope of the highly oscillatory electric field and n is the ion density perturbation.

In connection with our application we shall discuss only the so-called quasi-static limit of the Z eqs. neglecting the term n_{tt} (the limit of infinite ion mass). Then eq. (5.2) has the form

$$(n + |\mathscr{E}|^2)_{xx} = 0\,, \tag{5.3}$$

which implies $n = -|\mathscr{E}|^2$ if n and $|\mathscr{E}|^2$ are square-integrable.

The substitution of this expression for n into (5.1) yields the NLS eq. (3.1).

It is well known that eqs. (5.1), (5.2) have one-soliton wave solution

$$\mathscr{E} = \frac{1}{L}\operatorname{sech}\left[\frac{x - x_0 - vt}{L\sqrt{2(1 - \mathrm{v}^2)}}\right]\exp\left[\frac{1}{2}\,\mathrm{i}vx - \mathrm{i}\left(\frac{1}{4}v^2 - \frac{1}{2L^2(1 - v^2)}\right)t + \mathrm{i}\varphi\right], \tag{5.4}$$

$$n = -\frac{|\mathscr{E}|^2}{1 - v^2}\,. \tag{5.5}$$

where $L > 0$, v, x_0 and φ are constants.

It is clear that the solution (5.4) tends to the particular one-soliton solution (3.2)

$$\mathscr{E}(x, t) = \exp\left(\frac{it}{2L^2}\right)\frac{1}{L}\operatorname{sech}\frac{x}{L\sqrt{2}} \tag{5.5}$$

and $n(x) = -|\mathscr{E}|^2$, (5.6)

for $v = 0$, $x_0 = 0$, $\varphi = 0$.

Some experimental evidence of the existence of these solitary waves has been produced by Antipov [11].

If we put the solution (5.5) into eq. (3.1) we obtain

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{L^2\operatorname{ch}^2\frac{x}{L\sqrt{2}}}\right)\psi_1(E_1) = E_1\psi_1(E_1)\,, \tag{5.7}$$

which is the eigenvalue eq. $H_1\psi_1 = E_1\psi_1$, corresponding to eq. (3.1).

In eq. (5.7) we denote the eigenvalue $E_1 = -\gamma_1^2 = -\frac{1}{2L^2}$ and it corresponds to the eigenfunction

$$\psi_1 = \frac{1}{L}\,\text{sech}\,\frac{x}{L\sqrt{2}}.$$

Now we shall consider the ion density perturbation

$$n_1 = -|\psi_1|^2 = -\frac{1}{L^2}\,\text{sech}^2\,\frac{x}{L\sqrt{2}}$$

as the potential in the eigenvalue problem (5.7) and using SSQM we shall construct other reflectionless potentials as "supperpartners".

We can see that

$$H_1 = -\frac{d^2}{dx^2} + n_1$$

is the superpartner to the $H_0 = -\frac{d^2}{dx^2} + n_0$, where the potential n_0 does not support any bound states, meanwhile n_1 supports a simple bound state at the energy $E_1 = -\frac{1}{2L^2}$.

Choosing $n_0 = 0$, H_0 is then the free particle Hamiltonian and the reflection coefficient of n_0 is $R_0(k) = 0$ for the positive energies $E = k^2$.

$$R_1(k) = \frac{\gamma_1 - ik}{\gamma_1 + ik}\,R_0(k),$$

which is zero for $R_0(k) = 0$. But it is the case of the reflectionless potential in (4.11) for $N = 1$ and $|U_0| = L^{-2}$.

Let us suppose from SSQM

$$V_- = v^2 - v_x = \frac{1}{2L^2}. \tag{5.8}$$

Eq. (5.8) is a very simple Riccati eq., whose solution is given by substituting

$$v = -\frac{\psi_{0x}}{\psi_0} \tag{5.9}$$

and we have

$$\frac{\psi_{0xx}}{\psi_0} = \frac{1}{2L^2}. \tag{5.10}$$

Here ψ_0 is the solution of the zero-energy Schrödinger eq. with H^-

$$\left(-\frac{d^2}{dx^2} + v^2 - v_x\right)\psi_0 = 0$$

The solution ψ_0 from eq. (5.10) is:

$$\psi_0 = C\,\text{ch}\,\frac{x}{\sqrt{2}L} \tag{5.11}$$

and from (5.9)

$$V = -\frac{1}{\sqrt{2}L}\tanh\frac{x}{\sqrt{2}L}. \tag{5.12}$$

The superpartner to V_- has the form

$$V_+ = v^2 + v_x = \frac{1}{2L^2} - \frac{1}{L^2}\,\text{sech}^2\,\frac{x}{L\sqrt{2}}. \tag{5.13}$$

Now, if we denote

$$n_0(x) = v^2 - v_x - \frac{1}{2L^2} = 0,$$

$$n_1(x) = v^2 + v_x - \frac{1}{2L^2} = -\frac{1}{L^2}\,\text{sech}^2\,\frac{x}{L\sqrt{2}},$$

we can see that H_1 is the superpartner to H_0.

Using the receipt from SSQM shall now demonstrate how to construct a symmetric reflectionless $n_i(x)$, $i = 1, 2, \ldots N$.

For arbitrary i we may now assume that $n_{i-1}(x)$ is known and we define v_i by

$$n_{i-1} = v_i^2 - v_{ix} - E_i$$

and then the supersymmetric partner has the form $n_i = v_i^2 + v_{ix} - E_i$.

The crucial point, essential for the construction, is that the supersymmetric reflectionless partner can be expressed via the eigenfunctions of the corresponding Hamiltonian.

It can be see from the following ($i = 1$)

$$H^+ = A^+A^- = H^- + [A^+, A^-] = H^- + 2\frac{d}{dx}v = H^- - 2\frac{d^2}{dx^2}\ln\psi_0. \tag{5.14}$$

From this

$$\text{H}^+ = -\frac{d^2}{dx^2} + E_1 + n_0 - 2\frac{d^2}{dx^2}\ln\psi_0(E_1)$$

and

$$n_1 = n_0 - 2\frac{d^2}{dx^2}\ln\psi_0(E_1). \tag{5.15}$$

We can apply this procedure to the modified Z eqs.

It was shown by Sukumar in ref. [7] that the symmetric reflectionless $n_N(x)$ may be expressed in terms of the normalized bound states eigenfunctions in form

$$n_N^{(x)} = -4\sum_{i=1}^{N}[\gamma_i\psi_N^2(E_i)]. \tag{5.16}$$

There has been also demonstrated that $n_N(x)$ can be constructed via the normalized eigenfunctions $\psi_N(E_i)$.

Application of the formula (5.16) to the Z eqs. for the quasi-static case exactly fulfills the condition for the ion density perturbation to be a linear combination of the squares of eigenvalue solutions of the eq.

$$\left(-\frac{d^2}{dx^2} + n_N(x)\right)\psi_N(E_i) = E_i\psi_N(E_i)$$

with the weights γ_i, where $E_i = -\gamma_i^2$, $i = 1, 2, \dots N$.

We can define the modified Z eqs. for the quasi-static limit

$$i\partial t\psi_N + \psi_{Nxx} - n_N\psi_N = 0, \tag{5.18}$$

$$\left\{n_N + 4\sum_{i=1}^{N}[\gamma_i|\psi_N(E_i)|^2]\right\}_{xx} = 0, \tag{5.19}$$

where $\psi_N(E_i)$ are bound states by eq. (5.17) and n_N are reflectionless potentials.

For $N = 2$ there arises the interesting case when $\gamma_2^2 = 4\gamma_1^2$ and $n_2(x)$ is sysmmetric reflectionless well [7]

$$n_2(x) = -\frac{3}{L^2\,\mathrm{ch}^2\dfrac{x}{L\sqrt{2}}}. \tag{5.20}$$

The number 3 corresponds to $N = 2$ in the symmetric reflectionless potentials.

$$\sim\frac{N(N+1)}{2}\,\frac{1}{L^2\,\mathrm{ch}^2\dfrac{x}{L\sqrt{2}}},$$

which are discussed in sect. 3.

As regards the physical application there is an interesting case when

$$n_N(x) = -\frac{N(N+1)}{2}\,\frac{1}{L^2\,\mathrm{ch}^2\dfrac{x}{L\sqrt{2}}} = -\frac{N(N+1)}{2}\,n_1(x). \tag{5.21}$$

Mathematically the formula (5.11) denotes the case of the Lame-Ains N-zones elliptical potential [2].

Then the modified Z eqs. 5.18), (5.19) have the form

$$i\partial_t\psi_N + \psi_{Nxx} + \frac{N(N+1)}{2}|\psi_1|^2\psi_N = 0, \tag{5.22}$$

$$\left[n_N(x) + \frac{N(N+1)}{2}|\psi_1|^2\right]_{xx} = 0. \tag{5.23}$$

In these eqs. have the envelope solitary wave solutions

$$\psi_N = l^{\frac{it}{2L^2}}\frac{1}{L}\,\mathrm{sech}\,\frac{\sqrt{N(N+1)}}{2L}\,x,$$

for arbitrary $N = 1, 2, \dots$.

If we denote $|\psi_1|^2 = |\mathscr{E}|^2$, where $\mathscr{E}$ has physically the meaning of a slowly varying envelope of the high-frequency electric field, then (5.22), (5.23) represent the Z eqs. in the quasi-static limit, where the ion density perturbation is

$$n(x) = -\frac{N(N+1)}{2}|\mathscr{E}|^2.$$

In plasma physics experiments it could be interesting to verify whether the discrete values of the ion density perturbation — $|\mathscr{E}|^2$, $-3|\mathscr{E}|^2$, $-6|\mathscr{E}|^2$ are relevant to the existence of the Langmuir solitons.

Another useful application is the study of the effect of higher order nonlinearity on the propagation of nonlinear ion-acoustic waves in a collisionless plasma consisting of negative ions via the methods in SSQM.

It is known [13] that the basic set of fluid eqs. of a plasma consisting of negative cold ions, positive cold ions positive cold ions and hot electrons (non-isothermal and isothermal) reduces to the mKdV and the KdV eqs.

The close connection between the N-soliton solutions of the KdV eq. enables us makes it possible to obtain in this application N bound states exactly.

These states represent the multisoliton solutions and can be physicaly interpreted as ion-acoustic solitons in a multi-component plasma [14].

6. CONCLUSIONS

In the paper presented we applied after a short introduction to SSQM the methods of SSQM in nonlinear equations.

First we used the Grassmann variable for a toys extension of the NLS model to show the appearance of the reflectionless potentials and the connection with the N-soliton solution of the KdV eq., which corresponds with SSQM.

As interesting from the point of physics we presented here the application of SSQM to the Zakharov eqs. with the envelope solitons and to the KdV eq. and the mKdV eq. with the ion-acoustic solitons.

A possible experimental verification is also discussed.

Here we not that the N- bound states ψ_N which are bounded in the potential n_N and ψ_N is not the algorithm for the N-soliton solutions of the NLS eq.

It is only a coincidence that in the particular case the $|\mathscr{E}|^2$ in the NLS. eq. is the same as the instantaneous $t = 0$ two-soliton solution of the KdV eq.

The agorithm which is used in SSQM for constructing the reflectionless potential with N bound states is known to be related to the algorithm for the N-soliton solution of the KdV eq.

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СУПЕРСИММЕТРИШЕСКАЯ КВАНТОВАЯ МЕХАНИКА И НЕЛИНЕЙНЫЕ УРАВНЕНИЯ

В работе обсуждается метод получения суперпотенциалов в суперсимметрической квантовой механике в связи с нелинейными уравнениями и потенциалами без отражений. Кроме того, обсуждается применение суперсимметрической квантовой механики в физике плазмы.