ON THE TORSION OF INHOMOGENEOUS ELASTICALLY ANISOTROPIC (ORTHOTROPIC) BARS

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In the paper we solve the problem of torsion of the inhomogeneous elastically anisotropic (orthotropic) prismatic bar with a rectangular cross-section and with a fixed stress on the boundary. It is assumed that the material of the bar is weakly inhomogeneous. The problem is solved by the method of the small parameter and is reduced to the recurrent set of boundary problems of the homogeneous theory. For a given inhomogeneity (2.1), (3.1) we solve in (3.2) the problem (i.e. (the solution of) the boundary problems (2.3), (1.12)) — the Prandtl function of tension Ψ.

I. INTRODUCTION

In the past few years the theory of elasticity of inhomogeneous and anisotropic bodies had been rapidly developing. This theory is the generalization of the elasticity theory of the homogeneous isotropic bodies. The theory was developed by many authors. For example, V. A. Lomakin contributed in [3] to the theory of elasticity of the inhomogeneous body and S. G. Lechnickij [2] to the theory of the anisotropic body. Lomakin formulated the constitutive equations and showed the method of their solution. He also showed that problems in the theory of the linear viscoelasticity, or the problems of small elastic-plastic deformations can be reduced, e.g., to the problems of elasticity of the inhomogeneous body. Similarly, the problems of thermoelasticity, in the case when the parameters of elasticity are weakly temperature dependent, can be reduced to the problems of the classical thermoelasticity.

In this connection we want to mention reference [6] where the problems of weakly inhomogeneous bodies are formulated, both for the statics and the dynamics of the classical inhomogeneous and anisotropic bodies. In ref. [6] are also studied the mathematical aspects concerning the existence and uniqueness of the solutions and the convergence of small parameter expansions to the solutions of the problems.

Many papers are devoted to the investigation of problems of torsion of anisotropic both homogenenous and inhomogenenous bodies among them especially [1, 3] are significant. Lists of references devoted to the investigation of the stress and deformation states of the inhomogenenous bodies are in refs. [4, 5]. Some interesting results can be find in refs. [7—9, 11, 12]. In paper [11] there is solved the problem of torsion of the inhomogeneous anisotropic (non-orthotropic) prismatic bars in the case when the inhomogeneity functions are linear functions in two variables.

In the present paper we follow the ideas presented in [11]. We investigate here the pure torsion of the inhomogenenous and anisotropic (orthotropic) bar under the assumption of a weak inhomogeneity of the material of the bar.

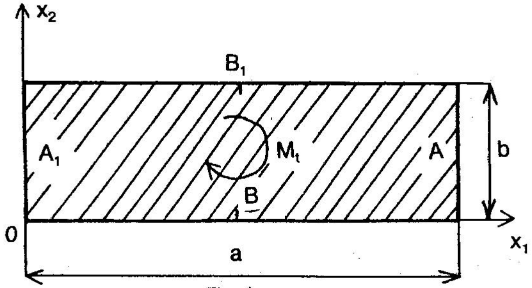

Fig. 1

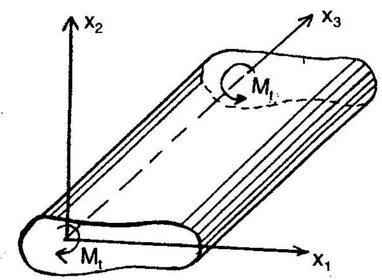

Fig. 2

II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Let us investigate the pure torsion of a prismatic bar of a rectangular cross-section with sides a, b (Fig. 1). We assume the bar to be made of the inhomogenenous anisotropic material and the anisotropy to be such that through any point of the body there pass three planes of elastic symmetry orthogonal to each other, perpendicular to the coordinate axes x_1, x_2, x_3. Such body is called othotropic.

The body is investigated in an orthogonal frame defined by the axes x_1, x_1, x_3. The origin of the system is located at one of the end points of the bar and the axis x_3 is identical with the geometrical axis of the bar considered.

Let the elastic characteristics a_{ij} be differentiable in the plane transversal to the bar and constant along it, i.e. $a_{ij} = a_{ij}(x_1, x_2)$.

One usually assumes that the body is loaded only on the end surfaces, and the forces acting at the ends are statically equivalent to the momentum of torsion $\boldsymbol{M}_t$. $\boldsymbol{M}_t$ is parallel with the axis of the bar (Fig. 2), the lateral side of

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which is free of the acting forces. In this case the state of stress in the bar is determined by the two components σ_{13} and σ_{23} on the stress tensor; the othrs are equal to zero

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0 \tag{1.1}$$

In this case the following components of the deformation tensor are equal to zero

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = 0\,, \tag{1.2}$$

where both ε_{13} and ε_{23} are non vanshing.

The stress tensor components satisfy the equations of the equilibrium. If we omit the volume forces, then the equations of equilibrium for the element of the bar take the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{1.3}$$

$(i, j = 1, 2, 3)$ or explicitly

$$\frac{\partial \sigma_{13}}{\partial x_3} = 0\,, \quad \frac{\partial \sigma_{23}}{\partial x_3} = 0\,, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0\,. \tag{1.4}$$

The equations are identically satisfied by putting

$$\sigma_{12} = \frac{\partial \Psi}{\partial x_2}\,, \qquad \sigma_{23} = -\frac{\partial \Psi}{\partial x_1} \tag{1.5}$$

where $\Psi(x_1, x_2)$ is the Prandtl function of stress, depending on x_1 and x_2 only.

One also assumes that deformation of the body is governed by general Hook's law, which in the case considered can be written in the form

$$\begin{aligned} \varepsilon_{23} &= a_{44}(x_1, x_2)\, \sigma_{23} \\ \varepsilon_{13} &= a_{55}(x_1, x_2)\, \sigma_{13}\,. \end{aligned} \tag{1.6}$$

The components of the deformation tensor satisfy the equations of continuity of deformations (Saint-Vernant's law) [6]

$$\varepsilon_{ilm}\varepsilon_{jrs} \frac{\partial^2 \varepsilon_{lr}}{\partial x_m \partial x_s} = 0 \tag{1.7}$$

(where ε_{jrs} is the Levi-Civita antisymmetric tensor, $i, j, l, m, r, s = 1, 2, 3$), which in our case can be written in the form

$$\frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{23}}{\partial x_1} = -2\vartheta \tag{1.8}$$

where ϑ is the torsion.

By substituting the relative shears from (1.6) into (1.8) and with respect to (1.5) we get the following differential equation for the determination stress function

$$a_{44}(x_1, x_2)\frac{\partial^2 \Psi}{\partial x_1^2} + a_{55}(x_1, x_2)\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial a_{44}(x_1, x_2)}{\partial x_1}\cdot\frac{\partial \Psi}{\partial x_1} + \frac{\partial a_{55}(x_1, x_2)}{\partial x_2}\cdot\frac{\partial \Psi}{\partial x_2} = -2\vartheta\,. \tag{1.9}$$

The torsion ϑ (i.e. the angle of torsion per unit of the length of the bar) is given by the relation [2, 7]

$$\vartheta = \frac{M_t}{O_t} \tag{1.10}$$

where

$$C_t = 2\int\int_D \Psi(x_1, x_2)\,dx_1\,dx_2 \tag{1.11}$$

is stiffness in the torsion.

Assuming that on the lateral end surface of the bar external forces do not act (Γ is the boundary of the simply connected region D of the cross section of the bar), then

$$\Psi|_\Gamma = 0\,. \tag{1.12}$$

Thus, the problem of torsion of the inhomogeneous anisotropic (orthotropic) prismatic bar leads to the solution of the differential equation (1.9) with the boundary condition (1.12).

III. TORSION OF ORTHOTROPIC BARS WITH WEAK INHOMOGENEITY

In what follows we shall investigate orthotropic bodies with weak inhomogeneity [6, 7, 11]. Their elastic characteristics can be written in the form

$$\begin{aligned} a_{44}(x_1, x_2) &= a_{44}^0[1 + \delta f_1(x_1, x_2)] \\ a_{55}(x_1, x_2) &= a_{55}^0[1 + \delta f_2(x_1, x_2)] \end{aligned} \tag{2.1}$$

where a_{ij}^0 are constants satisfying the inequalities

$$a_{44}^0 > 0\,, \qquad a_{55}^0 > 0\,, \tag{2.2}$$

$f_i(x_1, x_2)$ $(i = 1, 2)$ are known differentiable functions and δ is the small parameter (referred to in literature as the small physical parameter), characterizing inhomogeneity, $\delta \ll 1$. For orthotropic bodies $a_{45}^0 = 0$. Substituting a_{ij} from (2.1) into (1.9) we obtain the equation

$$A_0[\Psi] + \delta H_0[\Psi] = -2\vartheta \tag{2.3}$$

where the operators $A_0[\;]$, $H_0[\;]$ have the form

$$A_0[\;] = a_{44}^0 \frac{\partial^2}{\partial x_1^2} + a_{55}^0 \frac{\partial^2}{\partial x_2^2} \tag{2.4}$$

$$H_0[\;] = a_{44}^0 f_1 \frac{\partial^2}{\partial x_1^2} + a_{55}^0 f_2 \frac{\partial^2}{\partial x_2^2} + a_{44}^0 \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial}{\partial x_1} + a_{55}^0 \frac{\partial f_2}{\partial x_2}\cdot\frac{\partial}{\partial x_2} \tag{2.5}$$

We look for the solution of the boundary problem (2.3), (1.12) in the form of a power expansion in the small physical parameter δ

$$\Psi(x_1, x_2) = \Psi_0(x_1, x_2) + \sum_{p=1}^{\infty} \Psi_p(x_1, x_2)\,\delta^p$$

As regards the convergence questions of this expansion see ref. [6]. Substituting (2.6) into equation (2.3) we obtain a system of recurrent boundary problems determining the unknown functions $\Psi_p(x_1, x_2)$ (p = 0, 1, 2, ...)

$$A_0[\Psi_0] = -g_0 \qquad \Psi_0|_\Gamma = 0 \tag{2.7}$$

$$A_0[\Psi_p] = -Q_p(x_1, x_2;\ \Psi_{p-1}) \qquad \Psi_p|_\Gamma = 0\,. \tag{2.8}$$

Here

$$g_0 = -2\vartheta,\ Q_p(x_1, x_2;\ \Psi_{p-1}) = H[\Psi_{p-1}]\,.$$

So the problem of torsion of the inhomogenenous orthotropic prismatic bars leads to the solution of the recurrent sequence of problems (2.7), (2.8), consistent with the problem of torsion of the homogenenous orthotropic bars, which can be solved iteratively [6, 11].

IV. TORSION OF ORTHOTROPIC BARS IN THE CASE WHEN ITERATIVELY FUNCTIONS OF INHOMOGENEITY ARE IN TWO VARIABLES LINEAR

In what follows we shall investigate the torsion of the orthotropic prismatic bars of the rectangle cross section with weak inhomogeneity in the case when the functions of inhomogeneity in (2.1) are the linear in two variables

$$f_i + a_i x + c_i y \qquad (i = 1, 2)\,. \tag{3.1}$$

To simplify the notation we use x, y instead of x_1, x_2. We use the method of the small physical parameter to find the solution $\Psi(x, y)$ of the boundary problem (2.3), (1.12).

Using some results given in paper [11] one can easily show that the solution of the boundary problem (2.3), (1.12) taking only two terms in the infinite series has the form

$$\Psi(x, y) = -\frac{4\varrho_0 a^2 \vartheta}{3} \sum_{k=1,3}^{\infty} k^{-\frac{1}{3}} \left[1 - \frac{\operatorname{ch}\dfrac{\lambda k\left(\pi - 2\frac{\pi}{b}y\right)}{2}}{\operatorname{ch}\dfrac{\lambda k \pi}{2}}\right] . \sin\frac{\pi}{a}x +$$

$$+\,\delta\left[\frac{8\varrho_0 a^2\vartheta}{\lambda^2\pi^4}\sum_{l=2,4}^{\infty}\sum_{k=1,3}^{\infty}\left\{a_{lk}\frac{\operatorname{sh}\lambda l\frac{\pi}{b}y - \operatorname{sh}\lambda l^{\pi}\left(\frac{y}{b}-1\right)}{\operatorname{sh}\lambda l\pi}b_{lk} + \right.\right.$$

$$\left. + c_{lk}\frac{\operatorname{ch}\dfrac{\lambda k\pi\left(\frac{2}{b}y-1\right)}{2}}{\operatorname{ch}\dfrac{\lambda k}{2}}\right\}\sin l\frac{\pi}{a}x + \frac{8\varrho_0 a^2\vartheta}{\pi^4}\left\{\sum_{l=1,3}^{\infty}\frac{\pi^2 A_1}{4\lambda^2 l^3} + \right.$$

$$+ d_l \operatorname{th}\frac{\lambda l\pi}{2}\cdot\frac{\operatorname{sh}\lambda l\frac{\pi}{b}y - \operatorname{sh}\lambda l\pi\left(\frac{y}{b}-1\right)}{\operatorname{sh}\lambda l\pi} + e_l \operatorname{th}\frac{\lambda l\pi}{2}\cdot\frac{\operatorname{sh}\lambda l\frac{\pi}{b}y}{\operatorname{sh}\lambda l\pi} +$$

$$+ f_1\frac{\operatorname{sh}\lambda l\frac{\pi}{b}y}{\operatorname{sh}\lambda l} - \frac{\pi^2\left(A_1 + 2B_1\frac{y}{b}\right)}{4\lambda^2 l^3} + \frac{\pi^2 B_4\frac{y}{b}}{4l^3}\cdot\frac{\operatorname{ch}\dfrac{\lambda l\pi\left(\frac{2y}{b}-1\right)}{2}}{\operatorname{ch}\dfrac{\lambda l\pi}{2}} +$$

$$+ g_1\left(\frac{\pi}{b}y - \frac{1}{2\lambda l}\right) + h_1\left(\frac{\pi^2}{b^2}y^2 - \frac{\frac{\pi}{b}y}{\lambda l} + \frac{1}{2\lambda^2 l^2}\right) - \frac{\pi B_4}{8\lambda l^4}.$$

$$\cdot \frac{\operatorname{sh} \frac{\lambda l \pi\left(\frac{2y}{b}-1\right)}{2}}{\operatorname{ch} \frac{\lambda l \pi}{2}} \Bigg\} \sin l \frac{\pi}{a} x \Bigg] + \ldots \tag{3.2}$$

where

$$a_{lk} = \frac{4A_1}{(l^2-k^2)^3} - \frac{2(A_1-A_4)k^2+2A_4l^2}{lk^2(l^2-k^2)^2} - \frac{2(2\lambda^2A_3+A_4)k^2l-2A_4l^3}{k^2(l^2-k^2)^3},$$

$$b_{lk} = \frac{2(2A_1-A_4)k^2+2A_4l^2}{lk^2(l^2-k^2)^2},$$

$$c_{lk} = \frac{2(2\lambda^2A_3+A_4)k^2l-2A_4l^3}{k^2(l^2-k^2)^3} - \frac{4(A_1+B_1)l}{(l^2-k^2)^3},$$

$$d_l = \frac{\pi^2(\lambda^2A_3-A_1)}{16\lambda^2l^3} - \frac{\pi(\lambda^2B_3+B_1)}{16\lambda^3l^4} + \frac{2\pi B_4}{16\lambda l^4}, \tag{3.3}$$

$$e_l = \frac{\pi^3(\lambda^2A_3-A_1)}{8\lambda l^2} + \frac{\pi^2(\lambda\pi-1)(\lambda^2B_3+B_1)}{8\lambda^2l^3},$$

$$f_l = \frac{\pi^2B_1}{2\lambda^2l^3} - \frac{\pi^2B_4}{4l^3}, \quad g_l = \frac{\pi^2(\lambda^2A_3-A_1)}{8\lambda l^2}, \quad h_l = \frac{\pi(\lambda^2B_3+B_1)}{8\lambda l^2}.$$

The constants A_1, A_3, A_4, B_1, B_3, B_4 are given by the expressions [11 (3.18)]

$$A_1 = -\frac{a_1b_1^2}{a\pi}, \quad A_3 = -a_2\frac{a}{\pi}, \quad A_4 = -a_1\frac{b_1^2}{a\pi},$$

$$B_1 = -\sqrt{\frac{a_{55}^0}{a_{44}^0}}\cdot\frac{c_1b_1^3}{a^2\pi}, \quad B_3 = -\sqrt{\frac{a_{55}^0}{a_{44}^0}}\cdot c_2\frac{b_1}{\pi},$$

$$B_4 = -\sqrt{\frac{a_{55}^0}{a_{44}^0}}\cdot c_2\frac{b_1}{\pi},$$

and the constants b_1, λ are given by the expressions [11 (3.10)]

$$b_1 = b\sqrt{\frac{a_{44}^0}{a_{55}^0}}, \qquad \lambda = \frac{b_1}{a} = \frac{b}{a}\sqrt{\frac{a_{44}^0}{a_{55}^0}}$$

They are expressed in terms of the elastic properties and the transversal dimension a, b of the bar.

The correctness of the solution of the boundary problem (2.3), (1.12), in the space L_2 follows from known theorems [6].

V. CONCLUSION

From the known Prandtl stress function $\Psi(x, y)$ (3.2) the tangential components of stress can be determined by means of the expressions (1.5). Using the derived relations wa can investigate the behaviour of the maximal tangential stresses, which depends also on the elastic constants of the material. The stiffness in torsion C_t can be determined by the known formula (1.11) and the torsion angle ϑ is given by (1.10). In this way there would not only the problem of the theory of elasticity be solved but also that of practical — application of finding shape of the body with the best mechanical properties by the torsion deformation, i.e. to find the shape of its cross-section when the elastic properties of material are given, in order to gain the maximal stiffness in torsion and minimal tangential stress.

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Received May 22nd, 1987
Accepted for publication September 2nd, 1987

К ВОПРОСУ О КРУЧЕНИИ НЕОДНОРОДНЫХ УПРУГОАНИЗОТРОПНЫХ (ОРТОТРОПНЫХ) СТЕРЖНЕЙ

В работе решена проблема кручения неоднородных упругоанизотропных (ортотропных) призматических стержней прямоугольного поперечного сечения с заданным напряжением на границе. Предполагается, что материал стержня является слабо неоднородным. Проблема решается методом малого параметра и сводится к рекуррентной последовательности краевых задач однородной теории. Для заданной неоднородности (2.1), (3.1) решение проблемы (т. е. решение краевой задачи (2.3), (1.12)) дается соотношением (3.2) при помощи функции напряжения Прандтля Ψ.