acta phys. slov. 38 (1988), No. 3

PATH INTEGRALS FOR A NONRELATIVISTIC SPIN 1/2 PARTICLE MOVING IN AN ELECTROMAGNETIC FIELD IN A GENERAL CONFIGURATION SPACE

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The nonrelativistic Pauli equation for a spin 1/2 particle moving in an electromagnetic field in a general configuration space is found by using the path integrals formalism of quantum mechanics. This work is based on a classical model for the spin 1/2 particle, using commuting spin variables, which has been developed recently by Petráš.

1. INTRODUCTION

The path integrals formalism of quantum mechanics has many successful applications in several branches of physics based on nonrelativistic quantum mechanics. The first who generalized this formalism in the case of a curved configuration space was De Witt [1] by studying, in a fundamental paper, the motion of a nonrelativistic scalar particle moving in a general curved configuration space. In his paper he obtained the Schrödinger equation in a curved configuration space with an extra term $\hbar^2 R/12$, where R is the scalar curvature of the configuration space. Next Cheng [2] obtained the same result using the path integrals formalism. Then Ben-Abraham and Lonke [3] proved for a general scalar dynamical system moving in a Riemann configuration space that both the covariant canonical and the path integrals quantization procedure yield in the Schrödinger equation a correction term proportional to the scalar curvature of the configuration space, where the coefficient of proportionality is $\hbar^2/12$.

In the present work we shall pursue the work of De Witt, Cheng and Ben-Abraham and Lonke in the case of a nonrelativistic spin 1/2 moving in an electromagnetic field in a three-dimensional curved configuration space. As a special case of this work in a flat space we shall obtain the path integrals formalism of the motion of spin 1/2 particle in an electromagnetic field as done by Petráš [4].

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It is well known that to construct the path integrals for the quantum motion of a nonrelativistic spin 1/2 particle one needs to give a consistent description of the classical spin degrees of freedom taking into account the quantum nature of the spin. A solution of this problem was originally suggested by Martin [5], who showed that the anticommuting (Grassmann) variables can be used for a consistent description of the classical spin degrees of fredom for a particle.

The introducing of the anticommuting variables into classical physics for the purpose of describing the classical spin angular momentum was due to the postulate of the canonical quantization procedure and according to Pauli, who noticed that the quantum spin angular momentum operators $\hat{S}_i$ admit a complex half-integer representation in which the quantum operators $\hat{S}_i$ (in the case of spin 1/2) satisfy the following anticommutation relations

$$\hat{S}_i \hat{S}_j + \hat{S}_j \hat{S}_i = \frac{\hbar^2}{2} \delta_{ij} \qquad (1.1)$$

it was suggested to associate with the particle three real anticommuting variables $\Theta_i(t)$, on the classical level, which satisfy the following anticommutation relations

$$\Theta_i \Theta_j + \Theta_j \Theta_i = 0\,, \qquad \Theta_i^2 = 0 \qquad (1.2)$$

in addition to the position coordinates $x_i(t)$ of the particle, where $i, j = 1, 2, 3$. This gave rise to the so-called *"pseudoclassical"* mechanics which has been studied by Casalbouni [6], [7] and Berezin and Marinov [8].

Recently Petráš [9] succeeded in formulating a consistant description for the classical spin degrees of freedom in terms of six canonically conjugated ordinary (commuting) variables ξ_i, η_j by studying the dynamics of the so called *"point top"* moving in an electromagnetic field.

The present work can be considered as a prototype of the work which has been done by Balek, Petráš and Melek in [10] and [11]. In section 2 the classical action of the nonrelativistic point top in a three-dimensional curved configuration space will be carried out. In section 3 the classical orbital and spin equations of motion for a point top will be found. In section 4 the equivalence of the nonrelativistic point top model and the Berezin and Marinov model for spinning particles will demonstrate that both models lead to the same classical equations of motion. In section 5 the equation for the propagator of the nonrelativistic spin 1/2 particle is derived by using the path integrals procedure.

2. THE ACTION OF THE POINT TOP IN A CURVED CONFIGURATION SPACE

According to Petráš [9], the spin part of the action of the nonrelativistic point top in a flat space, can be written as follows

$$S_s = \int_{t_0}^{t_N} \left[\eta_i \dot{\xi}_i + \frac{e}{mc} s_i B_i \right] \mathrm{d}t - (\xi_i \eta_i)_{t = t_N} \,, \tag{2.1}$$

where ξ_i, η_i are canonically conjugated variables used for the description of the spin degrees of freedom of the point top and satisfying the following ordinary Poisson brackets

$$\{\xi_i, \eta_j\} = \delta_{ij} \quad \text{and} \quad \{\xi_i, \xi_j\} = \{\eta_i, \eta_j\} = 0 \tag{2.2}$$

where $i, j = 1, 2, 3$, s_i is the spin vector constructed as a function of the variables ξ_i and η_i

$$s_i = \varepsilon_{ijk} \xi_j \eta_k + a_i \xi_k \eta_k \tag{2.3}$$

$$\xi_x = \xi \sin\Theta \sin\varphi, \quad \xi_y = -\xi \sin\Theta \cos\varphi, \quad \xi_z = \xi \cos\Theta \tag{2.4}$$

$$\xi = (\xi_x^2 + \xi_y^2 + \xi_z^2)^{1/2} = \mathrm{e}^{\Psi} \tag{2.5}$$

$$\dot{\xi}_i = \frac{\mathrm{d}\xi_i}{\mathrm{d}t}$$

$$\boldsymbol{a} = \frac{\xi}{\xi_x^2 + \xi_y^2} (\xi_x, \xi_y, 0) \tag{2.6}$$

and φ, Θ, Ψ are Euler angles.

Now, we would like to study in a three-dimensional curved configuration space the most general nonrelativistic classical dynamics of the point top, which can be derived from the classical nonrelativistic Petráš dynamics for a spin 1/2 particle. This means that we have to write the action (2.1) in a three-dimensional general curved configuration space.

The action (2.1) contains the terms $\xi_i \eta_i$, $\varepsilon_{ijk} \xi_j \eta_k$ and $\eta_i \dot{\xi}_i$, and since ξ_i and η_i are the variables that describe the spin degrees of freedom, and due to the fact that spinors in curved spaces can be defined only with respect to the vierbein basis [12], [13], we have to define the components of ξ_i and η_i with respect to three independent and orthonormal vectors (dreibein) $\lambda_{i\nu}$ which are erected at each point of the configuration space, where $i = 1, 2, 3$ is the local (dreibein) index and $\nu = 1, 2, 3$ is the coordinate index. The two terms $\xi_i \eta_i$ and $\varepsilon_{ijk} \xi_j \eta_k$ can be easily written with respect to the dreibein vectors. To write the term $\eta_i \dot{\xi}_i$ in a curved configuration space, we have to find the time derivative of the variable $\xi_i (\mathrm{D}\xi_i/\mathrm{d}t)$ in the curved configuration space. To do that, we have to use the well-known relation

$$\mathrm{D}\xi_i = \mathrm{d}\xi_i - \delta\xi_i \tag{2.7}$$

where

$$\mathrm{d}\xi_i = \frac{\mathrm{d}\xi_i}{\mathrm{d}t} \mathrm{d}t$$

is the difference between the components of ξ_i at the two points $x^\mu + \mathrm{d}x^\mu$ and x^μ, which are located on the world line of the particle, $\delta\xi_i$ is the variation of the components of ξ_i due to its parallel displacement between the two points x^μ and $x^\mu + \mathrm{d}x^\mu$, and

$$\mathrm{D}\xi_i = \frac{\mathrm{d}\xi_i}{\mathrm{d}t} \mathrm{d}t$$

is the difference between the components of ξ_i at the point $x^\mu + \mathrm{d}x^\mu$ and the components of the parallel displaced ξ_i at the same point $x^\mu + \mathrm{d}x^\mu$. Hence instead of the term $\eta_i \dot{\xi}_i$ in the action (2.1) one has to insert the term $\eta_i \dfrac{\mathrm{D}\xi_i}{\mathrm{d}t}$, which is the same term $\eta_i \dot{\xi}_i$ but is written in a curved space.

Now, we need to calculate $\delta\xi_i$, which can be considered as the change in ξ_i due to an infinitesimal local rotation of the triad vectors. The standard way to calculate $\delta\xi_i$ is to calculate the following Poisson brackets

$$\delta\xi_i = \{F, \xi_i\} \tag{2.8}$$

where

$$F = -s_i \delta\varphi_i$$

is the generating function of the infinitesimal rotation of the coordinate system with an angle $\delta\varphi_i$. The angle $\delta\varphi_i$ can be expressed as follows

$$\delta\varphi_i = \frac{1}{2} \varepsilon_{ijk} \omega_{jk} \,, \tag{2.9}$$

where ε_{ijk} is the fully antisymetric unit tensor and ω_{jk} is the matrix of the infinitesimal rotation. Using the Poisson brackets (2.2) one can prove that

$$\delta\xi_i = -\varepsilon_{ijk} (\delta\varphi_j)\, \xi_k - (\delta\varphi_j)\, a_j \xi_i \,. \tag{2.10}$$

To express the matrix ω_{jk} as a function of $\mathrm{d}x^\mu$, let us choose two very close points A, x^μ, and B, $x^\mu + \mathrm{d}x^\mu$ in the curved configuration space, and define

$$(\lambda_{i\nu})_{\mathrm{A} \to \mathrm{B}} = \lambda''_{i\nu}$$

the dreiben vectors that are in parallel displaced from the point A to the point B, $(\lambda_{i\nu})_{\mathrm{A}}$ and $(\lambda_{i\nu})_{\mathrm{B}}$ are the dreibein vectors at the points A and B, respectively. One can express $(\lambda_{i\nu})_{\mathrm{B}}$ as

$$(\lambda_{i\nu})_B = \Lambda_{ij}(\lambda_{j\nu})_{A\to B}, \tag{2.11}$$

where Λ_{ij} is the rotation matrix can be expressed as follows

$$\Lambda_{ij} = \delta_{ij} + \omega_{ij}. \tag{2.12}$$

By inserting the expression of Λ_{ij} from (2.12) into (2.11) one would get

$$\lambda_{i\nu}(x^\mu + dx^\mu) - \lambda''_{i\nu}(x^\mu + dx^\mu) = \omega_{ij}\lambda''_{j\nu}(x^\mu + dx^\mu)$$

$$D\lambda_{i\nu} = \lambda_{i\nu;\mu}dx^\mu = \omega_{ij}\lambda_{j\nu}$$

$$\omega_{ik} = \lambda_{i\nu;\mu}\lambda^\nu_k dx^\mu$$

$$\omega_{ik} = -\gamma_{ik\mu}\, dx^\mu \tag{2.13}$$

where

$$\gamma_{ik\mu} = -\lambda_{i\nu;\mu}\lambda^\nu_k \tag{2.14}$$

are the Ricci coefficients of rotation. Hence from (2.9) and (2.13) one gets

$$\delta\varphi_i = -\frac{1}{2}\,\varepsilon_{ijk}\gamma_{jk\mu}\, dx^\mu \tag{2.15}$$

therefore (2.10) becomes

$$\delta\xi_i = \left[\frac{1}{2}\,\varepsilon_{jlm}\gamma_{lm\mu}\alpha_j\xi_i - \gamma_{il\mu}\xi_l\right] dx^\mu. \tag{2.16}$$

Hence from (2.16) and (2.7) one could obtain

$$\frac{D\xi_i}{dt} = \frac{d\xi_i}{dt} + \left[\gamma_{ij\mu}\xi_j - \frac{1}{2}\,\varepsilon_{jlm}\gamma_{lm\mu}\alpha_j\xi_i\right]v^\mu, \tag{2.17}$$

where

$$v^\mu = \frac{dx^\mu}{dt}.$$

Hence, the spin part of the action of the nonrelativistic point top moving in a magnetic field in a three-dimensional curved configuration space is given by

$$S_s = \int_{t_0}^{t_N}\left\{\eta_i\dot{\xi}_i + \left[\frac{e}{mc}\,B_j - \frac{1}{2}\,\varepsilon_{jlm}\gamma_{lm\mu}v^\mu\right]s_j\right\} dt - (\xi_i\eta_i)_{t=t_N}. \tag{2.18}$$

The orbital part of the action is given by

$$S_O = \int_{t_0}^{t_N}\left\{\frac{m}{2}\,g_{\mu\nu}v^\mu v^\nu + \frac{e}{c}\,g_{\mu\nu}A^\mu v^\nu - e\varphi\right\} dt, \tag{2.19}$$

where $g_{\mu\nu}$ is the metric tensor of the three-dimensional curved configuration

space, m is the mass of the point top, A^μ is the vector potential of the electric field, φ is the scalar potential, e is the charge of the point top and c is the velocity of light.

Therefore, the total action of a point top moving in an electromagnetic field in a three-dimensional curved configuration space is given by

$$S = S_O + S_s = \int_{t_0}^{t_N}\{\eta_i\dot{\xi}_i - R_T\}\, dt - (\xi_i\eta_i)_{t=t_N} \tag{2.20}$$

where R_T is the total Routh function of the point top given by

$$R_T = -\frac{m}{2}\,g_{\mu\nu}v^\mu v^\nu - \frac{e}{c}\,g_{\mu\nu}A^\mu v^\nu + e\varphi - \left[\frac{e}{mc}\,B_j - \frac{1}{2}\,\varepsilon_{jlm}\gamma_{lm\mu}v^\mu\right]s_j. \tag{2.21}$$

Here, the Routh function plays the role of the Lagrangian for the orbital variables and the role of the Hamiltonian for the spin variables of the point top. In the appendix we shall prove that the action (2.20) is invariant with respect to a general local rotation of the dreibein vectors.

3. CLASSICAL EQUATIONS OF MOTION

From the spin part of the action (2.18), the interacting Hamiltonian of the spin of the point top with an external magnetic field in a curved configuration space is given by

$$H_{int} = \left[-\frac{e}{mc}\,B_j + \frac{1}{2}\,\varepsilon_{jlm}\gamma_{lm\mu}v^\mu\right]s_j. \tag{3.1}$$

Hence, the time development equation for the spin vector s_i is given by

$$\frac{ds_i}{dt} = \{H_{int}, s_i\}, \tag{3.2}$$

where

$$\{H_{int}, s_i\} = \frac{\partial H_{int}}{\partial\xi_k}\cdot\frac{\partial s_i}{\partial\eta_k} - \frac{\partial H_{int}}{\partial\eta_k}\cdot\frac{\partial s_i}{\partial\xi_k}$$

is the standard definition of the Poisson bracket. By inserting the expression of H_{int} from (3.1) into (3.2) and using the fact that

$$\{s_i, s_j\} = -\varepsilon_{ijk}s_k \tag{3.3}$$

one should get

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = \varepsilon_{ijk}\left[\frac{1}{2}\,\varepsilon_{jlm}\,\gamma_{lm\mu}v^{\mu} - \frac{e}{mc}\,B_j\right]s_k\,. \tag{3.4}$$

By using the least action principle for the action (2.20), one would obtain the following orbital equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R_T}{\partial v^{\varrho}}\right) - \frac{\partial R_T}{\partial x^{\varrho}} = 0\,, \tag{3.5}$$

which will lead to the following equations

$$\frac{\mathrm{d}^2x^{\varrho}}{\mathrm{d}t^2} + \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t} = \frac{e}{mc}\,E^{\varrho} + \frac{e}{mc}\,\varepsilon^{\varrho}{}_{\mu\nu}\,\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\,B^{\nu} + \frac{e}{m^2c}\,B_{i;\mu}g^{\mu\varrho}s_i + $$
$$+\frac{1}{2m}\,\varepsilon_{ijk}\,s_i R_{jk\mu}{}^{\varrho}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\,, \tag{3.6}$$

where

$$\begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix} = \frac{1}{2}\,g^{\varrho\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$$

E_{ϱ} is the electric field intensity vector, $B_i = \lambda_i^{\mu}B_{\mu}$

$$B_{i;\varrho} = (\lambda_i^{\mu}B_{\mu})_{;\varrho} = \lambda^{\mu}_{i;\varrho}B_{\mu} + \lambda_i^{\mu}B_{\mu;\varrho}$$

$$\lambda^{\mu}_{i;\varrho} = \lambda^{\mu}_{i,\varrho} + \begin{Bmatrix}\mu\\ \nu\varrho\end{Bmatrix}\lambda_i^{\nu}$$

$$B_{\mu;\varrho} = B_{\mu,\varrho} - \begin{Bmatrix}\nu\\ \mu\varrho\end{Bmatrix}B_{\nu}$$

$$R_{jk\mu\nu} = \lambda_j^{\varrho}\lambda_k^{\sigma}R_{\varrho\sigma\mu\nu} = \gamma_{jk\nu,\mu} - \gamma_{jk\mu,\nu} + \gamma_{lk\nu}\gamma_{lj\mu} - \gamma_{lk\mu}\gamma_{lj\nu}\,.$$

In the case of the absence of the external electromagnetic field, the equations (3.6) will be

$$\frac{\mathrm{d}^2x^{\varrho}}{\mathrm{d}t^2} + \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t} = \frac{1}{2m}\,\varepsilon_{ijk}\,s_i R_{jk\mu}{}^{\varrho}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\,. \tag{3.7}$$

4. THE EQUIVALENCE OF THE NONRELATIVISTIC CLASSICAL MODEL OF THE POINT TOP AND THE GRASSMANN VARIABLES MODELS

Berezin and Marinov [8] studied the classical nonrelativistic spin dynamics by using three real Grassmann dynamical variabes $\Theta_i(t)$. They defined the Poisson brackets for any pair of functions of $\Theta_i(t)$ as follows

$$\{f(\Theta),\, g(\Theta)\} = \mathrm{i}\,(f\overleftarrow{\partial}_k)(\overrightarrow{\partial}_k g) \tag{4.1}$$

where the left-hand and right-hand partial derivative ($\overleftarrow{\partial}_k$) and ($\overrightarrow{\partial}_k$) are defined as a derivative with respect to Θ_k f a monomial as follows

$$(\Theta_{l_1}\ldots\Theta_{l_n})\,\frac{\overleftarrow{\partial}}{\partial\Theta_k} = \delta_{l_nk}\,\Theta_{l_1}\ldots Q_{l_{n-1}} -$$
$$-\,\delta_{l_{n-1}k}\,\Theta_{l_1}\Theta_{l_2}\ldots\Theta_{l_{n-2}}\Theta_{l_n} + \ldots + (-1)^n\,\delta_{l_1k}\,\Theta_{l_2}\ldots\Theta_{l_n}\,. \tag{4.2}$$

$$\frac{\overrightarrow{\partial}}{\partial\Theta_k}\,(\Theta_{l_1}\ldots\Theta_{l_n}) = \delta_{l_1k}\,\Theta_{l_2}\ldots\Theta_{l_n} -$$
$$-\,\delta_{l_2k}\,\Theta_{l_1}\Theta_{l_3}\ldots\Theta_{l_n} + \ldots + (-1)^n\,\delta_{l_nk}\,\Theta_{l_1}\Theta_{l_2}\ldots\Theta_{l_{n-1}}\,. \tag{4.3}$$

Thus the variables Θ_k are fulfilling the following Poisson brackets

$$\{\Theta_l,\, \Theta_k\} = \mathrm{i}\delta_{lk}\,. \tag{4.4}$$

The spin part of the classical action of a nonrelativistic spinning particle in an external magnetic field B_k in a flat space is given by

$$A_s = \int_{t_0}^{t_N}\left[\frac{\mathrm{i}}{2}\,\Theta_k\,\frac{\mathrm{d}\Theta_k}{\mathrm{d}t} + \frac{e}{mc}\,s_k(\Theta)\,B_k(x)\right]\mathrm{d}t + \frac{\mathrm{i}}{2}\,\Theta_k(t_N)\,\Theta_k(t_0)\,, \tag{4.5}$$

where

$$s_k(\Theta) = -\frac{\mathrm{i}}{2}\,\varepsilon_{klm}\Theta_l\Theta_m \tag{4.6}$$

is the spin vector which generates the group SO(3) in the Grassmann phase space, k, l, $m = 1, 2, 3$, and one can prove that

$$\{s_k,\, s_j\} = -\,\varepsilon_{kjl}s_l\,. \tag{4.7}$$

Here, we have to point out that the presence of the term $\left[\frac{\mathrm{i}}{2}\,\Theta_k(t_N)\,\Theta_k(t_0)\right]$ (boundary term) in the action (4.5) is due to the fact that the equation of motion for the variable $\Theta_i(t)$ is a first order differential equation, as it will be shown later, and therefore one cannot fix Θ_k at the initial and final points. Berezin and Marinov did not take into account this term in their work [8]. The classical equations of motion of the variable Θ_k can be found by varying the action (4.5) with respect to Θ_k and applying the least action principle with the following boundary condition

$$\delta\Theta_k(t_N) + \delta\Theta_k(t_0) = 0\,, \tag{4.8}$$

which will give

$$\frac{d\Theta_k}{dt} = \frac{e}{mc}\,\varepsilon_{kjl} B_j \Theta_l . \tag{4.9}$$

Applying the same procedure as in Sect. 2, the spin part of the action of the nonrelativistic spinning particle moving in an external magnetic field in a three-dimensional curved configuration space is given by

$$A_s = \int_{t_0}^{t_N} \left\{ \frac{i}{2}\,\Theta_k \dot{\Theta}_k + \left[\frac{e}{mc} B_k - \frac{1}{2}\,\varepsilon_{klm} \gamma_{lm\mu} v^{\mu} \right] s_k \right\} dt + \frac{i}{2}\,\Theta_k(t_N)\,\Theta_k(t_0) . \tag{4.10}$$

One can prove that the equations of motion of the spin vector s_k (4.6) of a spinning particle moving in an external magnetic field in a curved configuration space are the same as (3.4) and the orbital equations of motion are the same as (3.6).

Therefore, one can conclude the equivalence of the point top model and the Grassmann variables models for spinning particles on the classical level. This equivalence is valid, also, on the quantum level as it will be shown in the next section, in the sense that both classical theories will lead to the standard quantum theory of the nonrelativistic spin 1/2 particle.

5. PATH INTEGRALS FOR THE NONRELATIVISTIC PAULI EQUATION IN THE CURVED CONFIGURATION SPACE

The amplitude $K_{N,0}$ of the transition of the point top from the initial state

$$\Phi(\boldsymbol{x}_0, t_0; \xi_0) = \zeta^{+}(\xi_0)\,\Psi(\boldsymbol{x}_0, t_0) , \tag{5.1}$$

to the final state

$$\Phi(\boldsymbol{x}_N, t_N; \xi_N, \boldsymbol{\eta}_N) = \zeta^{+}(\xi_N, \boldsymbol{\eta}_N)\,\Psi(\boldsymbol{x}_N, t_N) \tag{5.2}$$

is defined by

$$K_{N,0} = \int \exp\left(\frac{i}{\hbar} S_O\right) \mathscr{D}\boldsymbol{x}(t) \times$$
$$\times \int \exp\left(\frac{i}{\hbar} S_s\right) \zeta(\xi_N, \boldsymbol{\eta}_N)\,\zeta^{+}(\xi_0)\, d^3\xi_0\, d^3\xi_N\, d^3\boldsymbol{\eta}_N \mathscr{D}\boldsymbol{\eta}(t) \mathscr{D}\xi(t) \tag{5.3}$$

where S_0, S_s are given by (2.18) and (2.19), respectively, Ψ is a Pauli spinor, ζ is a two-component spinor chosen by Petráš [9]

$$\zeta = \begin{pmatrix} \exp\left[-\frac{i}{2}(\psi + \varphi)\right] \cos\frac{\Theta}{2} \\ -i\exp\left[-\frac{i}{2}(\varphi - \psi)\right] \sin\frac{\Theta}{2} \end{pmatrix} \tag{5.4}$$

ζ^{+} is the hermitian conjugate of the spinor ζ

$$\mathscr{D}\boldsymbol{x}(t) = \frac{\left[1 + \frac{1}{12} R_{\mu\nu}(\boldsymbol{x}_N)\, Z^{\mu}_N Z^{\nu}_N\right]}{(2\pi i\hbar\, \Delta_N)^{3/2}} \times$$
$$\times \frac{\left[1 + \frac{1}{12} R_{\varrho\sigma}(\boldsymbol{x}_{N-1})\, Z^{\varrho}_{N-1} Z^{\sigma}_{N-1}\right] [g(\boldsymbol{x}_{N-1})]^{1/2}}{(2\pi i\hbar\, \Delta_{N-1})^{3/2}}\, d^3x_{N-1} \times \ldots$$
$$\times \frac{\left[1 + \frac{1}{12} R_{\alpha\beta}(\boldsymbol{x}_2)\, Z^{\alpha}_2 Z^{\beta}_2\right] [g(\boldsymbol{x}_2)]^{1/2}}{(2\pi i\hbar\, \Delta_2)^{3/2}}\, d^3x_2 \times$$
$$\times \frac{\left[1 + \frac{1}{12} R_{\varepsilon\delta}(\boldsymbol{x}_1)\, Z^{\varepsilon}_1 Z^{\delta}_1\right] [g(\boldsymbol{x}_1)]^{1/2}}{(2\pi i\hbar\, \Delta_1)^{3/2}}\, d^3x_1$$

$$\mathscr{D}\boldsymbol{\eta}(t) \mathscr{D}\xi(t) = \frac{d^3\boldsymbol{\eta}_0\, d^3\xi_1}{(2\pi\hbar)^3} \cdot \frac{d^3\boldsymbol{\eta}_1\, d^3\xi_2}{(2\pi\hbar)^3} \ldots \frac{d^3\boldsymbol{\eta}_{N-1}\, d^3\xi_{N-1}}{(2\pi\hbar)^3}$$

$$\Delta_{i+1} = t_{i+1} - t_i; \qquad Z^{\mu}_{i+1} = x^{\mu}_{i+1} - x^{\mu}_i; \qquad \boldsymbol{\eta}_i = \boldsymbol{\eta}(t_i);$$
$$\xi_i = \xi(t_i) \text{ and } R_{\mu\nu} = -g^{\varrho\sigma} R_{\mu\varrho\nu\sigma} .$$

To perform the integration of (5.3), we have to discretize both parts of the action (2.18) and (2.19). For the orbital part of the action, we shall follow De Witt's discretization [1] with some modification which is neccessary to obtain the correct form of the orbital terms in the nonrelativistic Pauli equation for the amplitude $K_{N,0}$. The modified discretized action is given by

$$S_O = \sum_{i=0}^{N-1} \frac{m}{2\Delta_{i+1}} \left\{ g_{\mu\nu}(\boldsymbol{x}_{i+1})\, Z^{\mu}_{i+1} Z^{\nu}_{i+1} - \frac{1}{2} g_{\mu\nu,\varrho}(\boldsymbol{x}_{i+1})\, Z^{\mu}_{i+1} Z^{\nu}_{i+1} Z^{\varrho}_{i+1} + \right.$$
$$+ \frac{1}{36}\big(6 g_{\mu\nu,\varepsilon\delta}(\boldsymbol{x}_{i+1}) - g^{\alpha\beta}(\boldsymbol{x}_{i+1}) ([\mu\nu, \alpha][\varepsilon\delta, \beta] +$$
$$\left. + [\mu\varepsilon, \alpha][\delta\nu, \beta] + [\mu\delta, \alpha][\nu\varepsilon, \beta])\big)\, Z^{\mu}_{i+1} Z^{\nu}_{i+1} Z^{\varepsilon}_{i+1} Z^{\delta}_{i+1} \right\} +$$
$$+ \sum_{i=0}^{N-1} \frac{e}{c} \left\{ -g_{\mu\nu}(\boldsymbol{x}_{i+1})\, A^{\mu}(\boldsymbol{x}_{i+1})\, Z^{\nu}_{i+1} + \frac{1}{2} A_{\mu,\nu}(\boldsymbol{x}_{i+1})\, Z^{\mu}_{i+1} Z^{\nu}_{i+1} \right\} -$$
$$- \sum_{i=0}^{N-1} e\varphi(\boldsymbol{x}_{i+1})\, \Delta_{i+1} . \tag{5.5}$$

The appropriate discretization of the spin part of the action, leading to the correct form of the spin terms in the nonrelativistic Pauli equation, is given by

$$S_s = \sum_{i=0}^{N-1} \eta_n(t_i)\,[\xi_n(t_{i+1}) - \xi'_n(t_i)] - \xi_n(t_N)\,\eta_n(t_N) \tag{5.6}$$

where

$$\xi'_n(t_i) = \xi_n(t_i) + \Delta_{i+1}(a_n)_{i+1} + \frac{1}{2}\,\varepsilon_{jlm}(b_{jn})_{i+1}\,\gamma_{lm\mu}(\boldsymbol{x}_{i+1})\,Z^\mu_{i+1} -$$
$$- \frac{1}{4}\,\varepsilon_{jlm}(b_{jn})_{i+1}\,\gamma_{lm\mu,\nu}(\boldsymbol{x}_{i+1})\,Z^\mu_{i+1} Z^\nu_{i+1} -$$
$$- \frac{1}{8}\,\varepsilon_{jlm}\varepsilon_{kl_1m_1}(C_{jkn})_{i+1}\,\gamma_{lm\mu}(\boldsymbol{x}_{i+1})\,\gamma_{l_1m_1\nu}(\boldsymbol{x}_{i+1})\,Z^\mu_{i+1} Z^\nu_{i+1} \tag{5.7}$$

$$a_n = -\frac{e}{mc}\frac{\partial}{\partial \eta_n}\,[s_j(\xi,\,\boldsymbol{\eta})\,B_j(\boldsymbol{x})] \tag{5.8}$$

$$b_{jn} = \varepsilon_{jkn}\xi_k + a_j\xi_n \tag{5.9}$$

$$C_{jkn} = S_j b_{kn} \tag{5.10}$$

$$S_j = \left(\frac{\partial s_j}{\partial \eta_k}\right)\frac{\partial}{\partial \xi_k} \tag{5.11}$$

$$S_j\eta^+(\xi,\,\boldsymbol{\eta}) = \frac{i}{2}\,\zeta^+(\xi,\,\boldsymbol{\eta})\,\sigma_j \tag{5.12}$$

$$\begin{aligned}(a_n)_{i+1} &= a_n(\xi_{i+1},\,\boldsymbol{\eta}_{i+1};\,\boldsymbol{x}_{i+1})\\ (b_{jn})_{i+1} &= b_{jn}(\xi_{i+1},\,\boldsymbol{\eta}_{i+1})\\ (C_{jkn})_{i+1} &= C_{jkn}(\xi_{i+1},\,\boldsymbol{\eta}_{i+1})\,.\end{aligned} \tag{5.13}$$

The integration with respect to the spin variables in (5.3) can be calculated and will lead to the following result

$$\int \exp\left(\frac{i}{\hbar}S_s\right)\zeta(\xi_N,\,\boldsymbol{\eta}_N)\,\zeta^+(\xi_0)\,d^3\xi_0\,d^3\xi_N\,d^3\boldsymbol{\eta}_N\,\mathscr{D}\boldsymbol{\eta}(t)\,\mathscr{D}\xi(t) =$$
$$= (2\pi)^{3/2}\left\{1 + \left[\Delta_N\left(\frac{ie}{2mc}\,B_j(\boldsymbol{x}_N)\right) - Z^\mu_N\left(\frac{i}{4}\,\varepsilon_{jlm}\gamma_{lm\mu}(\boldsymbol{x}_N)\right) +\right.\right.$$
$$\left. + Z^\mu_N Z^\nu_N\left(\frac{i}{8}\,\varepsilon_{jlm}\gamma_{lm\mu,\nu}(\boldsymbol{x}_N)\right)\right]\sigma_j +$$
$$+ \frac{1}{2}\,Z^\mu_N Z^\nu_N\left(\frac{i}{4}\,\varepsilon_{jlm}\gamma_{lm\mu}(\boldsymbol{x}_N)\,\sigma_j\right)\left(\frac{i}{4}\,\varepsilon_{kl_1m_1}\gamma_{l_1m_1\nu}(\boldsymbol{x}_N)\,\sigma_k\right)\bigg\} \times \ldots$$
$$\ldots \times \left\{1 + \left[\Delta_1\left(\frac{ie}{2mc}\,B_{j_1}(\boldsymbol{x}_1)\right) - Z^\varrho_1\left(\frac{i}{4}\,\varepsilon_{j_1pq}\gamma_{pq\varrho}(\boldsymbol{x}_1)\right) +\right.\right.$$
$$\left. + Z^\varrho_1 Z^\sigma_1\left(\frac{i}{8}\,\varepsilon_{j_1pq}\gamma_{pq\varrho,\sigma}(\boldsymbol{x}_1)\right)\right]\sigma_{j_1} +$$
$$+ \frac{1}{2}\,Z^\varrho_1 Z^\sigma_1\left(\frac{i}{4}\,\varepsilon_{j_1pq}\gamma_{pq\varrho}(\boldsymbol{x}_1)\,\sigma_{j_1}\right)\left(\frac{i}{4}\,\varepsilon_{k_1p_1q_1}\gamma_{p_1q_1\sigma}(\boldsymbol{x}_1)\,\sigma_{k_1}\right)\bigg\}. \tag{5.14}$$

By inserting the result (5.14) into (5.3) and applying the same procedure as in [14] to get the Kein-Gordon equation; one will get the following partial differential equation for the amplitude $K_{N,0}$

$$i\hbar\,\frac{\partial K_{N,0}}{\partial t} = HK_{N,0} \tag{5.15}$$

where

$$H = \frac{1}{2m(g)^{1/2}}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - \left(\frac{e}{c}\,A_\mu + \frac{\hbar}{4}\,\varepsilon_{jlm}\gamma_{lm\mu}\sigma_j\right)\right] \times\right.$$
$$\left.\times \left[(g)^{1/2}g^{\mu\nu}\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\nu} - \left(\frac{e}{c}\,A_\nu + \frac{\hbar}{4}\,\varepsilon_{kl_1m_1}\gamma_{l_1m_1\nu}\sigma_k\right)\right]\right]\right\} -$$
$$- \frac{e}{2mc}\,\boldsymbol{\sigma}.\boldsymbol{B} + e\varphi + \frac{\hbar^2}{12m}\,R. \tag{5.16}$$

R is the scalar curvature of the configuration space and σ_i, $i = 1, 2, 3$ are the Pauli spin matrices.

In the case of a flat configuration space, equation (5.15) becomes the nonrelativistic Pauli equation for the spin 1/2 particle.

6. CONCLUSIONS AND DISCUSSIONS

From the orbital equations of motion (3.6), one can conclude that particles with the same spin and different masses — and vice versa — will move on different trajectories. From Sect. 3, one can conclude that it is not neccessary to use the Grassmann variables for the description of the classical nonrelativistic spin degrees of freedom, but it is enough to use the commuting (ordinary) variables. This conclusion was obtained by Schulman [15] and Bezák [16]. From Sect. 5, one can conclude that the point top as a well defined classical object from both kinematical and dynamical points of view is a suitable candidate for the spin 1/2 particles.

APPENDIX

The aim of this appendix is to prove that the action of the nonrelativistic point top moving in an external electromagnetic field in a curved configuration space (2.20) is invariant with respect to a general local rotation of the dreibein vectors. From the Routh function (2.21), one can see that the orbital part of the action is invariant with respect to a general local rotation of the dreibein vectors. Also from the equation of motion of the spin s_i in a flat space and the Poisson bracket (3.3), one can conclude that the spin s_i behaves as an ordinary vector with respect to rotation. Hence, the term $\frac{e}{mc} B_j s_j$ is a scalar. Therefore, to prove that the action (2.20) is invariant, one has to prove that

$$\eta_i' \dot{\xi}_i' - \frac{1}{2} \varepsilon_{ijk} \gamma'_{jk\mu} s_i' v^\mu = \eta_i \dot{\xi}_i - \frac{1}{2} \varepsilon_{ijk} \gamma_{jk\mu} s_i v^\mu \tag{A.1}$$

which is equivalent to prove that

$$\delta\eta_i \dot{\xi}_i + \eta_i \delta\dot{\xi}_i = \frac{1}{2} \varepsilon_{ijk} [\gamma_{jk\mu} \delta s_i + \delta(\gamma_{jk\mu}) s_i] v^\mu \tag{A.2}$$

where

$$\dot{\xi}_i' = \dot{\xi}_i + \delta\dot{\xi}_i, \qquad \eta_i' = \eta_i + \delta\eta_i$$
$$s_i' = s_i + \delta s_i, \qquad \gamma'_{jk\mu} = \gamma_{jk\mu} + \delta(\gamma_{jk\mu})$$

$\delta\dot{\xi}_i$, $\delta\eta_i$, δs_i and $\delta(\gamma_{jk\mu})$ are the change in the quantities $\dot{\xi}_i$, η_i, s_i and $\gamma_{jk\mu}$, respectively, due to an infinitesimal local rotation of the dreibein vectors. The standard way to find the quantities $\delta\xi_i$, $\delta\eta_i$ and δs_i is to calculate the following Poisson brackets

$$\delta\xi_i = \{F, \xi_i\}, \qquad \delta\eta_i = \{F, \eta_i\}$$
$$\delta s_i = \{F, s_i\},$$

where

$$F = -s_i \delta\varphi_i$$

is the generating function of the infinitesimal rotation and $\delta\varphi_i$ is the infinitesimal angle of rotation. From these calculations one gets

$$\delta\xi_i = -[\varepsilon_{ijk} \xi_k + \alpha_j \xi_i] \delta\varphi_j \tag{A.3}$$

$$\delta\eta_i = \left[\varepsilon_{ijk} \eta_k + \alpha_j \eta_i + \frac{\partial \alpha_j}{\partial \xi_i} \eta_k \xi_k\right] \delta\varphi_j \tag{A.4}$$

$$\delta s_i = -\varepsilon_{ijk} s_k \delta\varphi_j . \tag{A.5}$$

By differentiating (A.3) with respect to the time one gets

$$\delta\dot{\xi}_i = -(\varepsilon_{ijk} \dot{\xi}_k + \dot{\alpha}_j \xi_i + \alpha_j \dot{\xi}_i) \delta\varphi_j - $$
$$- (\varepsilon_{ijk} \xi_k + \alpha_j \xi_i) v^\mu (\delta\varphi_j)_{,\mu} . \tag{A.6}$$

From the definition of $\gamma_{jk\mu}$ (2.14) one can prove that

$$\delta(\gamma_{jk\mu}) = (\varepsilon_{jlm} \gamma_{lm\mu} + \varepsilon_{klm} \gamma_{jl\mu}) \delta\varphi_m - \varepsilon_{jkl} (\delta\varphi_l)_{,\mu} . \tag{A.7}$$

By inserting the results of (A.4), (A.5), (A.6) and (A.7) into (A.2), one can prove that the left- and the right-hand side of (A.2) is equal to

$$-s_j v^\mu (\delta\varphi_j)_{,\mu}$$

which means that our aim has been achieved.

Also, using the previous calculations, one can prove that the action (4.10) is invariant with respect to a general local rotation of the dreibein vectors.

REFERENCES

[1] De Witt, B. S.: Rev. Mod. Phys. *29* (1957), 377.
[2] Cheng, K. S.: J. of Math. Phys. *13* (1972), 1723.
[3] Ben-Abraham, S. L., Lonke, A.: J. of Math. Phys. *14* (1973), 1935.
[4] Petráš, M.: *unpublished.*
[5] Martin, J. L.: proc. Roy. Soc. A *251* (1959), 536.
[6] Casalbouni, R.: Il Nuovo Cimento *33 A* (1976).
[7] Casalbouni, R.: Il Nuovo Cimento *33 A* (1976).
[8] Berezin, F. A., Marinov, M. S.: Ann. of Phys. *104* [1977), 336.
[9] Petráš, M.: *to be published.*
[10] Balek, V., Petráš, M., Melek, M.: *11th Int. Conf. on General Relativity and Gravitation,* Stockholm, Sweden (1986).
[11] Balek, V., Melek, M., Petráš, M.: *to appear in* Czech. J. of Phys. (1987).
[12] Utiyama, R.: Phys. Rev. *101* (1956), 1597.
[13] Kibble, T. W. B.: J. of Math. Phys. *2* (1961), 212.
[14] Melek, M.: Phys. Rev. D *29* (1984), 1081.
[15] Schulman, L.: Phys. Rev. *176* (1968), 1558.
[16] Bezák, V.: Int. J. of Theo. Phys. *11* (1974), 321.

Received September 23rd, 1987.
Accepted for publication November 18th, 1987.

ИНТЕГРАЛЫ ПО ТРАЕКТОРИЯМ ДЛЯ НЕРЕЛЯТИВИСТСКОЙ ЧАСТИЦЫ СО СПИНОМ 1/2, ДВИЖУЩЕЙСЯ В ЭЛЕКТРОМАГНИТНОМ ПОЛЕ В ОБЩЕМ КОНФИГУРАЦИОННОМ ПРОСТРАНСТВЕ

На основе квантовомеханического формализма интегралов по траекториям найденонерелятивистское уравнение Паули для частицы со спином 1/2, движущейся в электромагнитном поле в общем конфигурационном пространстве. Эта работа основана на классической модели для частицы со спином 1/2, в которой используются коммутирующие спиновые переменные, недавно введенные Петрашом.