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A STOCHASTIC THEORY OF WEAR WITH A QUADRATIC ANNIHILATION RATE PART II

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The continuation of the preceding contribution to this issue of APS (Part I) is presented. The theory of Part I is generalized for the case when $V(x) = V_0 + \lambda x^2$ with $V_0 = \text{const.} > 0$.

IV. THE CASE WHEN $V(x) = V_0 + \lambda x^2$, $V_0 > 0$, $\lambda > 0$

IV.1. The parallel between $C(x, \beta; x_0)$ and $g(x, t; x_0)$

When including a constant V_0 into the potential energy in equation (9), we only shift the energy scale. The corresponding transformation of the quantum-statistical density matrix is simple:

$$C(x, \beta; x_0) = \exp(-\beta V_0)\, C_0(x, \beta; x_0), \qquad \beta > 0. \tag{47}$$

For $C_0(x, \beta; x_0)$, we have already derived expression (12).

Similarly, for Green's function satisfying equation (3), we may write the result

$$g(x, t; x_0) = \exp(-tV_0)\, g_0(x, t; x_0), \qquad t > 0 \tag{48}$$

where $g_0(x, t; x_0)$ is given by expression (17) (or (19)). We may also write

$$\Phi(x_0, t) = \exp(-tV_0)\, \Phi_0(x_0, t), \qquad t > 0 \tag{49}$$

using expression (24) for $\Phi_0(x_0, t)$.

IV.2. The lifetime distribution $\varphi_s(\tau)$

It is convenient to measure the time in units of ω^{-1} (cf. definition (18)). Then we introduce the dimensionless parameter

$$s = V_0/\omega > 0 \tag{50}$$

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and write

$$\Phi(x_0, t) = \Phi_s(x_0, t) = e^{-s\omega t}\Phi_0(x_0, t). \tag{51}$$

($\omega t > 0$ is then the dimensionless time variable.) Thus, according to formula (2), we obtain the lifetime density

$$\varphi_s(x_0, \tau) = -\frac{\partial \Phi_s(x_0, \tau)}{\partial \tau} = [\varphi_0(x_0, \tau) + s\omega\Phi_0(x_0, \tau)]\, e^{-s\omega\tau}. \tag{52}$$

The statistical moments $\langle \tau^n \rangle$ calculated with this probability density will be denoted as $\langle \tau^n \rangle_s$, m_s, σ_s^2. (This is in consistency with the subscript zero in expressions of Sections III.2 and III.3.)

IV.3. The mean value and the r.m.s. deviation of the lifetime τ for $x_0 = 0$ in the wearless case

Using expressions (24) and (28), (29) in formula (52), we can — for any value of $\mu > 0$ — numerically calculate the statistical moments

$$\langle \tau^n \rangle_s = -\int_0^\infty d\tau \tau^n \frac{\partial \Phi_s(x_0, \tau)}{\partial \tau} = n \int_0^\infty d\tau \tau^{n-1} \Phi_s(x_0, \tau) \tag{53}$$

($n = 1, 2, \ldots$). In our further analysis, similarly as in Section III.3, we shall only consider the value $x_0 = 0$.

To calculate the moments $\langle \tau^n \rangle_s$ by the perturbation technique of Section III.3.2. we must cope with the integrals

$$A_n(s) = \frac{1}{2}\int_0^\infty dy y^n \frac{\tanh y}{(\cosh y)^{1/2}}, \tag{54}$$

$$B_n(s) = \frac{1}{2}\int_0^\infty dy y^n \frac{(\tanh y)^2}{(\cosh y)^{1/2}} \tag{55}$$

($n = 1, 2, \ldots$). (In Section III.3.2 and in the Appendices we use the denotation A_n, B_n for $A_n(0)$, $B_n(0)$.) Note that

$$A_n(s) = \left(-\frac{d}{ds}\right)^{n-1} A_1(s), \qquad B_n(s) = \left(-\frac{d}{ds}\right)^{n-1} B_1(s) \tag{56}$$

for $n = 2, 3, \ldots$.

For simplicity, we will only discuss the wearless case here (when $\mu = 0$). Then the integrals $B_n(s)$ drop out in futher calculations. On interating $A_n(s)$ by parts one finds out that

$$A_n(s) = nC_{n-1}(s) - sC_n(s) \qquad (n = 1, 2, \ldots) \tag{57}$$

where

$$C_m(s) = \int_0^\infty dy\, y^m \frac{e^{-sy}}{(\cosh y)^{1/2}} \qquad (m = 1, 2, \ldots). \tag{58}$$

These integrals are derivatives of $C_0(s)$:

$$C_m(s) = \left(-\frac{d}{ds}\right)^m C_0(s). \tag{59}$$

For $s = 0$, we have the values

$$C_m(s) = \frac{1}{m+1} A_{m+1} \qquad (m = 0, 1, \ldots).$$

IV.3.1. General series for $\langle \tau^n \rangle_s^{(0)}$

For the integral $C_0(s)$, we can, following the way indicated in Appendix A.1, directly derive the series

$$C_0(s) = \int_0^\infty dy \frac{e^{-sy}}{(\cosh y)^{1/2}} =$$

$$= \sqrt{2}\left[\frac{1}{s+\frac{1}{2}} + \sum_{\infty}^{\infty} k = 1\ (-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{1}{s + 2k + \frac{1}{2}}\right]. \tag{60}$$

This series is so simple and easy to be programmed that it need not be transformed into anything else. We could, anyway, write in general

$$C_0(s) = \frac{\sqrt{2}}{s+\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}\left(s+\frac{1}{2}\right); \frac{1}{2}\left(s+\frac{5}{2}\right); -1\right)$$

where $F(a, b; c; \xi)$ is the Gauss hypergeometric function (usually denoted as ${}_2F_1$ by mathematicians [10—13]).

Note that we can calculate the integral (60) directly for half-integer values of s. So we obtain the values:

$$C_0\left(\frac{1}{2}\right) = \sqrt{2}\,\text{Arsihn}\,1 = \sqrt{2}\ln(1+\sqrt{2}),$$

$$C_0\left(\frac{3}{2}\right) = 2, \text{ etc.}$$

Thus, taking for instance $s = \frac{3}{2}$, we can easily estimate the precision with which we may replace the infinite series (60) by partial sums (fixing some index $l > 1$ and dropping terms with $k > l$).

When bearing in mind formula (53) and expressions (4), (51), we can write the moments of τ in the form

$$\langle \tau^n \rangle_s^{(0)} = \frac{n}{\omega^n} C_{n-1}(s). \tag{61}$$

Thus, by using the relations (59) and (60), we can express each moment $\langle \tau^n \rangle_s^{(0)}$ as an infinite series. For the first and second moment, we have the results:

$$\langle \tau \rangle_s^{(0)} = \frac{1}{\omega} C_0(s), \tag{62}$$

$$\langle \tau^2 \rangle_s^{(0)} = \frac{2}{\omega^2} C_1(s) = -\frac{2}{\omega^2}\frac{dC_0(s)}{ds}. \tag{63}$$

One property of the moments $\langle \tau^n \rangle_s^{(0)}$ is obvious; they decrease as functions of s. (Indeed, e.g., for $s = 0$, we have found the value $\omega\langle \tau \rangle_0^{(0)} = 2.622$ and for $s = \frac{3}{2}$ the value $\omega\langle \tau \rangle_{3/2}^{(0)} = 2.$)

Instead of a further commentary on the numerical aspects (which do not imply serious difficulties) of employing formulae (62) and (63), we prefer to discuss the asymptotic cases when either $0 < s \ll 1$ or $2.\ 1 \ll s$.

IV.3.1. The case of small values s $(0 < s \ll 1)$

If s is small, we may write

$$e^{-sy} = 1 - sy + \frac{1}{2}s^2y^2 + O(s^3).$$

Formula (53) then gives the results

$$\langle \tau \rangle_s^{(0)} = \frac{1}{\omega}\left(A_1 - \frac{1}{2}A_2\, s\right) + O(s^2),$$

$$\langle \tau^2 \rangle_s^{(0)} = \frac{2}{\omega^2}\left(\frac{1}{2}A_2 - \frac{1}{3}A_3 s\right) + O(s^2).$$

(Note that $C_{n-1}(0) = (1/n)A_n$, $n = 1, 2, \ldots$; with the values of A_1, A_2, A_3 given in Appendix A.1.) Thus, in the approximation linear in s, we have the results:

$$m_s^{(0)} = \langle \tau \rangle_s^{(0)} \doteq \frac{1}{\omega} (2.622 - 5.558s), \tag{64}$$

$$\sigma_s^{(0)} = \{\langle \tau^2 \rangle_s^{(0)} - [\langle \tau \rangle_s^{(0)}]^2\}^{1/2} \doteq \frac{1}{\omega}\left[(A_2 - A_1^2)^{1/2} - \left(\frac{1}{3} A_3 - \frac{1}{2} A_1 A_2\right) s\right] =$$
$$= \frac{1}{\omega}(4.240 - 7.974s) = \sigma_0^{(0)}(1 - 1.881s). \tag{65}$$

IV.3.2. The case of large values of s ($1 \ll s$)

If the parameter $\lambda > 0$ of the annihilation rate $V(x)$ is small, the parameter s may be large even when V_0 (the "background annihilation rate") is small (recall definitions (18) and (50)). In the limiting case when $s \to \infty$ (or $\lambda \to 0$), the dimensionless lifetime $y = \omega\tau$ becomes a Poissonian random variable. Namely, then the mean value and r.m.s. deviation of y become equal, $m_s = \sigma_s$, and we have the result:

$$\langle y \rangle_s = \{\langle y^2 \rangle_s - [\langle y \rangle_s]^2\}^{1/2} = \frac{1}{s}.$$

Now, our aim is to show how the moments $\langle y \rangle_s$, $\langle y^2 \rangle_s$ (or the cumulants m_s, σ_s^2) get different from the Poissonian result if s is large but not tending to infinity yet.

It is clear that for $s \gg 1$ the main contribution to the value of the integral (60) is due to values of y close to zero. We may, therefore, write the development

$$\frac{1}{(\cosh y)^{1/2}} = 1 - \frac{1}{4} y^2 + \ldots$$

and insert it into the integral (60). So we obtain the approximative function

$$C_0(s) = \frac{1}{s}\left(1 - \frac{1}{2s^2}\right) + O\left(\frac{1}{s^5}\right). \tag{66}$$

Then the formulae (62) and (63) give us the moments

$$m_s^{(0)} = \langle \tau \rangle_s^{(0)} = \frac{1}{\omega s}\left(1 - \frac{1}{2s^2}\right) + O\left(\frac{1}{s^5}\right), \tag{67}$$

$$\langle \tau^2 \rangle_s^{(0)} = \frac{1}{\omega^2 s^2}\left(2 - \frac{3}{s^2}\right) + O\left(\frac{1}{s^6}\right). \tag{68}$$

Hence, we obtain also the corresponding approximation for the r.m.s. deviation of the lifetime τ:

$$\sigma_s^{(0)} = \frac{1}{\omega s}\left(1 - \frac{1}{s^2}\right) + O\left(\frac{1}{s^5}\right). \tag{69}$$

V. CONCLUDING REMARKS

In the present paper (Parts I and II) we have shown that utterly distant theories — such as the probabilistic theory of wear (that for someone might essentially be the same as the reliability theory) and the quantum theory — may, to a great extent, have common mathematical features. We have especially called attention to analogies between a specified theory of wear (reliability) making use of an annihilation rate of a quadratic form $V(x) = V_0 + \lambda x^2$ and the quantum-statistical theory related to the one-dimensional harmonic oscillator.

The theory of wear (reliability), as it was set out here in accordance with an idea of Ref. [1], may find various applications in technology, traffic, health matters, planning of industrial outputs, etc. Nowadays, undoubtedly, demands on service life of such appliances as, say, the microelectronic circuitry in computers or cosmic stations are extreme. Of course, the service life is not insignificant even for such traditional goods as, for instance, water-supply conduits which, unfortunately, are apt to corrode. The same statement holds, and is ominous, if one speaks of the arteries of the human body.

Any success of our theory — if a trustworthy prognosis is required of the expected lifetime of anything (anyone) — is, however, much dependent on our capability to be realistic enough in contriving a good expression for the annihilation rate function $V(x)$. In real situations, this function need not always be quadratic; in fact, it may be an arbitrary non-negative function. Nevertheless, in such a case, the reasoning, as expounded in this paper, may be pursued further, owing to its methodological generality.

Mathematically speaking, the theory has three inputs: 1. the parameter $\gamma > 0$ characterizing the intensity with which the system in question (described by a state variable x) is exposed to random shocks (cf. formula (1)); it is a notion defining a Gaussian white-noise process, 2. the parameter $\mu \geq 0$ characterizing an average wear rate of the system, and 3. the annihilation rate function $V(x) \geq 0$.

Having chosen γ and μ as constants and $V(x)$ as a quadratic function, we have calculated the lifetime probability density of the system and, with the aid of it, the mean value and the r.m.s. deviation of the lifetime.

Naturally, there are many possibilities of generalizing our theory. For example, one may take $x(t)$ as a non-Gaussian process, or at least one need not take γ as a constant; the theory then becomes much more complicated. The generalization would be simpler if we allowed μ alone to be non-constant: in a Brownian-motion analogy, this would mean the occurrence of a non-constant drifting force. Finally we could take the annihilation rate as a time-dependent function.

In any case, we considered it useful and interesting to elucidate the parallel between the theory of wear (or reliability) and the quantum theory. Confident of this parallel, we may try to apply techniques of one theory to the other. Particularly, we believe, functional (path-integral) techniques — which were recently in the centre of attention of modern physical theories [14—17] — may here find some new applications.

APPENDIX A.1

On substituting $y = \omega t$ into formula (34) we obtain the ω-dependence

$$\langle \tau^2 \rangle_0^{(0)} = \frac{1}{\omega^2} A_2$$

where

$$A_2 = \frac{1}{2}\int_0^\infty \mathrm{d}y\, y^2 \frac{\tanh y}{(\cosh y)^{1/2}} = 2\int_0^\infty \mathrm{d}y \frac{y}{(\cosh y)^{1/2}}. \qquad (*)$$

(The identity of these integrals can be proved with the aid of integration by parts). Using the denotation $u = \mathrm{e}^{-2y}$, we may write:

$$\frac{1}{(\cosh y)^{1/2}} = \sqrt{2}\, u^{-1/4} \frac{1}{(1+u)^{1/2}}.$$

As u is smaller than unity, we may use the McLaurin development:

$$\frac{1}{(1+u)^{1/2}} = 1 - \frac{1}{2}u + \frac{1.3}{2.4}u^2 - \frac{1.3.5}{2.4.6}u^3 + \dots .$$

After inserting this into the second integral (*) and integrating it term by term, we obtain the series

$$A_2 = 8\sqrt{2}\left[1 - \frac{1}{50} + \sum_{k=2}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{1}{(4k+1)^2}\right].$$

It is useful to group the terms in pairs; using $n = 2k$ (for even values of k), we obtain the series

$$A_2 = 8\sqrt{2}\left(0.98 + \sum_{n=1}^{\infty} R_n\right)$$

where all the terms R_n are positive:

$$R_n = \frac{(4n-1)!!}{(4n)!!}\left[1 - \frac{4n+1}{2(2n+1)}\left(\frac{8n+1}{8n+5}\right)^2\right] \cdot \frac{1}{(8n+1)^2}.$$

It is sufficient to take into account $n \leq 10$ in order to obtain the value

$$A_2 = 11.115 .$$

Note that if we used the same method in calculating the integral

$$A_1 = \frac{1}{2}\int_0^\infty \mathrm{d}y\, y \frac{\tanh y}{(\cosh y)^{1/2}} = \int_0^\infty \mathrm{d}y \frac{1}{(\cosh y)^{1/2}},$$

we could arrive at the same value, $A_1 = 2.622$, as in Section III.3.1, although we should have to sum up considerably more terms than in case of A_2.

On the other hand, when we apply the method to calculating the integral

$$A_3 = \frac{1}{2}\int_0^\infty \mathrm{d}y\, y^3 \frac{\tanh y}{(\cosh y)^{1/2}} = 3\int_0^\infty \mathrm{d}y\, y^2 \frac{1}{(\cosh y)^{1/2}} \qquad (**)$$

we obtain the series

$$A_3 = 48\sqrt{2}\left[1 - \frac{1}{250} + \sum_{k=2}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{1}{(4k+1)^3}\right],$$

which is rapidly convergent. The terms with $k \geqslant 6$ are negligible. So we obtain the value

$$A_3 = 67.638 .$$

APPENDIX A.2

Now our problem is to calculate the integral

$$B_1 = \frac{1}{2}\int_0^\infty \mathrm{d}y\, y \frac{(\tanh y)^2}{(\cosh y)^{1/2}}. \qquad (***)$$

Putting $U_1(y) = y \tanh y$, $V_1(y) = -(\cosh y)^{-1/2}$, we can transform B_1 into the form

$$B_1 = -\int_0^\infty \mathrm{d}y\, U_1'(y)\, V_1(y) = 2 + \int_0^\infty \mathrm{d}y \frac{y}{(\cosh y)^{5/2}}.$$

On the other hand, when taking $(\tanh y)^2 = 1 - (\cosh y)^{-2}$, we can also write

$$2B_1 = \int_0^\infty \mathrm{d}y \frac{y}{(\cosh y)^{1/2}} - \int_0^\infty \mathrm{d}y \frac{y}{(\cosh y)^{5/2}}.$$

Thus, using definition (*) of Appendix A.1, we derive the relationship

$$3B_1 = 2 + \frac{1}{2} A_2$$

from which we obtain the value

$$B_1 = 2.519 .$$

Similarly, we can also calculate the integral

$$B_2 = \frac{1}{2} \int_0^\infty \mathrm{d}y y^2 \frac{(\tanh y)^2}{(\cosh y)^{1/2}} . \qquad (****)$$

Integrating it by parts, with $U_2(y) = y^2 \tanh y$, $V_2(y) = -(\cosh y)^{-1/2}$, and using the denotation (*), we can write B_2 in the form

$$B_2 = -\int_0^\infty \mathrm{d}y U_2'(y) V_2(y) = 4A_1 + \int_0^\infty \mathrm{d}y \frac{y^2}{(\cosh y)^{5/2}} .$$

Expressing $(\tanh y)^2$ as $1 - (\cosh y)^{-2}$, we can also write, using definition (**):

$$2B_2 = \frac{1}{3} A_3 - \int_0^\infty \mathrm{d}y \frac{y^2}{(\cosh y)^{5/2}} .$$

Hence, we arrive at the equation

$$3B_2 = 4A_1 + \frac{1}{3} A_3$$

which gives us the value

$$B_2 = 11.011 .$$

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СТОХАСТИЧЕСКАЯ ТЕОРИЯ ИЗНАШИВАНИЯ С КВАДРАТИЧНОЙ СКОРОСТЬЮ УНИЧТОЖЕНИЯ. ЧАСТЬ II

Предлагается продолжение предыдущей статьи в этом выпуске APS (Часть II). Теория, приведенная в Части I, обобщена на случай $V(x) = V_0 + \lambda x^2$ с $V_0 = \mathrm{const} > 0$.