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A STOCHASTIC THEORY OF WEAR WITH A QUADRATIC ANNIHILATION RATE. PART I.

BEZÁK, V.,[1] Bratislava

Some mathematical problems common to the quantum-statistical theory of the harmonic oscillator and the theory of a Brownian process $x(t)$ with a quadratic annihilation rate $V(x) = \lambda x^2$ are reviewed. For $\lambda > 0$, taken as a deterministic constant, the lifetime τ of the process $x(t)$ is finite. With a constant deterministic wear parameter $\mu \geqslant 0$ the stochastic properties of τ are analysed. In particular, the mean lifetime $\langle\tau\rangle$ and the variance $\langle(\tau - \langle\tau\rangle)^2\rangle$ (or the root of the latter) are calculated. The calculations are first carried out for $\mu = 0$ and afterwards — in the style of a perturbation theory (with respect to μ) — for $\mu > 0$. To stress a potential utility of the calculations, the author presents his exposition with considerations of a probabilistic view on the reliability of the human cardiovascular system.

I. INTRODUCTION

It is quite amusing, but also useful, to compare methods by which mathematicians on the one hand and theoretical physicists on the other, solve analogous problems. Recently WENOCUR, in mathematical paper [1], treated the problem of the Brownian motion with a quadratic killing rate. The author of the present paper[2] was attracted by this problem not only because of its significance, but rather for its affinity with a standard problem of the quantum theory. It has been well known for a long time (see e.g. [2]) that the mathematical aspects of the theory of the Brownian motion could be studied as something very skin to quantum mechanics. Aware of this, one can at once infer a formal correspondence between Wenocur's quadratic killing rate and a quantum-mechanical potential energy due to a one-dimensional harmonic oscillator. Throughout the present paper, we will consistently take advantage of this parallel.

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[1] Matematicko-fyzikálna fakulta UK, Mlynská dolina F-2, 842 15 BRATISLAVA, Czechoslovakia

[2] This paper consists of two related contributions considered from the editorial standpoint as two separate items in this issue of APS (Part I and Part II). Part II is a generalization of Part I and involves some computational details (including two Appendices).

We shall simply take x as a state of a system. Wenocur's problem consists in an analysis of a stochastic process $x(t)$ of the Brownian type subject to externally generated shocks, some of them fatal in the sense that they can stop ("kill") the process. Following LANGEVIN [3, 4], we may relate any Brownian process to a Gaussian white-noise velocity $\dot{x}(t)$. On assuming that the process $\dot{x}(t)$ is stationary, with a constant mean value $\mu \geqslant 0$, we define its auto-correlation function

$$\langle [\dot{x}(t_1) - \mu][\dot{x}(t_2) - \mu] \rangle = \gamma^2 \delta(t_1 - t_2) \tag{1}$$

with a constant amplitude γ^2 $(\gamma > 0)$. In agreement with [1], we do not put a restriction on the variable x; in our formulation x is allowed to acquire any real value. The basic problem is to calculate the probability density $p(x, t; x_0)$ of that the system avoids all fatal shocks within a finite duration of time from $t_0 = 0$ to $t > 0$, and finds itself in the state x at t if it was in state x_0 at t_0. The calculation must respect the danger of the shocks measured by a non-negative function $V(x)$. We take $V(x)\,\mathrm{d}t$ as the probability for the system, being in state x at t, to be annihilated (we prefer, for linguistic grounds, the term annihilation instead of killing) within the time interval $(t, t + \mathrm{d}t)$; $V(x)$ was called the "killing rate" ("annihilation rate"). Wenocur's choice for $V(x)$ was a quadratic function: $V(x) = \lambda x^2$, $\lambda > 0$, $-\infty < x < \infty$. Then, indeed, the calculation of $p(x, t; x_0)$ is the same as the well-known quantum-statistical calculation of the canonical density matrix for the harmonic oscillator [5].

In order to explain our mathematical approach in a clear way, we keep in mind throughout this paper, starting in Section II, the same vital example — certainly of real concern to everybody — that was discussed in [1]: we try to suggest a way for elaborating a statistical theory relating to a set of living persons, all born at $t_0 = 0$, with the aim to predict how they will be distributed according to their blood pressure x at $t > 0$, provided that their initial blood pressure was x_0. As the blood pressure increases with age, we define some possibly positive average velocity of this increase $\mu = \langle \dot{x}(t) \rangle \geqslant 0$. The function $V(x)$ is defined as a rate of heart failures (of course, in a statistical sense). (The statistics should be done by retrospection of medical mortality records.) We shall pay attention to a function $\varphi(\tau) = p_{lifetime}(\tau) \geqslant 0$, the probability density for the span of life of an individual to reach the value τ. If $\Phi(t) = P_{survival}(t)$ means the probability that the individual will live longer than t, then obviously

$$\varphi(\tau) = -\frac{\mathrm{d}\Phi(\tau)}{\mathrm{d}\tau}, \qquad \tau > 0 \tag{2}$$

$(p_{lifetime}(\tau) = -\mathrm{d}P_{survival}(\tau)/\mathrm{d}\tau)$.

Section III is devoted to the explicit juxtaposition of the quantum oscillator theory with Wenocur's problem with $V(x) = \lambda x^2$. Contrary to [1], we present detailed numerical calculations of the statistical moments $\langle \tau \rangle$, $\langle \tau^2 \rangle$ and the r.m.s deviation of τ. We perform these calculations first for $\mu = 0$ and afterwards modify them, in the sense of a perturbation theory, to the case of small positive values of μ.

In Section IV, we repeat the calculations using the annihilation rate $V(x) = V_0 + \lambda x^2$, $V_0 > 0$, $\lambda > 0$. This generalization would be, mathematically speaking, trivial for any constant value V_0 if we took into account the functions $p(x, t; x_0)$, $\varphi(\tau)$, $\Phi(t)$ (cf. formula (1)) alone. Nevertheless, when calculating the moments $\langle \tau^n \rangle$, or the corresponding cumulants, one does find that the presence of $V_0 \neq 0$ in the quadratic annihilation rate implies new computational problems.

Finally, in Section V, we give some concluding remarks.

II. INCIPIENT THEORETICAL PROPOSITIONS

Abstractly saying, we shall investigate a Brownian process, with a diffusion constant $\gamma^2/2$ and drift $\mu \geqslant 0$, "living" under a certain risk of killing influences. This risk is selective — measured by the non-negative function $V(x)$. If the process starts from state x at t, the probability that it will survive even the time instant $t + \mathrm{d}t$ is $1 - V(x)\,\mathrm{d}t$.

To interpret this more explicitly, we will now comment on the example mentioned above somewhat more at length. We consider the human cardiovascular system and define x as the deviation of the systolic blood pressure from its optimum value. As already mentioned in the Introduction, we assume that the cardiovascular system wears out — this means that x tends to grow with age. (The average velocity of this growth is μ. Thus the drift parameter μ and the wear parameter are synonyms.) The actual velocity $\dot{x}(t)$ is, however, stochastic; its dispersion (variance) is denoted as γ^2 (in suitable units). The human heart must incessantly endure shocks (stresses from diseases, accidents, irregularities of training, fluctuations in the way of living, errors in diet, negative influences of medicaments or drugs including smoking and the inhalation of various pollutants, sudden physical and emotional strains, etc.), and its ability to recuperate depends on x. If x is small, the shocks — even though always present — are not so fatal in contrast to the case when someone suffers from hypertony (then x is large). Moreover, even the occurrence rate of fatal shocks is smaller among younger people (when the overall health, partially just with regard to the small value of x, is better) than among older people. These two reasons allow to conclude that the function $V(x)$ is not only increasing, for $x > 0$, but also convex. Besides, it was assumed in [1] that $V(x)$ should behave similarly for $x < 0$ if $|x|$ grows. This idea is acceptable, since if the blood pressure is low, it may also have for a person fatal consequences. (There is a higher probability of injuries in consequence of vertigos, and if the blood pressure gets too low it simply means heart insufficiency. Of course, there is also another side of the problem, but we will not enlarge upon it: we will not discuss here why the function $V(x)$ should be symmetric with respect to $x = 0$.) Thus, the quadratic function $V(x) = \lambda x^2$, $\lambda > 0$ (Section III), or more generally, $V(x) = V_0 + \lambda x^2$, $V_0 = \text{const.} > 0$, $\lambda > 0$ (Section IV) seems to be substantiated. In any case, it is the simplest model for the annihilation rate.

We define $p(x, t)\,dx$ as the probability that the heart maintains its operation until the time t when the blood pressure falls to the interval $(x, x + dx)$. The probability density $p(x, t)$ fulfils the equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\gamma^2 \frac{\partial^2 p}{\partial x^2} - \mu \frac{\partial p}{\partial x} - V(x)p, \qquad t > 0. \tag{3}$$

The probability for the lifetime τ of a person to exceed a given value t is determined by the integral

$$P_{survival}(t) = \Phi(t) = \int dx\, p(x, t). \tag{4}$$

The probability that the person passes alive the time instant t and then dies during the infinitesimal time dt is $dt \int dx\, V(x) p(x, t)$. Hence, the average human lifetime is

$$\langle \tau \rangle = \int_0^\infty d\tau\, \tau \int dx\, V(x) p(x, t). \tag{5}$$

This implies that the expression

$$P_{lifetime}(\tau) = \varphi(\tau) = \int dx\, V(x) p(x, \tau) \tag{6}$$

is the probability density for the lifetime distribution. (Compare it with equation (2).) It should also be pointed out that after integrating equation (2) with respect to τ from zero to infinity, and noting that $\Phi(0) = 1$, we readily prove that $\varphi(\tau)$ is properly normalized:

$$\int_0^\infty d\tau\, \varphi(\tau) = 1. \tag{7}$$

Then we may use the function $\varphi(\tau)$ in the general definition of the lifetime moments:

$$\langle \tau^n \rangle = \int_0^\infty d\tau\, \tau^n \varphi(\tau), \qquad n = 1, 2, \ldots. \tag{8}$$

Naturally, the function $p(x, \tau)$ does depend on the initial function $p(x, 0)$ which is assumed to be known at the outset.

III. THE CASE WHEN $V(x) = \lambda x^2$, $\lambda > 0$

III.1. Comparison of the quantum-statistical theory of the harmonic oscillator and the theory of the Brownian motion with the quadratic annihilation rate

The one-particle canonical density matrix $C(x, \beta; x_0)$ satisfies the Bloch equation

$$\frac{\partial C}{\partial \beta} = \frac{\hbar^2}{2m} \frac{\partial^2 C}{\partial x^2} - V(x)C, \qquad \beta = \frac{1}{k_B T} > 0, \tag{9}$$

and the "initial" condition

$$C(x, 0; x_0) = \delta(x - x_0). \tag{10}$$

In case when

$$V(x) = \lambda x^2, \qquad \lambda > 0, \tag{11}$$

the solution to equations (9), (10) was found many years ago [5]:

$$C(x, \beta; x_0) = C_0(x, \beta; x_0) =$$

$$= \left(\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega\beta)}\right)^{1/2} \exp\left\{-\frac{m\omega}{2\hbar \sinh(\hbar\omega\beta)}[(x^2 + x_0^2)\cosh(\hbar\omega\beta) - 2xx_0]\right\} \tag{12}$$

where $\omega = (\lambda/2m)^{1/2}$.

For equation (3), we may write (and this is possible for any function $V(x)$) the solution $p(x, t)$ in the form

$$p(x, t) = \int dx_0\, g(x, t; x_0) P(x_0, 0), \tag{13}$$

assuming that we know the initial function $p(x, 0)$ a priori. The Green function $g(x, t; x_0)$ itself obeys equation (3) and the initial condition

$$g(x, 0; x_0) = \delta(x - x_0). \tag{14}$$

On substituting

$$g(x, t; x_0) = \exp\left[\frac{\mu}{\gamma^2}\left(x - x_0 - \frac{1}{2}\mu t\right)\right] f(x, t; x_0) \tag{15}$$

into equation (3) we obtain the equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\gamma^2 \frac{\partial^2 f}{\partial x^2} - V(x)f. \tag{16}$$

The function $f(x, t; x_0)$ again obeys the initial condition

$$f(x, 0; x_0) = \delta(x - x_0).$$

Equation (16) is formally equal to equation (9) and therefore, by utilizing formulae (12) and (15), we can write, for $t > 0$, the result:

$$g(x, t; x_0) = g_0(x, t; x_0) = \left(\frac{\omega}{2\pi\gamma^2 \sinh(\omega t)}\right)^{1/2} \exp\left[\frac{\mu}{\gamma^2}\left(x - x_0 - \frac{1}{2}\mu t\right)\right] \times$$
$$\times \exp\left\{-\frac{\omega[(x^2 + x_0^2)\cosh(\omega t) - 2xx_0]}{2\gamma^2 \sinh(\omega t)}\right\} \tag{17}$$

with a "frequency"

$$\omega = (2\lambda)^{1/2}\gamma. \tag{18}$$

Expression (17) represents a Gaussian:

$$g(x, t; x_0) = g_0(x, t; x_0) = \left(\frac{\omega}{2\pi\gamma^2 \sinh(\omega t)}\right)^{1/2} \exp\left[\frac{-\mu^2}{2\gamma^2}\left(t - \frac{\tanh(\omega t)}{\omega}\right)\right] \times$$
$$\times \exp\left[-\frac{\omega \tanh(\omega t)}{2\gamma^2} x_0^2\right] \exp\left[-\frac{\mu}{\gamma^2}\left(1 - \frac{1}{\cosh(\omega t)}\right) x_0\right] \times$$
$$\times \exp\left[-\frac{\omega}{2\gamma^2 \tanh(\omega t)}\left(x - \frac{\omega x_0 + \mu \sinh(\omega t)}{\omega \cosh(\omega t)}\right)^2\right]. \tag{19}$$

In particular, for $x_0 = 0$, we obtain the solution

$$p(x, t) = p_0(x, t) = \left(\frac{\omega}{2\pi\gamma^2 \sinh(\omega t)}\right)^{1/2} \exp\left[\frac{\mu}{\gamma^2}\left(x - \frac{1}{2}\mu t\right)\right] \times$$
$$\times \exp\left\{-\frac{x^2\omega}{2\gamma^2 \tanh(\omega t)}\right\} \tag{20}$$

which can also be rewritten into the centred Gaussian form:

$$p_0(x, t) = \left(\frac{\omega}{2\pi\gamma^2 \sinh(\omega t)}\right)^{1/2} \exp\left[-\frac{\mu^2}{2\gamma^2}\left(t - \frac{\tanh(\omega t)}{\omega}\right)\right] \times$$
$$\times \exp\left[-\frac{\omega}{2\gamma^2 \tanh(\omega t)}\left(x - \mu\frac{\tanh(\omega t)}{\omega}\right)^2\right]. \tag{21}$$

In the limiting case when $\omega \to 0$ (i.e. when $\lambda \to 0$) formula (17) yields the Gaussian

$$g_0(x, t; x_0) \to \frac{1}{(2\pi\gamma^2 t)^{1/2}} \exp\left[-\frac{(x - x_0 - \mu t)^2}{2\gamma^2 t}\right]. \tag{22}$$

It should be pointed out that WENOCUR [1] preferred to analyse another function — the function

$$P_{survival}(x_0, t) = \Phi(x_0, t) = \int dx g(x, t; x_0), \qquad t > 0. \tag{23}$$

After performing the integration of expression (19) with respect to x, we obtain Wenocur's result:

$$\Phi(x_0, t) = \Phi_0(x_0, t) = \frac{1}{(\cosh(\omega t))^{1/2}} \exp\left[-\frac{\mu^2}{2\gamma^2}\left(t - \frac{\tanh(\omega t)}{\omega}\right)\right] \times$$
$$\times \exp\left\{-\frac{\omega \tanh(\omega t)}{2\gamma^2} x_0^2\right\} \exp\left\{-\frac{\mu}{\gamma^2}\left(1 - \frac{1}{\cosh(\omega t)}\right) x_0\right\}. \tag{24}$$

The probability $\Phi(x, t)$ is the solution to the equation

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2}\gamma^2 \frac{\partial^2 \Phi}{\partial x^2} + \mu \frac{\partial \Phi}{\partial x} - V(x)\Phi \tag{25}$$

respecting the condition $\Phi(x, 0) = 1$ (valid independently of x). Note the difference in the signs of the μ-terms in equations (3) and (25). It is easily explained: x represents the "final" state in case of equation (3) but, on the other hand, the "initial" state in case of equation (25). (Equations (3), (25) simply differ in the sense as the "forward" Kolmogorov equation differs from the "backward" one [6].)

III.2. The lifetime distribution

Now we are ready to apply formula (5) to calculating the lifetime probability density

$$\varphi(\tau) = \int dx V(x) \int dx_0 g(x, \tau; x_0)\, p(x_0, 0). \tag{26}$$

For $p(x, 0) = \delta(x - x_0)$, we write

$$\varphi(x_0, \tau) = \int dx V(x) g(x, \tau; x_0) \tag{27}$$

and in the particular case when $x_0 = 0$:

$$\varphi(0, \tau) = \int dx V(x) g(x, \tau; 0). \tag{28}$$

After inserting expression (19) into formula (27), we can easily perform the integration, noting that

$$\int_{-\infty}^{\infty} dx x^2 \exp[-a(x - b)^2] = \left(\frac{\pi}{a}\right)^{1/2}\left[\frac{1}{2a} + b^2\right], \qquad a > 0.$$

So we arrive at the distribution function

$$\varphi(x_0,\ \tau) = \varphi_0(x_0,\ \tau) = \varphi_0(0,\ \tau) \times$$
$$\times \frac{\gamma^2\omega \cosh(\omega\tau) + (\sinh(\omega\tau))^{-1}(\omega x_0 + \mu \sinh(\omega\tau))^2}{\gamma^2\omega \cosh(\omega\tau) + \mu^2 \sinh(\omega\tau)} \times$$
$$\times \exp\left[-\frac{\mu}{\gamma^2}\left(1 - \frac{1}{\cosh(\omega\tau)}\right)x_0\right], \tag{29}$$

where

$$\varphi(0,\ \tau) = \varphi_0(0,\ \tau) =$$
$$\frac{\omega \tanh(\omega\tau)}{2(\cosh(\omega\tau))^{1/2}}\left(1 + \frac{\mu^2}{\gamma^2}\frac{\tanh(\omega\tau)}{\omega}\right)\exp\left[-\frac{\mu^2}{2\gamma^2}\left(\tau - \frac{\tanh(\omega\tau)}{\omega}\right)\right]. \tag{29.1}$$

In this way we have derived an explicit expression for the lifetime probability density $\varphi(x_0,\ \tau)$. Of course, expressions (29), (29.1) could equally well be derived by using the formula

$$\varphi(x_0,\ \tau) = -\frac{\partial \Phi(x_0,\ \tau)}{\partial \tau}. \tag{30}$$

If the drift parameter $\mu > 0$ is small, we may develop $\varphi(x_0,\ \tau)$ into the Taylor series with respect to μ and confine ourselves to some lowest-order terms:

$$\varphi(x_0,\ \tau) = \varphi^{(0)}(x_0,\ \tau) + \varphi^{(1)}(x_0,\ \tau) + \varphi^{(2)}(x_0,\ \tau) + \ldots \tag{31}$$

(where $\varphi^{(n)}(x_0,\ \tau) = O(\mu)$). For $\mu > 0$, satisfying the conditions

$$\mu|x_0| \ll \gamma^2, \tag{32}$$
$$\mu^2 \ll \gamma^2\omega, \tag{33}$$

we neglect terms of a higher order than the second.

III.3. The mean value and the r.m.s. deviation of the lifetime τ for small values of μ and $x_0 = 0$

In general, even if we choose $x_0 = 0$, the integral

$$\langle \tau^n \rangle = \int_0^\infty \mathrm{d}\tau \tau^n \varphi(0,\ \tau) \tag{34}$$

cannot be calculated analytically; however, its numerical evaluation is not difficult.

For values of μ that are small enough in the sense of inequality (33), we may use, to the second order in μ, the distribution function

$$\varphi(0,\ \tau) \approx \varphi_0^{(0)}(0,\ \tau) + \varphi_0^{(2)}(0,\ \tau) = \frac{\omega \tanh(\omega\tau)}{2(\cosh(\omega\tau))^{1/2}}\left[1 - \frac{\mu^2}{2\gamma^2\omega}(\omega\tau - 3\tanh(\omega\tau))\right]. \tag{35}$$

Note that for $x_0 \neq 0$, the function $\varphi(x_0,\ \tau)$ involves a term linear in μ; this term vanishes for $x_0 \to 0$.

The subscript zero in $\varphi_0^{(2)}$ — and in other functions of Section III — suggests our choice $V_0 = 0$ here in contrast to Section IV where $V_0 > 0$.

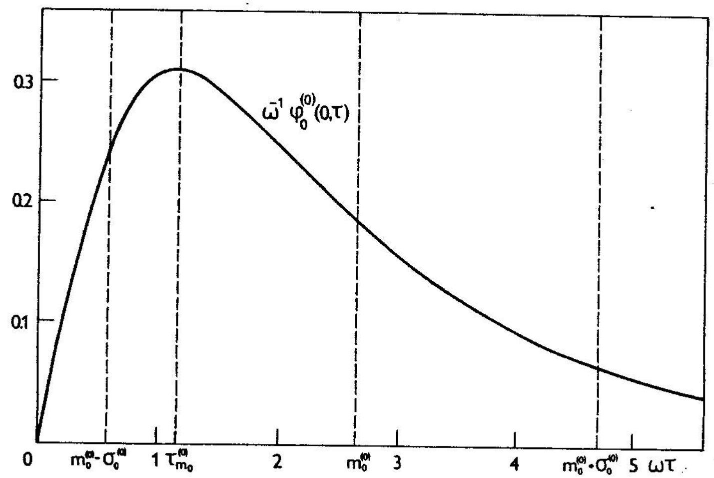

Fig. 1. The probability density $p_{lifetime}(\tau)$ in the case when $x_0 = 0$, $\mu = 0$ and $V_0 = 0$ (calculated from formula (36)). It is the negative derivative of the curve of Fig. 2 (in accordance with formula (2)).

III.3.1. The wearless case, $\mu = 0$

The function

$$\varphi_0^{(0)}(0,\ \tau) = \frac{\omega \tanh(\omega\tau)}{2(\cosh(\omega\tau))^{1/2}} \tag{36}$$

has the maximum value at

$$\tau_{m0}^{(0)} = \frac{\mathrm{Arcosh}\sqrt{3}}{\omega} = \frac{\ln(\sqrt{3}+\sqrt{2})}{\omega} = \frac{1.146}{\omega} \tag{37}$$

(see Fig. 1).

The mean lifetime $m_0^{(0)}$ is determined by the integral

$$m_0^{(0)} = \langle \tau \rangle_0^{(0)} = \frac{1}{2}\,\omega \int_0^\infty \mathrm{d}\tau\tau \frac{\tanh(\omega\tau)}{(\cosh(\omega\tau))^{1/2}} \tag{38}$$

which can be solved as follows. After substituting $\cosh y = \eta$ and integrating by parts, we can prove the validity of the identities

$$\frac{1}{2}\int_0^\infty \mathrm{d}y \, \frac{y \tanh y}{(\cosh y)^{1/2}} = \frac{1}{2}\int_1^\infty \mathrm{d}\eta \, \frac{\operatorname{Arcosh} \eta}{\eta^{3/2}} = \int_1^\infty \frac{\mathrm{d}\eta}{[\eta(\eta^2-1)]^{1/2}}.$$

The last integral can be transformed into the form of the complete elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{[1 - k^2 \sin^2 \vartheta]^{1/2}}, \qquad 0 \le k \le 1;$$

namely:

$$\int_1^\infty \frac{\mathrm{d}\eta}{[\eta(\eta^2-1)]^{1/2}} = \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

(Cf. [7], p. 234.)

So we have obtained the value

$$m_0^{(0)} = \frac{\sqrt{2}\,K(1/\sqrt{2})}{\omega} = \frac{2.622}{\omega}. \tag{39}$$

(We have used a Table from [8]; a less accurate Table for $K(k)$, but sufficient for us, is also in [9]. We must warn here that our denotation $K(k)$ corresponds to that used in [7] and [9]; what ABRAMOWITZ and STEGUN [8] denote as $K(k)$ should be written otherwise — say, $K_{AS}(k)$; then the relationship $K(k) = K_{AS}(k^2)$ has to be used when the Tables are compared.)

The calculation of the second-order moment

$$\langle \tau^2 \rangle_0^{(0)} = \frac{1}{2}\,\omega \int_0^\infty \mathrm{d}\tau \tau^2 \, \frac{\tanh(\omega\tau)}{(\cosh(\omega\tau))^{1/2}} \tag{40}$$

gives the result (cf. Appendix A.1):

$$\langle \tau^2 \rangle_0^{(0)} = \frac{11.115}{\omega}. \tag{41}$$

Thus, for the r.m.s. deviation of the lifetime τ, we obtain the result

$$\sigma_0^{(0)} = (\langle(\tau - \langle\tau\rangle_0^{(0)})^2\rangle)^{1/2} = \frac{2.059}{\omega}. \tag{42}$$

Under the neglection of wear (as we still take $\mu = 0$ in this Section), the survival probability corresponding to the probability density $\varphi_0^{(0)}(0, \tau)$ is:

$$\Phi_0^{(0)}(0, t) = \frac{1}{(\cosh(\omega t))^{1/2}}. \tag{43}$$

This function is shown in Fig. 2. Using this function, we can answer the following question: If any wear of organism is neglected, what is the probability $P_{death}^{(0)}(t_1, t_2)$ that an individual born at $t_0 = 0$ with $x_0 = 0$ will die (in consequence of the heart attacks modelled by the function $V(x)$) during some given time interval (t_1, t_2)? The probability is determined by the simple formula

$$P_{death}^{(0)}(t_1, t_2) = P_{survival}^{(0)}(t_1) - P_{survival}^{(0)}(t_2). \tag{44}$$

In Figs. 1 and 2 we have chosen two characteristic times:

$$t_1 = m_0^{(0)} - \sigma_0^{(0)}, \qquad t_2 = m_0^{(0)} + \sigma_0^{(0)}.$$

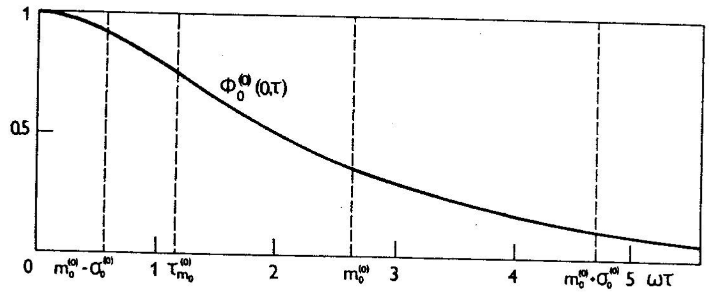

Fig. 2. The probability $P_{survival}(t)$ for $x_0 = 0$, $\mu = 0$ and $V_0 = 0$. It is given by formula (43)).

In retrospective statistics concerning sets of dead persons who all were born at the same time t_0 (and were selected for the analysis with a proper regard to what their killing rate function $V(x) = \lambda x^2$ would mean), we may ascertain that the percentage of those persons who died: i) before the time $t_0 + m_0^{(0)} - \sigma_0^{(0)}$, ii) between the times $t_0 + m_0^{(0)} - \sigma_0^{(0)}$, $t_0 + m_0^{(0)} + \sigma_0^{(0)}$, and iii) after the time $t_0 + m_0^{(0)} + \sigma_0^{(0)}$, is, respectively, 7.3%, 79.1% and 13.6%. These three numbers correspond to the values

$$P_{death}^{(0)}(0, m_0^{(0)} - \sigma_0^{(0)}) = 1 - \frac{1}{(\cosh 0.563)^{1/2}} = 0.073, \tag{i}$$

$$P_{death}^{(0)}(m_0^{(0)} - \sigma_0^{(0)}, m_0^{(0)} + \sigma_0^{(0)}) = \frac{1}{(\cosh 0.563)^{1/2}} - \frac{1}{(\cosh 4.681)^{1/2}} = 0.791, \tag{ii}$$

$$P_{death}^{(0)}(m_0^{(0)} + \sigma_0^{(0)}, \infty) = \frac{1}{(\cosh 4.681)^{1/2}} = 0.136. \tag{iii}$$

What is also interesting, is the asymmetry of the distribution function $\varphi_0^{(0)}(0, \tau)$.

As seen in Figs. 1 and 2 (and we can easily derive it from formulae (36), (43)), the asymptotic behaviour of the functions $\varphi_0^{(0)}(0, \tau)$, $\Phi_0^{(0)}(0, t)$ is exponential:

$$\varphi_0^{(0)}(0, \tau) \approx \frac{\omega}{\sqrt{2}} e^{-\omega\tau/2} \quad \text{for } \tau \gg m_0^{(0)}, \tag{36.a}$$

$$\Phi_0^{(0)}(0, t) \approx \sqrt{2} e^{-\omega t/2} \quad \text{for } t \gg m_0^{(0)}. \tag{43.a}$$

This suggests that the problem involves the time constant $2/\omega$. This constant would equal the mean lifetime if the exponential function (36.a) might be extrapolated into the region of short lifetimes τ. Fortunately, the mean lifetime $m_0^{(0)}$ is markedly longer than $2/\omega$ (by 31.1 per cent).

III.3.2. The wear correction quadratic in μ

We shall use formula (34) for $n = 1$ and $n = 2$ in case when $\varphi(x_0, \tau) \approx \varphi_0^{(0)}(x_0, \tau) + \varphi_0^{(0)}(x_0, \tau)$ expression (30)), $\mu > 0$. Ti distinguish the denotation for the mean value and the r.m.s. deviation of τ from the wearless case we shall now, respectively, write $m_0^{(2)}$, $\sigma_0^{(2)}$.

Using the integrals

$$A_n = \frac{1}{2}\int_0^\infty dy y^n \frac{\tanh y}{(\cosh y)^{1/2}},$$

$$B_n = \frac{1}{2}\int_0^\infty dy y^n \frac{(\tanh y)^2}{(\cosh y)^{1/2}},$$

we can write the mean value $\langle\tau\rangle_0^{(2)} = m_0^{(2)}$ and the variance

$$(\sigma_0^{(2)})^2 = \langle\tau^2\rangle_0^{(2)} - (m_0^{(2)})^2$$

in the form:

$$m_0^{(2)} = \frac{1}{\omega}\left[A_1 - \frac{1}{2}(A_2 - 3B_1)\frac{\mu^2}{\gamma^2\omega}\right],$$

$$(\sigma_0^{(2)})^2 = \frac{1}{\omega^2}\left[A_2 - \frac{1}{2}(A_3 - 3B_2)\frac{\mu^2}{\gamma^2\omega}\right] - (m_0^{(2)})^2.$$

Taking the square root of the last expression and remembering inequality (33) we obtain the expression

$$\sigma_0^{(2)} = \frac{1}{\omega}\left[(A_2 - A_1^2)^{1/2} - \frac{A_3 - 3B_2 - 2A_1(A_2 - 3B_1)}{4(A_2 - A_1^2)^{1/2}}\frac{\mu^2}{\gamma^2\omega}\right].$$

The values A_1, A_2, A_3, B_1, B_2 are calculated in Appendices A.1, A.2. Thus we obtain the results:

$$m_0^{(2)} = \frac{1}{\omega}\left(2.622 - 1.779\frac{\mu^2}{\gamma^2\omega}\right), \tag{45}$$

$$\sigma_0^{(2)} = \frac{1}{\omega}\left(2.059 - 1.936\frac{\mu^2}{\gamma^2\omega}\right). \tag{46}$$

The wear, as it is seen, not only shortens the mean lifetime but it shortens also the r.m.s. deviation of τ. This means, naturally, that the distribution function $\varphi(0, \tau)$ gets narrower for $\mu > 0$ than for the wearless case when $\mu = 0$.

Appendices and References are printed within Part II (next contribution in this issue of APS).

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СТОХАСТИЧЕСКАЯ ТЕОРИЯ ИЗНАШИВАНИЯ С КВАДРАТИЧНОЙ СКОРОСТЬЮ УНИЧТОЖЕНИЯ. ЧАСТЬ I

Рассмотрены некоторые математические проблемы, общие для квантовостатистической теории гармонического осциллятора и теории броуновского движения $x(t)$ с квадратичной скоростью уничтожения $V(x) = \lambda x^2$. Для $\lambda > 0$, взятой в качестве детерминистской постоянной, время жизни τ процесса $x(t)$ конечно. Анализируются стохастические свойства τ с детерминистской постоянной параметра изнашивания $\mu \geq 0$. В частности, вычислены среднее значение времени жизни $\langle\tau\rangle$ и вариация $\langle(\tau - \langle\tau\rangle)^2\rangle$ (или корень последней). Сначала проведены вычисления для $\mu = 0$, а затем — по теории возмущений (по отношению к μ) — для $\mu > 0$. Чтобы подчеркнуть возможную применимость вычислений, автор излагает свои предположения с вероятностной точки зрения на надежность кардиоваскулярной системы человека.