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ON LOCALIZABILITY OF PARTICLES

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We present reasons which suggest that operators proportional to the generators of the Lorentz boosts play the role of the position operators in relativistic quantum mechanics.

I. INTRODUCTION

One generally believes that particles are localizable. By this we usually mean that for a given particle there exist operators q_i $(i = 1, 2, 3)$, eigenfunctions and eigenvalues which can be interpreted as localized states and possible values of position (the components of the position vector) of a particle in question.

In non-relativistic quantum mechanics the construction of the position operators is quite wellknown: The position is represented by three operators canonically conjugate to momenta. The eigenfunctions of these operators in the x-representation are δ-functions. The position operators are commuting, hermitian, possess desirable transformation properties and the concept of localizability is invariant with respect to Galilei transformations.

In relativistic quantum mechanics the situation is more complicated despite of a considerable effort to solve the problem. A lack of basic experimental information forces physicists only to guess under what conditions we can construct position operators. For this reason several not equivalent approaches have been developed and none of them seems to have been generally accepted. One can say that any approaches contains postulates which seem to be, more or less, in harmony with our everyday experience. However, theoretical considerations resulting from the postulates lead to serious difficulties.

Out of many approaches in constructing position operators the first systematic is that of Newton and Wigner (NW) [1]. In their work localized states (and then position operators) are deduced from general principles. If we denote by S_0 a set of states describing a particle localized at the origin 0 of space-time, then the NW postulates can be summarized as follows:

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NW1. S_0 is a linear set, i.e., the superposition principle holds.
NW2. S_0 remains invariant after the 3-rotations (rotations in the ordinary space) around 0 and after space and time reflections.
NW3. All states from S_0 are orthogonal to states obtained by 3-translations.
NW4. Certain mathematical conditions must be satisfied.

It follows from NW1.—4. that there exist hermitian position operators q_i for any particle with a rest mass $M \neq 0$ and an arbitrary spin but there are no position operators, e.g., for a neutrino or a photon. A serious difficulty of the NW approach consists in a relativistically non-invariant introduction of the localizability concept. By this is meant that the description of a localized particle by observes in different frames of reference is not physically consistent. Namely, if φ_0 is a state of a particle localized at the origin 0 and if we perform a Lorentz transformation preserving the point 0, then from a new frame of reference a particle is not localized [6].

There are several improvements of the NW approach [2—4]. However, localized states reamin non-covariant [4] or position operators are not unique [2, 3, 5].

The Lorentz invariance of the localizability concept is one of the starting points of Kálnay's approach [6]. Considering a spinless particle only, Kálnay's postulates can be formulated as follows:

K1. Each component of the 3-position is represented by an operator q_i.
K2. q_i $(i = 1, 2, 3)$ are conponents of a three-vector operator.
K3. The Lorentz invariance of localization. (If a particle is localized in a region R of the space-time and if we perform a Lorentz transformation leaving R invariant, then from a new frame of reference a considered particle must also be localized in R).
K4. The space translational operator transforms localized states into new localized states.

The main result of Kálnay's work is: If a localization is Lorentz invariant, then the position operators are not hermitian. (Moreover it is necessary to note that those position operators do not posses a proper behaviour under the Lorentz transformations as will be seen later).

In other works position operators also possess undesirable properties. For example, position operators for neutrinos do not form a 3-vector under 3-rotations [8, 9], only two components of the 3-vector position operator are commuting [10], the localizability of the photon has been achieved under the assumption (among others) that the photon can also occur in states with negative energy [11], the space-time position operators for a spinless particle are not defined in the Hilbert space spanned by solutions of the Klein—Gordon equation [12]. A

more detailed analysis of the above and other works relating to localizability can be found in [7].

The forementioned results are to a certain degree surprising, because conditions imposed on q_i seem to be in most cases natural. Moreover the non-relativistic version of those conditions for $M \neq 0$ leads to physically acceptable results. The question is whether (or in what sense) the localization of particles is a physically meaningful concept. To illustrate some problems in greater detail we present in the next section one of the possible constructions of the position operators for a free particle. In Sec. III we propose the localizability concept within the framework of relativistic quantum mechanics. Sec. IV contains several notes.

II. POSITION OPERATORS FOR A FREE PARTICLE

The dynamics of a particle can be formulated by means of the constraint formalism. In such a case a particle is described by the co-ordinates x^0, x^i $(i = 1, 2, 3)$ and the momenta p^0, p^i $(i = 1, 2, 3)$. The hamiltonian H is equal to zero and in the non-relativistic case the momenta satisfy the constraint

$$A = p^0 + \frac{p^k p^k}{2M} = 0 \tag{1}$$

where M is the mass of the particle in question and summation over repeated indices is understood throughout the paper. The equation of motion is of the form [13]

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \lambda(\tau)\{f, A\} \tag{2}$$

where τ is an evolution parameter, $\lambda(\tau)$ is an arbitrary function and $\{,\}$ denotes the Poisson bracket of quantities in question. For a free particle the non-zero Poisson brackets of the basic variables are

$$\{x^i, p^j\} = \delta^{ij}\{x^0, p^0\} = 1. \tag{3}$$

Different choices of $\lambda(\tau)$ corespond to different parametrizations of world-lines and A is the generator of those reparametrizations [13]. Since the physical quantities F must be invariant with respect to those reparametrizations, they have to satisfy

$$\{F, A\} = 0. \tag{4}$$

In the quantum theory the basic variables are replaced by operators satisfying the following commutation relations

$$[x^0, p^0] = i\hbar \quad [x^i, p^j] = i\hbar\delta^{ij} \tag{5}$$

and the physical states Ψ of the spinless particle are projected by the condition

$$A\Psi = 0. \tag{6}$$

Let us now observe the position operators q^i. The natural and physically acceptable conditions imposed on q^i seem to be the following:

1. The transformation properties of q^i are the same as those of x^i. (Here the standard transformation properties of x^0, x^i, p^0, p^i under the Galilei transformations, 3-rotations, 3-translations, space and time reflections are assumed i.e. (x^0, x^i) transform like (time, position) and (p^0, p^i) transform like (energy, momentum)).
2. $[q^i, A) = 0$ (in accordance with Eq. (4))
3. q^i are constructed by means of x^0, x^i, p^0, p^i, M and $\hbar$ only.
4. The description of a localized particle by observers in different frames of reference has to be physically consistent.

It follows from 1—3 that

$$q^i = x^i - \frac{x^0 p^i}{M} - i\hbar\alpha p^i (p^k p^k)^{-1} \tag{7}$$

where α is some dimensionless constant. Now if we put $p^0 = -p^k p^k/2M$ into a wave function in the p-representation, we can write

$$q^i = i\hbar \frac{\partial}{\partial p^i} - i\hbar\alpha p^i (p^k p^k)^{-1}. \tag{8}$$

Eigenstates of q^i corresponding to eigenvalues a^i are of the form

$$\varphi(p^k; a^k) = \text{const.}\ (p^k p^k)^{\alpha/2} \exp\left(-\frac{i}{\hbar} a^k p^k\right) \tag{9}$$

and the wave function

$$\Psi(p^k; a^k, t_0) = \varphi(p^k; a^k) \exp\left[-\frac{i}{\hbar}(t - t_0)\frac{p^k p^k}{2M}\right] \tag{10}$$

where t is time, can be interpreted as the wave function of a particle localized at the time t_0 at the point a^k.

As to condition 4 let us perform a homogeneous Galilei transformation. In a new frame of reference, which moves with the velocity v^k with respect to that

in which a considered particle is described by $\Psi_0(p^k, t; a^k, t_0)$, a particle will be described by Ψ_v given as

$$\Psi_v = \exp\left(\frac{i v^k G^k}{\hbar}\right) \quad \Psi_0(p^k, t; a^k, t_0)$$

where $G^k = tp^k - Mx^k$ are generators of Galilei boosts. It is easy to show that the condition

$$q^i \Psi_v|_{t=t_0} = (a^i - v^i t_0)\, \Psi_v|_{t=t_0}$$

will be satisfied only if $\alpha = 0$. Hence we obtain the standard result $q^i = i\hbar\,\partial/\partial p^i$.

In the relativistic spinless case the constraint $A = 0$ is of the form

$$A = p^\mu p_\mu - M^2 c^2 = 0 \tag{11}$$

where $\mu = 0, 1, 2, 3$, $p^\mu = g^{\mu\nu} p_\nu$, $g^{\mu\nu} = \text{diag}\,(+, -, -, -)$ and c is the velocity of light. The non-zero commutators of the basic variables are

$$[x^\mu, p^\nu] = -i\hbar g^{\mu\nu}. \tag{12}$$

The relativization of conditions 1—4 is not unambiguous but their straightforward generalization to the relativistic case can be as follows: First of all, the Galilei transformations must be replaced by the Lorentz ones. Moreover, in 1 we must assume (x^0, x^i), (p^0, p^i) to form a four-vector and the constant c must be added to 3.

Now the conditions 1—4 (modified in accordance with the previous consideration) reproduce Kálnay's result [6]

$$q^i = x^i - p^i (p^0)^{-1} x^0.$$

On the other hand, if we omit condition 4 and require q^i to be hermitian (with respect to the corresponding scalar product), we obtain the Newton—Wigner result [1]

$$q^i = x^i - p^i (p^0)^{-1} x^0 + \frac{i\hbar}{2} p^i (p^0)^{-2}.$$

Hence the considered relativistic version of conditions 1—4 and the requirement of hermiticity of q^i are not consistent.

III. „POSITION" IN RELATIVISTIC QUANTUM MECHANICS

A proper formalism of the localization would exhibit the Lorentz covariance. All physical quantities in the relativistic theory must posses a good behaviour under the Lorentz transformations. It means that they must be scalars, components of four-vectors, tensors, etc. Since there are difficulties in finding some four-vector the space components of which would represent three position operators q^i, we conjecture that three quantities closely related to the non-relativistic position operators are *oi*-components of some antisymmetric tensor. As will be seen later this approach represents one of the possible relativizations of the non-relativistic position operator properties. Hence we assume

i) Quantities playing the role of position in relativistic quantum mechanics are *oi*-components of some antisymmetric tensor $q^{\mu\nu}$.
ii) $q^{\mu\nu}$ is constructed by means of x^μ, p^μ, M, c and $\hbar$ only (we confine ourselves to the spinless case only).
iii) $[q^{\mu\nu}, A] = 0$
iv) In the non-relativistic approximation q^{oi} have to reduce to $q^i = x^i - x^0p^i/M$.

An antisymmetric tensor $q^{\mu\nu}$ can be written in the form

$$q^{\mu\nu} = f\frac{1}{Mc}(x^\mu p^\nu - x^\nu p^\mu)$$

where f is some Lorentz invariant operator which can be written in the form

$$f = f\left(\frac{M^2c^2}{\hbar^2}x^\mu x_\mu - \frac{1}{\hbar^2}(x^\mu p_\mu)^2\right).$$

(The tensor dual to $(x^\mu p^\nu - x^\nu p^\mu)$ is excluded from our considerations because it does not possess desirable transformation properties under discrete transformations).

Let us now consider the non-relativistic approximation. In this case we have to replace x^0 by cx^0, p^0 by $Mc - p^0/c$ and moreover we have to neglect terms of the type $(\ldots)\,p^0/Mc^2$ with respect to the terms $(\ldots)$. Then we obtain

$$p^\mu p_\mu - M^2c^2 = 0 \to p^0 + \frac{p^kp^k}{2M} = 0$$

$$[x^0, p^0] = -i\hbar \to [x^0, p^0] = i\hbar$$

$$-\frac{1}{Mc}(x^0p^i - x^ip^0) \to x^i - \frac{x^0p^i}{M}.$$

Since, in general, f does not reduce to a constant at the non-relativistic limit we choose $f = -1$. So we finally obtain

$$q^{\mu\nu} = -\frac{1}{Mc}(x^\mu p^\nu - x^\nu p^\mu).$$

This result does not seem to be unexpected. In the non-relativistic case the

position operators $q^i = x^i - x^0p^i/M$ are proportional to the generators of the Galilei boosts acting in the Hilbert space spanned by solutions of the equation $(p^0 + p^kp^k/2M)\,\Psi = 0$. Hence one may expect that in the relativistic theory the operators playing the role of the position operators will be proportional to the generators of the Lorentz boosts.

IV. CONCLUDING REMARKS

To summarize the results of our speculations we can say that quantities playing the role of position operators in the relativistic quantum mechanics are proportional to the generators of the Lorentz boosts.

The presented formalism possesses at least two satisfactory features. Firstly, it exhibits the explicit Lorentz covariance and secondly, the non-relativistic concept of the localization naturally restores at the non-relativistic approximation. On the other hand $q^{\mu\nu}$ cannot be introduced for massless particles and are not hermitian in general.

Concluding this section we note that we confined ourselves to such $q^i(q^{oi})$ which did not depend upon an evolution parameter τ explicitly. We conjecture that if we should consider $q^i(q^{oi})$ explicitly depending upon τ and if we should find (under some plausible assumptions) a relation between τ and time, we should obtain the position operators in the Heisenberg representation.

We think that the non-relativistic concept of localization (following our everyday experience) cannot be without drastic changes introduced in relativistic quantum mechanics. One of the reasons for supporting this conjecture is perhaps the following: In relativistic quantum mechanics there is no quantity representing the o-component of some four-vector j^μ satisfying $\partial_\mu j^\mu = 0$, which would be interpreted as the probability density in finding a given particle at a given point in general. However, this problem requires a more detailed investigation.

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О ВОЗМОЖНОСТИ ОПРЕДЕЛЕНИЯ МЕСТОНАХОЖДЕНИЯ ЧАСТИЦ

В работе приведены причины, на основе которых дается предпочтение величинам, пропорциональныи генераторам лоренцевых бустов и служащим в качестве операторов положения в релятивистской квантовой механике.