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ON A FACTORIZATION OF THE KINETIC ISING MODEL

B. MAMOJKA[1]), Bratislava

Using the principle of the maximal information entropy, a set of factorization of probabilities of states for kinetic Ising models is deduced. The properties of factorization are studied. We find the connection between the approximation based on our factorization and the Hamiltonian of the corresponding equilibrium Ising model.

О ФАКТОРИЗАЦИИ КИНЕТИЧЕСКОЙ МОДЕЛИ ИЗИНГА

В работе при использовании принципа максимальной энтропии информации выведена совокупность факторизации вероятностей состояний для кинетических моделей Изинга. При этом изучаются свойства данной факторизации. Найдена связь между приближением, основанном на данной факторизации, и гамильтонианом соответствующей равновесной модели Изинга.

I. INTRODUCTION

In order to solve the kinetic Ising model it is often necessary to factorize the probability distributions of the states of the whole system by means of the probability distributions of the states of its suitably chosen subsystems. There is a variety of more or less physically or mathematically justified factorizations (see e. g. [1]). In this article we study the factorization $P(M(S))$ of the probability distribution of the states $M(S)$ of a spin system ($s_i = 1$ or -1) on a lattice M. The distribution is factorized by means of the probability distribution $P(x(S))$ of the states $x(S)$ on the clusters x from the suitably chosen set X of clusters on the lattice M. The studied factorization is a unique consequence of the below formulated axioms based on the principle of the maximal entropy. It is also the consequence of the choice of the set X of clusters on the lattice M with respect to the structure of the corresponding equilibrium Hamiltonian.

Performing the calculations with out factorization we found two important results:

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[1]) Institute of Physics, Electro-Physical Research Centre, Slov. Acad. Sci., Dúbravská cesta, 842 28 BRATISLAVA, Czechoslovakia.

1. The necessary condition imposed on the structure of the set X which enables to obtain the correct equilibrium solution using the approximation based on our factorization.

2. The time development of the studied kinetic Ising model obtained by use of our factorization is such as if at any time t the system were in an instantaneous equilibrium state of a certain time-dependent Hamiltonian at a fixied time t which goes to the corresponding equilibrium Hamiltonian for t.

We have centred our attention on the kinetic Ising models in order to gain a deeper understanding and justification of some used factorizations. These models remain very important objects for the theoretical investigation of the Ising systems. Originally, some simple kinetic lattice models have been studied in order to describe adsorption and desorption of gases on the surface of crystals [2], [3]. Only since Glauber [17] formulated his wellknown simple kinetic models, the equilibrium states of which are equivalent to those of the corrresponding Ising models, the subject has been intensively studied. As even such simple version of kinetic models, except for the simplest possible one, were not solved exactly, several approximative procedures have been proposed for their study [4, 5, 6]. These procedures, as well as the other ones, based on Green's function method, are described in the survey [7]. The theoretical investigation of the kinetic lattice models has been carried out simultaneously with experimental observations of a surface diffusion, adsorption, desorption and chemical reactions. In order to describe these effects methods have been commonly used which are typical for the thermal equilibrium [8, 9]. Recently, the behaviour of kinetic models has been mainly studied in the critical region [10, 11, 12] by means of the method of the renormalization group as well as by means of the methods using computer simulations of the growth of a new phase during the phase transition [13, 14, 15].

II. FACTORIZATION

To obtain our factorization we postulate the following axioms:

A_1. The probabilities $P(x(S))$, $x \in X$ are known.

A_2. There are known all the momenta

$$m_z = \sum_{M(S)} \prod_{i \in z} s_i P(M(S)), \qquad z \in X' \tag{1.a}$$

$$X' = \{z \mid z \subseteq x, \ x \in X\} \tag{1.b}$$

generated by the probabilities $P(x(S))$, $x \in X$. The index i numbers the lattice points. The axiom A_2 is equivalent to A_1.

B. We have no information on the probabilities $P(y(S))$, $y \notin X'$.

C. The missing information on the probabilities $P(y(S))$, $y \notin X'$, is replaced by the demand for the maximum entropy of the factorized probability distribution $P(M(S))$.

These axioms can be applied to the approximative solution of the kinetic Ising models, the time development of which is described by the kinetic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(y(S))) = F_y(M(S)), \qquad y \subseteq M \tag{2.a}$$

Where $F_y(M(S))$ are functionals depending generally on all the probabilities $P(z(S))$, $z \subseteq M$. In the majority of cases we can not solve exactly this system of equations. To solve Eq. (2.a) approximately we must take only some subsystem of the system, for which we write

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(x(S))) = F_x(M(S)), \qquad x \in X, \tag{2.b}$$

where X is the suitable set of clusters on M mentioned in the axiom A_1. The functionals $F_x(M(S))$, $x \in X$ still depend on all the probabilities $P(y(S))$, $y \notin X'$ which do not appear on the left-hand sides of the Eqs. (2.b). In order to make the system (2.b) closed we have to express all the probabilities $P(y(S))$, $y \notin X'$ only by means of the probabilities $P(x(S))$, $x \in X$. Inserting these factorizations into Eq. (2.b) we obtain the system of equations for the calculation of the probabilities $P(x(S))$, $x \in X$. These are then used as an input in our factorization procedure through the axiom A_1. Thus we have obtained the selfconsistent method for an approximative solution of the system of the kinetic equations (2.a). The choice of factorization (e. g. using the axioms A.—C.) and the choice of the set X (e. g. the set of all pairs of nearest neighbours) create the basis of the approximation applied to the solution to the given problem.

In order to find the actual form of the wanted factorization, we use first axioms A_2., B., and C. We look for the maximum of the entropy

$$W = \sum_{M(S)} P(M(S)) \ln P(M(S)) \tag{3}$$

satisfying the condition (1.a). Then the variational method gives

$$P(M(S)) = \exp\left[\sum_{z \in X'} q_z \prod_{i \in z} s_i\right] \Big/ Z, \tag{4.a}$$

where Lagrange's multiplicators are given as follows

$$m_z = \frac{\partial \ln Z}{\partial q_z} \tag{4b}$$

$$Z = \sum_{M(S)} \exp\left[\sum_{z \in X'} q_z \prod_{i \in z} s_i\right]. \tag{4.c}$$

From the formula (4.a) it also follows directly that

$$\sum_{M(S)} \prod_{i \in z} s_i \ln P(M(S)) = \begin{cases} q_z & \text{if } z \in X' \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We can obtain the same equations by means of extremalization of entropy (3) with respect to the unknown momenta m_z, $z \notin X'$. In this procedure the following expression of the probability distributions is used

$$P(M(S)) = \sum_{M(T)} \prod_{i \in M} \left[\frac{1}{2}(1 + s_i t_i)\, P(M(T))\right] = \left(\frac{1}{2}\right)^N \left(1 + \sum_i m_i s_i + \sum_{i \neq j} m_{ij} s_i s_j + \dots\right), \tag{6}$$

where N is the number of lattice points.

The exponent in the formula (4.a) is a Hamiltonian of the Ising model with the interactions — kTq_z of the spins within the range of the cluster $z \in X'$. This fact points to the correspondence between the applied approximation — namely the choice of the set X — and the Hamiltonian of the studied system in its thermal equilibrium. Let this Hamiltonian be of the form

$$H = -\sum_{u \in U} K_u \prod_{i \in u} s_i. \tag{7}$$

The applied factorization (4.a) enables us to obtain an approximative solution achieving an equilibrium state only if the following condition is satisfied

$$U \subseteq X'. \tag{8}$$

Only in this case is it possible to require the satisfaction of the following conditions

$$\lim_{t \to +\infty} q_u(t) = -K_u/kT \text{ for all } u \in U \tag{9}$$

$$\lim_{t \to +\infty} q_z(t) = 0 \text{ otherwise.}$$

The condition (8) sets up the important requirements on the choice of the set X because the set X' is unambigously determined by the set X by means of the formula (1.b). The choice of a set X larger than required by conditions (8) can improve the approximation but simultaneously the calculations are more cumbersome.

It is sometimes more convenient to use another expression for the factorization (4) based on the axioms A_1, B and C. In this case we use the conditions

$$P(x(S)) = \sum_{\bar{x}(S)} P(M(S)), \quad x \in X \tag{10}$$

instead of the condition (1.b), $\bar{x}$ is the complement to the x in M. In this case the variational method gives

$$P(M(S)) = \exp\left[\sum_{x \in X} L_x(S)\right] \tag{11}$$

or

$$P(M(S)) = \prod_{x \in X} l_x(S), \quad l_x(S) = \exp[L_x(S)], \tag{12}$$

where Langrange's multiplicators $L_x(S)$ depend only on spins in the cluster x. Since, e. g.

$$L_x(S) + L_y(S) = L'_x(S) + L'_y(S)$$
$$L'_x(S) = L_x(S) + v_{x \cap y}(S)$$
$$L'_y(S) = L_y(S) - v_{x \cap y}(S),$$

the functions $L_x(S)$ are not unambiguous. Here $v_{x \cap y}(S)$ is an arbitrary function of spins on the interaction of sets x and y. The expressions (4), (11), (12) are equivalent to each other. That can be easily shown by expanding $L_x(S)$ into the spin series.

The most eirect way to find the explicit form of the factorization (4.a) is to calculate q_z, $z \in X'$ as functions of $P(x(S))$, $x \in X$ or m_z, $z \in X'$. As we shall see later on this procedure is too cumbersome.

We are able to solve it successfully only in one dimension and for special quasi-one dimensional models. In spite of this fact it is possible to derive some generally valid properties of the factorization for any dimension.

Let us define the neighbourhood B_u of the cluster u as follows

$$B_u = C_u - u \tag{13.a}$$

$$C_u = \bigcup_{z \in X_u} z \tag{13.b}$$

$$x_u = \{z \mid z \cap u \neq \emptyset,\ z \in X\}. \tag{13.c}$$

Thus the neighbourhood B_u is the set of all the lattice points which do not belong to u but belong at least to one cluster $z \in X$ such that $z \cap u \neq \emptyset$. This neighbourhood is unambigously determined by the set X. If the set X is chosen as the smallest, satisfying the requirement (8), then the neighbourhood B_u represents the range of interaction of spins belonging to the cluster u.

Let us study the properties of the conditional probability $P(u(S) | \overline{(u \cup v)}\ (S))$ where $u \cap v = \emptyset$. Using the definition (13) and the factorization (12) we obtain

$$P(u(S)|(\overline{u\cup v})(S)) = P(\bar{v}(S))/P(\overline{u\cup v}((S)) = \sum_{v(S)} P(M(S)) \Big/ \sum_{(u\cup v)(S)} P(M(S) =$$

$$= \sum_{v(S)} \prod_{x\in C_{u\cup v}} l_x(S) \Big/ \sum_{(u\cup v)(S)} \prod_{y\in C_{u\cup v}} l_y(S) = g((u\cup B_{u\cup v})(S)). \tag{14.a}$$

This formula can be written in the form

$$P(v(S)) = g((u\cup B_{u\cup v})(S))\ P((\overline{u\cup v})\ (S))$$

and by means of summation with respect to the spin states $\bar{C}_u\cup_v(S)$ we obtain

$$g((u_{u\cup v})(S)) = P((u\cup B_{u\cup v})\ (S))/P(B_{u\cup v})\ (S)) = P(u(S)|B_{u\cup v}(S)). \tag{14.b}$$

Inserting (14.b) into (14.a) we finally find

$$P(u(S)|(\overline{u\cup v}(S)) = P(u(S)|B_{u\cup v}(S)). \tag{15}$$

Thus the studied conditional probability does not depend on the state $(\overline{u\cup v})$ (S) on the whole residue of the lattice M but only on the states in the neighbourhood $B_{u\cup v}$. In the case $v=\emptyset$ formula (15) implies

$$P(u(S)|\bar{u}(S) = P(u(S)|B_u(S)). \tag{16}$$

Let us have the following decomposition

$$M = a\cup b\cup \ldots \cup u\cup v \tag{17.a}$$

of the lattice M where $a, b, \ldots, u, v$ are mutually disjunct sets of lattice points. Then we can write

$$P(M(S)) \equiv P(a(S)|\bar{a}(S)\ P(b(S)|(\overline{a\cup b})\ (S)) \ldots \times$$

$$\times\ P(u(S)\ (\overline{a\cup b\ \ldots\ \cup u})\ (S))\ P(v(S)) \tag{17.b}$$

Inserting the approximation (15) into the exact formula (17.b) we obtain

$$P(M(S)) = P(a(S)|B_a(S))\ P(b(S)|B_{a\cup b}(S)) \ldots$$

$$\ldots\ P(u(S)|B_{a\cup b\ldots\cup u}(S))P(v(S)). \tag{18}$$

Hence we see that in our approximative approach the factorized probability contains conditional probabilities depending on much smaller clusters than those in the exact expression (17.b). The factorization (18) does not depend on the used form of the decomposition (17.a) since it always is reduced to the mutually equivalent forms of factorization (4), (11), (12). But this reduction is almost always

the theoretical one which is achieved by means of the formula (14.a). This is due to the problems connected with the calculation of $l_x(S)$, resp. q_z. Only in a few very special cases the factorization (18) is the same as the formula (12) which can be effectively applied. The size of the neighbourhoods occurring in the expression (18) becomes obviously very quickly greater than the size of clusters from the set X.

A suitable choice of the decomposition (17.a) might sometimes lend to a useful form of the factorization (18). Let us consider a d-dimensional lattice. A position of a lattice point is described by the vector $(k_1, k_2, \ldots, k_d)$ where $k_1 = 1, \ldots,$ n. The set X is chosen so that any cluster $x\in X$ lies in a layer with $k+1$ lattice points in the direction of the axis k_1, i.e. $i = k_1 = k+1$ where i is fixed for a given cluster and k is fixed for all clusters $x\in X$. We use the following form of the decomposition (17.a)

$$M = \bigcup_{i=1}^{n} a_i \tag{19.a}$$

$$a_i = \{k_1, k_2, \ldots, k_d);\ \ k_1 = i\},$$

i. e. the lattice M is decomposed into monoatomic layers in the direction of the axis k_1. Then the needed neighbourhoods are of the form

$$B_{a_1\cup\ldots\cup a_i} \subseteq a_{i+1}\cup\ldots\cup a_{i+k};\ \ i = 1, \ldots, n-k. \tag{19.b}$$

In what follows we will suppose that there is valid only the sign of equality in the expression (19.b). Such a casual enlargement of neighbourhoods might not make the approximation worse. Inserting (19.b) into (18) we obtain

$$P(M(S)) = \prod_{i=1}^{n-k-1} P(a_i(S)\ (a_{i+1}\ \ldots\ a_{i+k})\ (S))\ P(a_{n-k}\cup\ldots\cup a_n)\ (S)) =$$

$$= \prod_{i=1}^{n-k} P(a_i\cup\ldots\cup a_{i+k})\ (S)) \Big/ \prod_{j=2}^{n-k} P((a_j\cup\ldots\cup a_{j+k-1})\ (S)). \tag{19.c}$$

This formula is convenient to apply in the calculations if the following condition is valid

$$a_i\cup a_{i+1}\cup\ldots\cup a_{i+k}\in X;\ \ i = 1, \ldots, n-k. \tag{19.d}$$

Namely in this case we know all factors appearing on the right-hand side of the euation (19.c) according to the axiom A_1.

In the one-dimensional case the set a_i is reduced to the ith lattice point and the set X contains all chains of $k+1$ neighbouring lattice points. Hence the formula (19.c) becomes as follows

$$P(s_1, \ldots, s_n) = \prod_{i=1}^{n-k} P(s_i, \ldots, s_{i+k}) \Big/ \prod_{j=2}^{n-k} P(S_j, \ldots, S_{j+k-1}). \tag{20}$$

All factors in this expression are known.

If the lattice has no free ends in the direction of the axis k_1 or it the set X does not contain "a sufficient number" of convenient clusters "spread" over the whole lattice in the directions perpendicular to the axis k_1, in order to fulfil the condition (19.d) we do not know the factors occuring on the right-hand side of (19c) without the combersome calculation of $l_x(S)$, $x\in X$ or q_z, $z\in X'$. These problems will be demonstrated by the following examples.

Example 1.: Let us have a one-dimensional lattice. Its end points are nearest neighbours (i. e. $k=1$). Then the needed neighbourhoods are as follows

$$B_{a_i\cup\ldots\cup a_i}=a_{i+1}\cup a_n,\ i=1,\ldots n-2$$

and from the formula (18) we have

$$P(s_1,\ldots,s_n)=\prod_{i=1}^{n-2}P(s_i,s_{i+1},s_n)\Big/\prod_{j=2}^{n-2}P(s_j,s_n). \tag{21}$$

There is not known any factor on the right-hand side of (21) according to axiom B. For instance in the simplest case of $n=3$ we obtain a cubic equation which is very complicated to solve.

Example 2.: Let us have a two dimensional square lattice with 3 time lattice points. Let the set X contain all pairs of nearest neighbours, i. e. $X=\{s_{11},s_{12}),(s_{11},s_{21}),\ldots\}$. The formula (18) then reads

$$\begin{aligned}P(s_{11},\ldots,s_{33})=&P(s_{11}|s_{12},s_{21})\,(P(s_{12}|s_{13},s_{21},s_{22})\,P(s_{13}|s_{21},s_{22},s_{23})\times\\&\times P(s_{21},s_{22},s_{23},s_{31})\,P(s_{22}|s_{23},s_{31},s_{32})\,P(s_{23}|s_{31},s_{32},s_{33})\times\\&\times P(s_{31}|s_{32},s_{33})\,P(s_{32},s_{33}).\end{aligned} \tag{22}$$

Also in this formula we do not know directly any factor except the $P(s_{32},s_{33})$, which cancels with the same term from the preceding factor according to the definition of the conditional probability. Thus it is necessary to solve 21 nonlinear equations (4.a).

III. TEST OF FACTORIZATION

Next we will test our factorization. For this purpose we use the exactly solvable version of the Glauber model [17]. It is the linear kinetic Ising model without external field with transition probability

$$w(-s_i|s_i,s_{i-1},s_{i+1})\sim\frac{1}{2}\left[1-\frac{1}{2}s_i(s_{i-1}+s_{i+1})\right]\qquad\gamma=\text{th}\,(2K) \tag{23a}$$

and equilibrium states equal to that of the corresponding Ising model with the Hamiltonian

$$H=-K\sum_i s_is_{i+1}. \tag{23b}$$

Its time development is given by the following system of kinetic equations

$$\frac{d}{dt}m(i)=-m(i)+\frac{1}{2}\gamma m(i-1)+\frac{1}{2}\gamma m(i+1) \tag{24}$$

$$\frac{d}{dt}m(i,j)=-2m(i,j)+\frac{1}{2}\gamma m(i-1,j)+\frac{1}{2}\gamma m(i+1,j)+\frac{1}{2}\gamma m(i,j-1)+\frac{1}{2}\gamma m(i,j+1)\ldots,$$

where

$$m(i)=\langle s_i\rangle,\ m(i,j)=\langle s_is_j\rangle.$$

The solution to this system is

$$\begin{aligned}m(i,t)&=e^{-t}\sum_{n=-\infty}^{\infty}m(n;0)I_{i-n}(\gamma t)\\m(i,j,t)&=m^e(i,j)+\\&e^{-2t}\sum_{p<r}m(p,r;0)-m^e(p,r)I_{i-p}(\gamma t)I_{j-r}(\gamma t)-\\&-I_{i-r}(\gamma t)I_{j-p}(\gamma t)\,m^e(p,r)=R^{|p-r|};\quad R=\text{th}\,K,\end{aligned} \tag{25}$$

where $m^e(p,r)$ is th equilibrium value of $m(p,r,t)$ and $I_k(x)$ is the Bessel function of an imaginary argument. Supposing that the initial values are of the form

$$m(i;0)=m_0;\ m(i,j;0)=(R')^{|i-j|};\quad R'=\text{th}\,K', \tag{26}$$

we obtain from (26)

$$m(i;t)=m_0\exp\left[-(1-\gamma)t\right] \tag{27.a}$$

$$m(i,i+1;t\to+\infty)=R+\frac{2(\gamma'-\gamma)}{(1-|\gamma|)(1-|\gamma'|)}(4|\gamma|t)^{-3/2} \tag{27.b}$$

$$\exp\left(-2(1-|\gamma|)t\gamma'\right)=\text{th}\,(2K). \tag{27.c}$$

we will compare the exponent

$$\alpha_\infty=2(1-|\gamma|) \tag{28}$$

of the exact solution (26.c) with the corresponding exponent α_k of the approximative solution

$$m_{12}^{(k)}=R+c_k\exp\left(-\alpha_kt\right)$$

obtained by means of the factorization (20) for given k.

We assume translational invariance of the momenta, i. e.

$$m(i,j,\ldots,k)=m(1,j-i+1,\ldots,k-i+1)\quad i<j\ldots<k. \tag{29}$$

Thus we write

$$m(i)=m(1),\ m(i,\ +i+1)=m(1,\ 2),\ \dots \tag{29.b}$$

The assumption (29) is justified namely for $t \to \infty$, i. e. in the neighbourhood of thermal equilibrium. The time development of this simplified model is given by the following equations

$$\frac{dm(1)}{dt}=(1-\gamma)\,m(1) \tag{30.a}$$

$$\frac{dm(1,\ l+1)}{dt}=-2m(1,\ l+1)+\gamma m(1,\ l)+\gamma m(1,\ l+2);\ m(1,\ 1)=1;\quad l=1,\ 2,\ \dots \tag{30.b}$$

The solution of (30.a) is exactly the same as that of (27.b).

The system (30.b) determines among other facts the exact form of $m(1,\ 2)$. In order to obtain an approximation of $m(1,\ 2)$ with the help of our factorization we consider first k equations of the system (30.b) for $l=1,\ \dots,\ k$. In this restricted system, there is only one undetermined quantity, namely $m(1,\ k+2)$, which must be factorized. In our choice the set X contains all chains of $k+1$ mutualy neighbouring lattice points. The expansion of nonlinear factorization of momentum $m(1,\ k+2)$ to the first order in deviations from equilibrium is as follows (Appendix)

$$m(1,\ k+2)=R^{k+1}-R^2[m(1,\ k)-R^{k-1}]+2R[m(1,\ k+1)-R^k]. \tag{31}$$

By use of (30.b) for $l=1,\ \dots,\ k$ and (31) we obtain

$$\frac{dx_1}{dt}=-2(1-\gamma R)x_1 \quad \text{for } k=1 \tag{32.a}$$

$$\begin{aligned}\frac{dx_1}{dt}&=-2x_1+\gamma x_2\\ \frac{dx_2}{dt}&=\gamma x_1-2x_2+\gamma x_3\\ &\vdots\\ \frac{dx_{k-1}}{dt}&=\gamma x_{k-2}-2x_{k-1}+\gamma x_k\end{aligned} \tag{32.b}$$

$$\frac{dx_k}{dt}=\gamma(1-R^2)x_{k-1}-2(1-\gamma R)x_k;\quad k=2,\ 3,\ \dots$$

$$x_l=m(1,\ l+1)-m^e(1,\ l+1;\ m^e(1,\ l+1)=R^l. \tag{32.c}$$

Now, we can easily find the following exponents α_k for $k=1,\ 2,\ 3$

$$\begin{aligned}\alpha_1&=2(1-R)=2\sqrt{1-\gamma^2}\\ \alpha_2&=2-\gamma R-|\gamma|=1-|\gamma|+\sqrt{1-\gamma^2}\\ \alpha_3&=2-\frac{1}{2}\gamma R-|\gamma|\sqrt{\frac{1}{4}r^2+2}\\ \alpha_\infty&=2(1-|\gamma|)\\ \alpha_1&\geqq\alpha_2\geqq\alpha_3\geqq\alpha_\infty\end{aligned} \tag{33}$$

where the equalities hold only it $_\gamma=0$ or $|\gamma|=1$ (see Fig. 1). We can infer from the form of the equations (32.b) and from the ineguality (33) that the exponents α_k converge to the exact value α_∞ with an increase of k i. e. with increase of size of clusters contained in the set X.

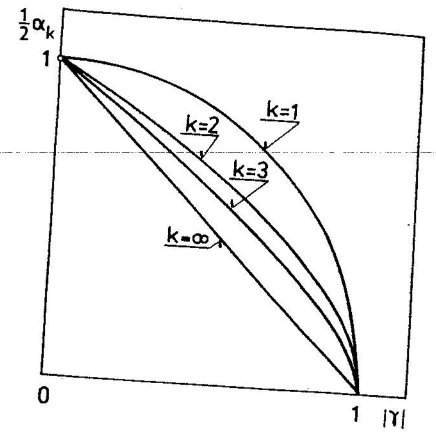

Fig. 1. The dependence of the parameter α_k on γ-th $2K$ in the approximations characterized by $k=1,\ 2,\ 3$. The straight line α_∞ corresponds to the exact solution.

IV. CONCLUSION

As has been shown, the studied factorization is actually convenient to apply only in the one-dimensional case. According to the formula (22) our factorization means the following approximation of the conditional probability

$$P(s_i|\,s_{i+1},\ \dots,\ s_{i+k},\ s_{i+k+1},\ \dots)=P(s_i|\,s_{i+1},\ \dots,\ s_{i+k}) \tag{34}$$

in the one-dimensional case.

An intuitively understandable and dimension independent result is the correspondence between the two following aspects of the problem. The first aspect is the Hamiltonian of the equilibrium counterpart of the studied kinetic model and the second is the structure of the set X which determines the probabilities according to which the factorization is to be done. This correspondence is the consequence of the physical requirements which requires that it must be possible to achieve the correct equilibrium state when the factorization is applied. It is represented by the necessary demands imposed on the set X by the formulas (7), (8) and (1.b). In an approximation based on such the factorization the behaviour of the studied kinetic model is at any fixed time such as if the system were in the state of an instantaneous equilibrium corresponding to the Ising model with the Hamiltonian

$$-kTH(t)=\sum_{z\in X} q_z(t)\prod_{i\in z} s_i. \tag{35}$$

Axioms A, B, and C and the requirement of a correct thermal equilibrium are evidently not sufficient to determine an applicable factorization in more than one dimension. There ought to be added some further physical requirements to be built into the probabilities of states on spatially separated clusters regarding the given type of the spin interaction.

Concluding I should like to express my gratitude to Dr. Anton Šurda for valuable suggestions and helpful discussions.

APPENDIX

The factorization of the momentum $m(j, j+p)$ with respect to the momenta on the nearest smaller clusters is given as follows

$$m(j, j+p)=\sum_{s_j,\dots,s_{j+p}} s_j s_{j+p} P(s_j, \dots, s_{j+p-1})P(s_{j+1}, \dots, s_{j+p})/P(s_{j+1}, \dots, s_{j+p-1}), \tag{A1}$$

where the probabilities on the right-hand side are expressed through momenta by means of formula (6). We have to expand the nonlinear quantity (A1) to the first order in the deviations

$$x_{i,j\dots}=m(i, j, \dots)-m^e(i, j, \dots) \tag{A2}$$

of momenta from their equilibrium values. The equilibrium values are easily calculated by means of the transfer matrix method [16]. This method also allows us to find the following equilibrium probabilities

$$p^e(s_i, \dots, s_j)=\frac{1}{2}(2\cosh K)^{-(j-i+1)}\exp\left(K(s_i s_{i+1}+\dots+s_{j-1}s_j)\right) \quad (j>i). \tag{A3}$$

The wanted expansion is of the following form

$$x_{j,j+p}=\sum_{i=1}^{3}\sum_{z\in M_i}\left(\frac{\partial m(j, j+1)}{\partial m_z}\right)_e x_z \tag{A4a}$$

where

$$\begin{aligned} M_1&=\{z\,|\,z=\{j\}\cup v, \quad v\in C\}\\ M_2&=\{z\,|\,z=\{j+p\}\cup v, \quad v\in V\}\\ M_3&=V=\{v\,|\,v\subseteq\{j+1, \dots, j+p-1\}\}. \end{aligned} \tag{A4b}$$

With the help of (A1), (A3), (A4) and the formula

$$\frac{\partial}{\partial m_z}P(s_i, \dots, s_j)=2^{-(j-i+1)}\prod_{l\in z} s_l, \tag{A5}$$
$$\text{if } z\subseteq\{i, \dots, j\}, \text{ othervise}=0$$

which is the consequence of the expression (6), we have

$$\begin{aligned} \left(\frac{\partial m(j, j+p)}{\partial m_z}\right)_{e,\,z\in M_1}&=R \text{ if } z=\{j, j+p-1\}\\ &=0 \text{ othervise}\\ \left(\frac{\partial m(j, j+p)}{\partial m_z}\right)_{e,\,z\in M_2}&=R \text{ if } z=\{j+1, j+p\}\\ &= \text{othervise}\\ \left(\frac{\partial m(j, j+p)}{\partial m_z}\right)_{e,\,z\in M_3}&=-R^2 \text{ if } z=\{j+1, j+p-1\}\\ &=0 \text{ othervise} \end{aligned} \tag{A6}$$

Thus we obtain

$$x_{j,j+p}=R^2 x_{j+1,j+p-1}+R(x_{j,j+p+1}+x_{j+1,j+p}) \quad (x_{j,j}=0) \tag{A7}$$

and in the case of translational invariance

$$x_{1,p+1}=-R^2 x_{1,p-1}+2Rx_{1,p} \quad (x_{1,1}=0). \tag{8}$$

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