acta phys. slov. 36 (1986), No. 6

PIEZOELECTRIC HALF SPACE PROBLEM WITH GENERALIZED THERMAL COUPLING

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A simple model of generalized thermopiezoelectricity is used to investigate one-dimensional disturbances in a piezoelectric half space under certain conditions. Short time approximate solutions are deduced and discontinuities in the mechanical, thermal and stress fields are analysed using the Laplace transform technique. Ultimately, some of the results obtained have been plotted graphically.

ПРОБЛЕМА ПЬЕЗОЭЛЕКТРИЧЕСКОГО ПОЛУПРОСТРАНСТВА С ОБОБЩЕННОЙ ТЕРМИЧЕСКОЙ СВЯЗЬЮ

Для исследования одномерных возмущений в пьезоэлектрическом полупространстве при определенных условиях использована простая модель обобщенного термопьезоэлектричества. Выведены кратковременные проближенные решения и при помощи преобразования Лапласа проанализированны разрывы в термическом поле и поле механических напряжений. Кроме того, некоторые полученные результаты представлены графически.

1. INTRODUCTION

The theory of thermoelasticity which takes into account the time required for the acceleration of the flow has aroused much interest in recent years. This theory is a generalization of the classical coupled thermoelasticity theory. Several authors, for example, Lord and Shulman [1], Green and Lindsay [2] etc. have derived the field equations of this theory on different bases taking into account one or two thermal relaxation parameters, respectively. After their pioneering contributions several other researchers have extended the classical problems of thermoelasticity to situations where such generalizations have been made. In this connection the studies undertaken by Chandrasekharaiah [3, 4], Bhatta [5], Agarwal [9] etc. deserve mention.

As far as the present author is aware very few attempts have been made to study polarizable media in general, and piezoelectrics in particular, taking into account

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generalized thermal coupling. In this derection, the two recent investigations due to Bassiony and Ghaleb [6], Pal and Ray [7] are worth mentioning.

The objective of the present paper is to attempt a similar problem of a piezoelectric half space D: $x \geq 0$ with generalized thermal coupling as the Lord and Shulman model [1], under the following conditions.

(i) Plane boundary is subjected to a step excitation of finite stress and the boundary surface is insulated.

(ii) Plane boundary is rigidly fixed and subjected to instantaneous heat flux.

2. FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS

The three-dimensional equations of generalized thermopiezoelectricity from which the one-dimensional equations are to be deduced, are the same as for the usual theory, with the expectation of Fourier's law for heat conduction. In a system of orthogonal cartesian coordinates, these equations are the following.

Consitutive equations

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - e_{kij}D_k - a_{ij}T \\ E_i &= -e_{ijk}\varepsilon_{jk} + b_{ij}D_j - c_iT \\ S &= a_{ij}\varepsilon_{ij} + c_iD_i + aT, \end{aligned} \tag{2.1}$$

Equation of motion

$$\varrho\ddot{u}_i = \sigma_{ij,\,j}. \tag{2.2}$$

Equations of electrostatics

$$D_{i,\,i} = 0, \quad E_i = -v_{,i}. \tag{2.3}$$

Equation for entropy production

$$T\dot{S} = -q_{i,\,i}. \tag{2.4}$$

Fourier's law for heat conduction

$$q_i + A_{ij}\dot{q}_j = -K_{ij}T_{,j} \tag{2.5}$$

where ϱ is the density of the piezoelectric material, u_i, σ_{ij}, ε_{kl}, E_i, D_k, q_i and $k_{i,j}$ are, respectively, the components of displacement, stress tensor, strain tensor, electric field, electric displacement heat flux vector and conductivity. S is the entropy and T is the temperature. c_{ijkl}, e_{kij}, b_{ij} are, respectively, the elastic stiffness components, piezoelectric constants and dielectric impermeability constants. a_{ij}, c_i and a etc are thermopiezoelectric constants. A_{ij} are the thermal relaxation constants.

One dimensional equations

Since the present problem is essentially one dimensional in nature, the corresponding one dimensional equations obtained from the above system of equations can be put in a very convenient form using the following dimensionless parameters

$$\xi=\left(\frac{c_{1111}}{\varrho_0}\right)^{1/2}\left(\frac{aT_0}{k_{11}}\right)X,\quad \tau=\left(\frac{c_{1111}}{\varrho_0}\right)\left(\frac{aT_0}{k_{11}}\right)t$$

$$\tau_1=\left(\frac{c_{1111}}{\varrho_0}\right)\left(\frac{At_0}{k_{11}}\right)A_{11}\qquad \Theta=\frac{(T-T_0)}{T_0}$$

$$u=\left(\frac{c_{1111}}{\varrho_0}\right)^{1/2}\left(\frac{aT_0}{k_{11}}\right)u_1\qquad \sigma=\frac{\sigma_{11}}{c_{1111}}\tag{2.5a}$$

$$E=\frac{E_1}{c_1T_0},$$

where ϱ_0 is the initial density of the piezoelectric half space.

After some manipulations, the linearised one-dimensional equations reduce to the following

$$\frac{\partial^2\Theta}{\partial\xi^2}=\frac{\partial\Theta}{\partial\tau}+\tau_1\frac{\partial^2\Theta}{\partial\tau^2}+g\left(\frac{\partial}{\partial\xi}\frac{\partial u}{\partial\tau}+\tau_1\frac{\partial^2}{\partial\tau^2}\frac{\partial u}{\partial\xi}\right)\tag{2.6}$$

$$\frac{\partial^2 u}{\partial T^2}=\frac{\partial^2 u}{\partial\xi^2}-\varepsilon\frac{\partial\Theta}{\partial\xi}\tag{2.7}$$

$$\sigma=\frac{\partial u}{\partial\xi}-\varepsilon\Theta\tag{2.8}$$

$$E=-e\frac{\partial u}{\xi}-\Theta\tag{2.9}$$

where $\varepsilon=a_{11}T_0/c_{1111}$, $g=a_{11}/aT_0$, $e=e_{111}/c_1T_0$.

Now the equations (2.6) to (2.9) will be solved under the following two sets of boundary conditions.

The limiting conditons corresponding to the two cases (i) and (ii) mentioned in the introduction are the following

(i)

$$\left.\begin{array}{l}\sigma(\xi,\tau)_{\xi=0}=\sigma_0H(\tau)\\ \sigma(\xi,\tau)_{\xi=\infty}=0\end{array}\right\}\quad \tau\geqslant 0$$

$$\left.\begin{array}{l}\frac{\partial\Theta}{\partial\xi}(\xi,\tau)|_{\xi=0}=0\\ \Theta(\xi,\tau)|_{\xi=\infty}\end{array}\right\}\quad \tau\geqslant 0$$

$$\left.\begin{array}{l}\sigma(\xi,\tau)\tau)_{\tau=0}=0\\ \frac{\partial\sigma}{\partial\tau}(\xi,\tau)|_{\tau=0}=0\end{array}\right\}\quad \xi>0$$

$$\left.\begin{array}{l}u(\xi,\tau)_{\tau=0}=0\\ \frac{\partial u}{\partial T}(\xi,\tau)|_{\tau=0}=0\end{array}\right\}\quad \xi>0$$

(ii)

$$u(\xi,\tau)=0\quad \text{at } \xi=0$$

$$\frac{\partial\Theta}{\partial\xi}(\xi,\tau)|_{\xi=0}=-\Theta_0'\delta(\tau)$$

$$\left.\begin{array}{l}\sigma(\xi,\tau)_{\tau=0}=0\\ \frac{\partial\sigma}{\partial\tau}(\xi,\tau)|_{\tau=0}=0\end{array}\right\}\quad \xi>0$$

$$\left.\begin{array}{l}u(\xi,\tau)_{\tau=0}=0\\ \frac{\partial u}{\partial\tau}(\xi,\tau)|_{\tau=0}=0\end{array}\right\}\quad \xi>0$$

3. SOLUTION OF THE PROBLEM

To facilitate the solution of the problem we introduce a thermoelastic potential Φ in dimensionless form such that

$$u=\frac{\partial\Phi}{\partial\xi}.\tag{3.1}$$

The other unknowns of the problem can be expressed through Φ as follows

$$\Theta=\left(\frac{1}{\varepsilon}\right)\left(\frac{\partial^2\Phi}{\partial\xi^2}-\frac{\partial^2\Phi}{\partial\tau_2^2}\right)\tag{3.2}$$

$$\sigma=\frac{\partial^2\Phi}{\partial\tau^2}\tag{3.3}$$

$$E=-\left(e+\frac{1}{\varepsilon}\right)\frac{\partial^2\Phi}{\partial\xi^2}+\frac{1}{\varepsilon}\frac{\partial^2\Phi}{\partial\tau^2}.\tag{3.4}$$

It may be easily verified that the function Φ satisfies the fourth order partial differential equation

$$\left[\frac{\partial^4}{\partial\xi^4}-(1+\tau_1+\varepsilon_1\tau_1)\frac{\partial^4}{\partial\xi^2\partial\tau^2}-(1+\varepsilon_1)\frac{\partial^3}{\partial\xi^2\partial\tau}+\frac{\partial^3}{\partial\tau^3}+\tau_1\frac{\partial^4}{\partial\tau^4}\right]\Phi=0 \qquad (3.5)$$

where $\varepsilon_1=g\varepsilon$. Now taking the Laplace transform of parameter p the above equation simplifies to the following

$$D^4\bar{\Phi}-\alpha_1 D^2\bar{\Phi}+\alpha_2\bar{\Phi}=0 \qquad (3.6)$$

where

$$\alpha_1=(1+\tau_1+\varepsilon_1\tau_1)p^2+(1+\varepsilon_1)p$$

$$\alpha_2=p^3+\tau_1 p^4.$$

$\bar{\Phi}$ is the Laplace transform of Φ and $D=\mathrm{d}/\mathrm{d}\xi$.

Since the medium is semi infinite in nature $\bar{\Phi}\to 0$ as $\xi\to\infty$. Consequently the solution to the equation (3.6) can be written in the form

$$\bar{\Phi}=A_1\,\mathrm{e}^{-m_1\xi}+A_2\,\mathrm{e}^{-m_2\xi} \qquad (3.7)$$

A_1 and A_2 are two constants to be obtained from boundary conditions of the problem. m_1 and m_2 are the roots of the equation

$$(m^2)^2-\alpha_1(m^2)+\alpha_2=0. \qquad (3.8)$$

Taking the Laplace transform of the equations (3.1) and (3.2) we find $\bar{u}=D\bar{\Phi}$ and $\bar{\Theta}=(c/\lambda)\left\{D^2-\frac{\varrho}{c}p^2\right\}\bar{\Phi}$ substituting the expression for $\bar{\Phi}$ given by equation (3.7) in the above two equations we find

$$\bar{u}=-m_1A_1\,\mathrm{e}^{-m_1\xi}-m_2A_2\,\mathrm{e}^{-m_2\xi} \qquad (3.9)$$

$$\bar{\Theta}=\frac{1}{\varepsilon}[(m_1^2-p^2)]\,A_1\,\mathrm{e}^{-m_1\xi}. \qquad (3.10)$$

Using the relevant boundary conditions corresponding to case (i), we find the following two equations for the constants A_1 and A_2.

$$\sigma_0/p^3=A_1+A_2 \qquad (3.11)$$

$$A_1m_1(m_1^2-p^2)+A_2m_2(m_2^2-p^2)=0. \qquad (3.12)$$

Solving the equations (3.11) and (3.12) we find

$$A_1=\frac{-\sigma_0 m_2(m_2^2-p^2)}{p^3(m_1-m_2)\{(m_1+m_2)^2-m_1m_2-p^2\}}$$

and

$$A_2=\frac{\sigma_0 m_1(m_1^2-p^2)}{p^3\{m_1(m_1^2-p^2)-m_2(m_2^2-p^2)\}}\,.$$

Similarly for the case (ii) we find the following two equations for the constants A_1 and A_2.

$$A_1m_1+A_2m_2=0 \qquad (3.13)$$

$$\Theta_0'\varepsilon=\{A_1m_1(m_1^2-p^2)+A_2m_2(m_2^2-p^2)\}. \qquad (3.14)$$

Solving the equations (3.13) and (3.14) we find

$$A_1=-\Theta_0'\varepsilon/m_1\{m_2^2-m_1^2\}\ \text{and}\ A_2=\Theta_0'\varepsilon/m_2\,\{m_2^2-m_1^2\}.$$

Since we restrict our analysis to small values of time, we determine the roots m_1, m_2 of equation (3.7) in the form of a series in ascending powers of $(1/p)$.

$$m_1=a_0p+a_1+a_2/p+\dots$$

$$m_2=b_0p+b_1+b_2/p+\dots$$

where

$$a_0=\frac{1}{2}\{(l_1+2\tau_1^{1/2})^{1/2}+(l_1-2\tau_1^{1/2})^{1/2}\} \qquad (3.15)$$

$$a_1=\frac{1}{4}\left[\frac{(l_0+1/\tau_1^{1/2})}{(l_1+2T_1^{1/2})^{1/2}}+\frac{(l_0-1/\tau_1^{1/2})}{(l1-2T_1^{1/2})^{1/2}}\right]$$

$$a_2=\frac{1}{16}\left[\frac{1}{\tau_1^{3/2}}\left\{\frac{1}{(l_1-2\tau_1^{1/2})^{1/2}}-\frac{1}{(l_1+2\tau_1^{1/2})^{1/2}}\right\}-\right.$$

$$\left.-\left\{\frac{\left(l_0+\frac{1}{T_1^{1/2}}\right)^2}{(l_1+2T_1^{1/2})^{3/2}}+\frac{\left(l_0-\frac{1}{T_1^{1/2}}\right)^2}{(l_1-2T_1^{1/2})^{3/2}}\right\}\right]$$

$$a_3=\frac{1}{32}\left[\frac{1}{\tau_1^{3/2}}\left\{\frac{1}{(l_1+2\tau_1^{1/2})^{1/2}}-\frac{1}{(l_1-2\tau_1^{1/2})^{1/2}}\right\}+\right.$$

$$\left.+\frac{1}{\tau_1^{3/2}}\left\{\frac{\left(l_0+\frac{1}{\tau_1^{1/2}}\right)}{(l_1+2\tau_1^{1/2})^{3/2}}-\frac{\left(l_0-\frac{1}{\tau_1^{1/2}}\right)}{(l_1-2\tau_1^{1/2})^{3/2}}\right\}\right.$$

$$b_0=\frac{1}{2}\{(l_1+2\tau_1^{1/2})^{1/2}-(l_1-2\tau_1^{1/2})^{1/2}\}$$

$$b_1=\frac{1}{4}\left\{\frac{\left(l_0+\frac{1}{\tau_1^{1/2}}\right)}{(l_1+2\tau_1^{1/2})^{1/2}}-\frac{\left(l_0-\frac{1}{\tau_1^{1/2}}\right)}{(l_1-2\tau_1^{1/2})^{1/2}}\right\}$$

$$b_2=\frac{1}{4}\left[\frac{1}{4}\left\{\frac{\left(l_0-\frac{1}{\tau_1^{1/2}}\right)^2}{(l_1-2\tau_1^{1/2})^{3/2}}-\frac{\left(l_0+\frac{1}{\tau_1^{1/2}}\right)^2}{(l_1+2\tau_1^{1/2})^{3/2}}\right\}-\right.$$

$$\left.-\frac{1}{4\tau_1^{3/2}}\left\{\frac{1}{(l_1+2\tau_1^{1/2})^{1/2}}+\frac{1}{(l_1-2\tau_1^{1/2})^{1/2}}\right\}\right]$$

$$b_3=\frac{1}{32}\left[\frac{1}{\tau^{5/2}}\left\{\frac{1}{(l_1+2\tau_1^{1/2})^{1/2}}+\frac{1}{(l_1-2\tau_1^{1/2})^{1/2}}\right\}+\right.$$

$$\left.+\frac{1}{\tau_1^{3/2}}\left\{\frac{\left(l_0+\frac{1}{\tau_1^{1/2}}\right)}{(l_1+2\tau_1^{1/2})^{3/2}}+\frac{\left(l_0-\frac{1}{\tau_1^{1/2}}\right)}{(l_1-2\tau_1^{1/2})^{3/2}}\right\}\right].$$

The values of the other constants have been mentioned as the final result contains only the above ones and l_0, l_1 given by the following relations.

$$l_0=1+\varepsilon$$

$$l_1=1+\tau_1+\varepsilon_1\tau_1.$$

Now to find out the displacement and temperature distributions etc., for the two cases we substitute the values of the constants in the equation (3.9) and (3.10), respectively. Since we limit ourselves to short time approximation, we expand the resulting functions in ascending powers of $1/p$ and retain only terms up to the order $1/p$.

The expressions for displacement and temperature distribution for the first case are the following:

$$\bar{u}(\xi,p)=\frac{\sigma_0}{\{(a_0+b_0)^2-(a_0b_0+1)\}\,(a_0-b_0)}\times \tag{3.16}$$

$$\times\left[\frac{e^{-a_0 B\xi}}{p^2}\{a_0b_0(b_0^2-1)\,e^{-a_1\xi}\}-\frac{e^{-B_0p\xi}}{p^2}\{a_0b_0(a_0^2-1)\,e^{-b_1\xi}\}+\right.$$

$$+\frac{e^{-a_1\xi}e^{-a_0p\xi}}{p^3}\{2a_0b_0b_1+(b_0^2-1)\,(a_1b_0+b_1a_0)-$$

$$-a_2\xi a_0b_0(b_0^2-1)\Big\}+\frac{e^{-b_0p\xi}}{p^3}\,e^{-b_1\xi}\{b_2\xi a_0b_0(a_0^2-1)-$$

$$\left.-(a_0^2-1)\,(a_1b_0+b_1a_0)-2a_0^2b_0b_1+k_0a_0b_0(a_0^2-1)\}\right]$$

and

$$\bar{\Theta}(\xi,p)=\frac{\sigma_0}{\varepsilon\,\{(a_0+b_0)^2-(a_0b_0+1)\}\,(a_0-b_0)}\left[\frac{e^{-b_0\xi p}}{p}\{e^{-b_1\xi}a_0\times\right. \tag{3.17}$$

$$\times(a_0^2b_0^2-b_0^2-1)\}-\frac{e^{-a_0\xi p}}{p}\{e^{-a_1\xi}b_0(a_0^2b_0^2-b_0^2-a_0^2-1)\Big\}+$$

$$+\frac{e^{-b_1\xi}e^{-b_0\xi p}}{p^2}\{2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-2a_0a_1)a_0+$$

$$+(a_0^2b_0^2-b_0^2-a_0^2-1)\,(a_1-b_2\xi a_0-k_0a_0)\}+\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^2}\times$$

$$\times\{(a_0^2b_0^2-b_0^2-a_0^2-1)\,(a_2\xi b_0+k_0b_0+b_1)-b_0(2a_0a_1b_0^2+$$

$$+2b_0b_1a_0^2-2b_0b_1-2a_0a_1)\}+\frac{e^{-b_1\xi}e^{-b_0\Rightarrow p}}{p^3}\{(2a_0a_1b_0^2-$$

$$2b_0b_1a_0^2-2b_0b_1-2a_0a_1)\,(a_1-a_0b_3\xi)+(a_0^2b_0^2-b_0^2-$$

$$-a_0^2-1)\,(a_2-k_0a_1+b_3\xi k_0a_0-b_3\xi a_1-b_3\xi a_0)\}+$$

$$+\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^3}\{(a_0^2b_0^2-b_0^2-a_0^2-1)\,(a_3\xi v_0-a_2\xi k_0b_0+$$

$$+a_2\xi v_1-k_0b_1+b_2)+(2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-2a_0a_1)\times$$

$$\times(a_2\xi b_0-b_1)\}$$

Similarly for the second case, the expressions for the displacement and the temperature distribution are found to be

$$\bar{u}(\xi,p)=\frac{\Theta_0'\varepsilon}{(b_0^2-a_0^2)}\left\{\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^2}-\frac{e^{-b_1\xi}e^{-b_0\xi p}}{p^2}-\right. \tag{3.18}$$

$$\left.-\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^3}(a_2\xi+k_0')-\frac{e^{-b_1\xi}e^{-b_0\xi p}}{p^3}(b_2\xi-k_0')\right\}.$$

$$\bar{\Theta}(\xi,p)=\Theta_0'\left[\frac{e^{-b_1\xi}e^{-b_0\xi p}}{p}\,\frac{(b_0^2-1)}{b_0(b_0^2-a_0^2)}-\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p}\,\frac{(a_0^2-1)}{a_0(b_0^2-a_0^2)}\right. \tag{3.19}$$

$$+\frac{e^{-b_1\xi}e^{b_0\xi p}}{p^2}\left\{\frac{2b_0b_1-k_0'(b_0^2-1)}{b_0(b_0^2-a_0^2)}-\frac{b_2\xi(b_0^2-1)}{b_0(b_0^2-a_0^2)}\right\}+$$

$$+\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^2}\left\{a_2\xi\,\frac{(a_0^2-1)}{a_0(b_0^2-a_0^2)}-\frac{2a_0a_1-A_0(a_0^2-1)}{a_0(b_0^2-a_0^2)}\right\}+$$

$$+\frac{e^{-b_1\xi}e^{-b_0\xi p}}{p^3}\left\{\frac{(b_1+2b_0b_2)-k_0b_0b_1+(k_0'^2-k_1'+k_1'^2)\,(b_0^2-1)}{b_0(b_0^2-a_0^2)}-\right.$$

$$\left.-b_2\xi\,\frac{2b_0b_1-k_0'(b_0^2-1)}{b_0(b_0^2-a_0^2)}-\frac{b_3\xi(b_0^2-1)}{b_0(b_0^2-a_0^2)}\right\}+\frac{e^{-a_1\xi}e^{-a_0\xi p}}{p^3}\times$$

$$\times\left\{a_3\xi\,\frac{(a_0^2-1)}{a_0(b_0^2-a_0^2)}+a_2\xi\,\frac{2a_0a_1-A_0'(a_0^2-1)}{a_0(b_0^2-a_0^2)}-\right.$$

$$\left.-\frac{(a_0^2+2a_0a_2)-A_0'a_0a_1+(A_0'^2-A_1'+A_1'^2)\,(a_0^2-1)}{a_0(b_0^2-a_0^2)}\right\}\Bigg]$$

where

$$K_0=\frac{(a_0-b_0)\,\{2(a0+b_0)\,(a_1+b_1)-(a_1b_0+b_1a_0)\}+\{(a_0+b_0)^2}{b_0(b_0^2-a_0^2)}-$$

$$-\frac{(a_0b_0-1)\,(a_1-b_1)\}}{b_0(b_0^2-a_0^2)}$$

$$K_0'=\frac{(2b_0b_1-2a_0a_1)b_0+b_1(b_0^2-a_0^2)}{b_0(b_0^2-a_0^2)}$$

$$K_1'=\frac{\{(b_0^2-a_0^2+2b_0b_2+2a_0a_2(b_0+(2b_0b_1-2a_0a_1)b_1\}}{b_0(b_0^2-a_0^2)}$$

$$A_0'=\frac{(2b_0b_1-2a_0a_1)a_0+a_1(b_0^2-a_0^2)}{a_0(b_0^2-a_0^2)}$$

and

$$A_1'=\frac{(b_0^2-a_0^2+2b_0b_2-2a_0a_2)a_0+(2b_0b_1-2a_0a_1)a_1}{a_0(b_0^2-a_0^2)}.$$

Now taking the inverse Laplace transform of the two equations (3.16) and (3.17) we find the final expressions for the displacement and temperature in terms of non-dimensional variables ξ and T for the situation in the first case.

$$u(\xi,T)=\frac{\sigma_0}{\{(a_0+b_0)-(a_0b_0+1)\}(a_0b_0)}\,[e^{-a_1\xi}(T-a_0\xi)\,H\,(\tau- \tag{3.20}$$

$$-a_0\xi)\,\{a_0b_0(b_0^2-1)-a_0b_0(b_0^2-1)a_2\xi(\tau-a0\xi)+$$

$$+(\tau-a_0\xi)\,(b_0^2-1)\,(a_1b_0+b_1a_0)+2a_0b_0^2b_1(\tau-a_0\xi)-$$

$$k_0a_0b_0(b_0^2-1)\,(\tau-a_0\xi)\}-(\tau-b_0\xi)\,e^{-b_1\xi}H(\tau-b_0\xi)\times$$

$$\times\{a_0b_0(a_0^2-1)+a_0b_0(a_0^2-1)b_2\xi(\tau-b_0\xi)+(a_0^2-1)\times$$

$$\times(a_1b_0+b_1a_0)\,(\tau-b_0\xi)+2a_0^2b_0b_1(\tau-b_0\xi)-$$

$$-k_0a_0b_0(a_0^2-1)\,(\tau-b_0\xi)\}]$$

and

$$\Theta(\xi,\tau)=\frac{\sigma_0H(\tau-b_0\xi)}{\varepsilon\{(a_0+b_0)^2-(a_0b_0+1)\}\,(a_0-b_0)}\times \tag{3.21}$$

$$\times[\{a_0(a_0^2b_0^2-b_0^2-a_0^2-1)\}\,\{e^{-b_1\xi}-b_2\xi\,e^{-b_1\xi}(\tau-b_0\xi)-$$

$$-b_3\xi\,e^{-b_1\xi}(\tau-b_0\xi)^2\}+\{a_0(2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-$$

$$-2a_0a_1)-(a_0^2b_0^2-b_0^2-a_0^2-1)\,(a_1-k_0a_0)\}\,\{e^{-b_1\xi}(\tau-b_0\xi)-$$

$$-b_2\xi e^{-b_1\xi}(\tau-b_0)^2\}+\{a_1(2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-$$

$$2a_0a_1)+(a_0^2b_0^2-b_0^2-a_0^2-1)\,(a_2-k_0a_1)\}\,\{e^{-b_1\xi}(\tau-b_0\xi)^2\}]-$$

$$-\frac{\sigma_0H(\tau-a_0\xi)}{(a_0-b_0)\varepsilon\{(a_0+b_0)^2-(a_0b_0+1\}}\,[\{b_0(a_0^2b_0^2-b_0^2-$$

$$-a_0^2-1)\}\,\{e^{-a_1\xi}-a_2\xi\,e^{-a_1\xi}(\tau-a_0\xi)-a_3\xi\,e^{-a_1\xi}(\tau-a_0\xi)^2\}+$$

$$+\{b_0(2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-2a_0a_1)-k_0b_0(a_0^2b_0^2-$$

$$-b_0^2-a_0^2-1)\}+b_1(a_0^2b_0^2-b_0^2-a_0^2-1)\}\,\{e^{-a_1\xi}(\tau-a_0\xi)-$$

$$a_2\xi\,e^{-a_1\xi}(\tau-a_0\xi)^2\}+\{b_1(2a_0a_1b_0^2+2b_0b_1a_0^2-2b_0b_1-$$

$$-2a_0a_1)-k_0b_1(a_0^2b_0^2-b_0^2-a_0^2-1)+b_2(a_0^2b_0^2-b_0^2a_0^2-1)\}\times$$

$$\times\{e^{-a_1\xi}(\tau-a_0\xi)^2\}].$$

Similarly taking the inverse Laplace transform of the two equations (3.18) and (3.19) we find the final expression for the displacement and temperature in terms of the non-dimensional variables ξ and τ for the situation in the second case.

$$u(\xi,\tau)=\frac{\Theta_0'\varepsilon}{(b_0^2-a_0^2)}\,[H\tau-a_0\xi)\,\{e^{-a_1\xi}(\tau-a_0\xi)-a_2\xi\,e^{-a_1\xi}(\tau-a_0\xi)^2-$$

$$-k_0'\,e^{-a_1\xi}(\tau-a_0\xi)^2\}+H(\tau-\beta_0\xi)\,\{b_2\xi\,ee^{-b_1\xi}(\tau-b_0\xi)^2- \tag{3.22}$$

$$e^{-b_1\xi}(\tau-b_0\xi)+K_0'\,e^{-b_1\xi}(\tau-b_0\xi)^2\}]$$

and

$$\Theta(\xi,\tau)=\frac{\Theta_0'H(\tau-b_0\xi)}{b_0(b_0^2-a_0^2)}\,[(b_0^2-1)\,e^{-b_1\xi}-(\tau-b_0\xi)\,(b_0^2-1) \tag{3.23}$$

$$b_2\xi\,e^{-b_1\xi}-(b_0^2-1)b_3\xi\,e^{-b_1\xi}(\tau-b_0\xi)^2+\{2b_0b_1-$$

$$-K_0'(b_0^2-1)\}\,e^{-b_1\xi}(\tau-b_0\xi)-\{2b_0b_1-K_0'(b_0^2-1)\}b_2\xi\times$$

$$\times e^{-b_1\xi}(\tau-b_0\xi)^2+\{(b_0^2+2b_0b_2)-K_0'b_0b_1+(K_0'^2+$$

$$+ K_1'^2 - K_1')(b_0^2-1)\} e^{-b_1\xi}(\tau - b_0\xi)^2] - \frac{\Theta_0' H(\tau - a_0\xi)}{a_0(b_0^2-a_0^2)} \times$$
$$\times [(a_0^2-1) e^{-a_1\xi} - (a_0^2-1) a_2 \xi e^{-a_1\xi}(\tau - a_0\xi) - a_3 \xi e^{-a_1\xi} \times$$
$$\times (a_0^2-1)(\tau - a_0\xi)^2 + \{2a_0a_1 - A_0'(a_0^2-1)\} e^{-a_1\xi}(\tau - a_0\xi) -$$
$$- a_2\xi e^{-a_1\xi}(\tau - a_0\xi)^2\{2a_0a_1 - A_0'(a_0^2-1)\} + \{(a_1^2 + 2a_0a_2) -$$
$$- A_0'a_0a_1 + (A_0'^2 - A_1' + A_1'^2)(a_0^2-1)\} e^{-a_1\xi}(\tau - a_0\xi)^2]$$

where $H(\tau - a_0\xi)$, $H(\tau - b_0\xi)$ are well-known Heaviside unit functions.

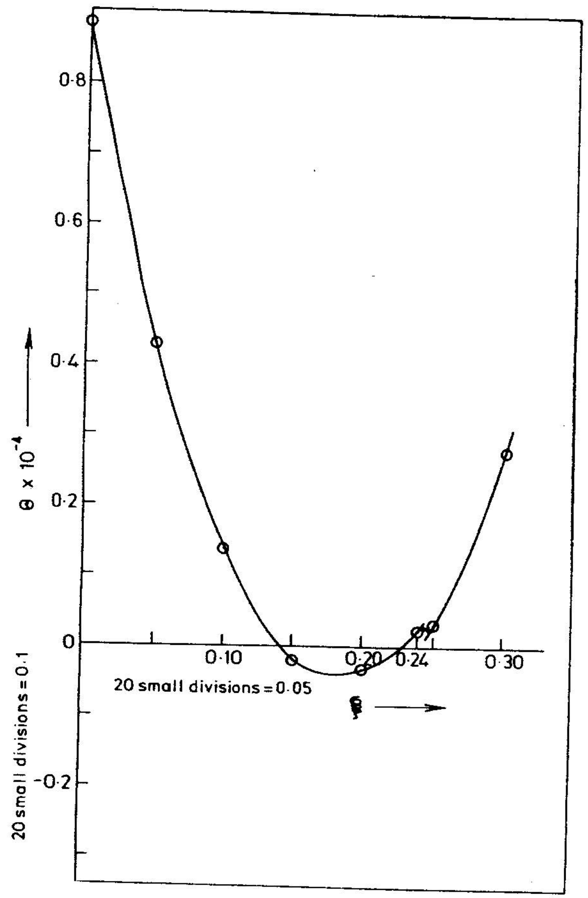

Fig. 1. Dependence of $\Theta(\tau)$ on ξ (see definitions in Rel. (2.5a)) with small divisions specified in the figure.

4. DISCONTINUITIES IN WAVE PROPAGATION

From the results obtained in the last section, we find that the expression for the displacement $u(\xi, \tau)$ and temperature $\Theta(\xi, \tau)$ contain terms involving the Heaviside functions $H(T - a_0\xi)$ and $H(\tau - b_0\xi)$. The probable points of discontinuity are $\xi = (\tau/a_0)\ (\tau/b_0)$. Again since $a_0 > b_0$, it is found that one of these two points of discontinuities moves faster than the other. The jumps in the displacement

$$\{u(\xi, \tau)^+ - u(\xi, \tau)^-\} = [u(\xi, \tau)]$$

and temperature

$$\{\Theta(\xi, \tau)^+ - \Theta(\xi, \tau)^-\} = [\Theta(\xi, \tau)]$$

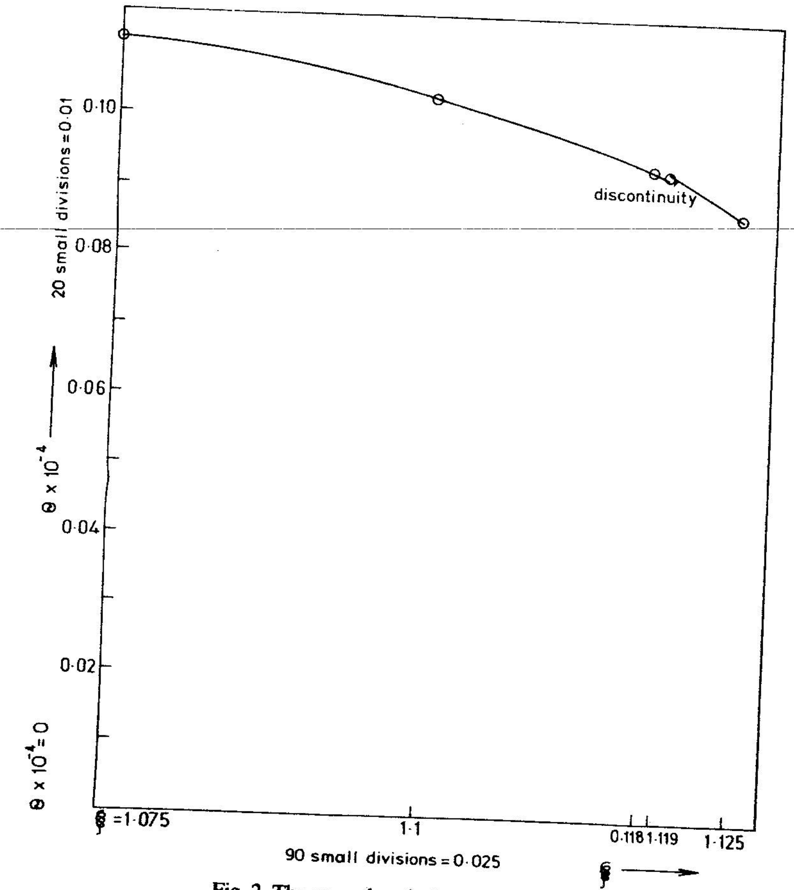

Fig. 2. The same description as in Fig. 1.

at these two points have been determined remembering that $a_0 > b_0$ vide, equation (3.15). Here $\{u(\xi, \tau), \Theta(\xi, \tau)\}^+$ and $\{u(\xi, \tau), \Theta(\xi, \tau)\}^-$ indicate the values of the displacement and temperature to the left and right side of the point. The jumps for the displacement are found to be zero at each of these two points, which indicate that no discontinuities exist in deformation.

Unlike deformation, the expressions for temperature were found to be discontinuous at each of these two points. In the first case

$$[\Theta(\xi, \tau)]_{\xi=\tau/a_0} = \frac{\sigma_0 b_0(a_0^2 b_0^2 - b_0^2 - a_0^2 - 1)\, e^{-a_1 \xi}}{\varepsilon\{(a_0+b_0)^2 - (a_0 b_0 + 1)\}(a_0 - b_0)}$$

$$[\Theta(\xi, \tau)]_{\xi=\tau/b_0} = \frac{\sigma_0 a_0(a_0^2 b_0^2 - b_0^2 - a_0^2 - 1)\, e^{-b_1 \xi}}{\varepsilon\{(a_0+b_0)^2 - (a_0 b_0 + 1)\}(a_0 - b_0)}. \tag{4.1}$$

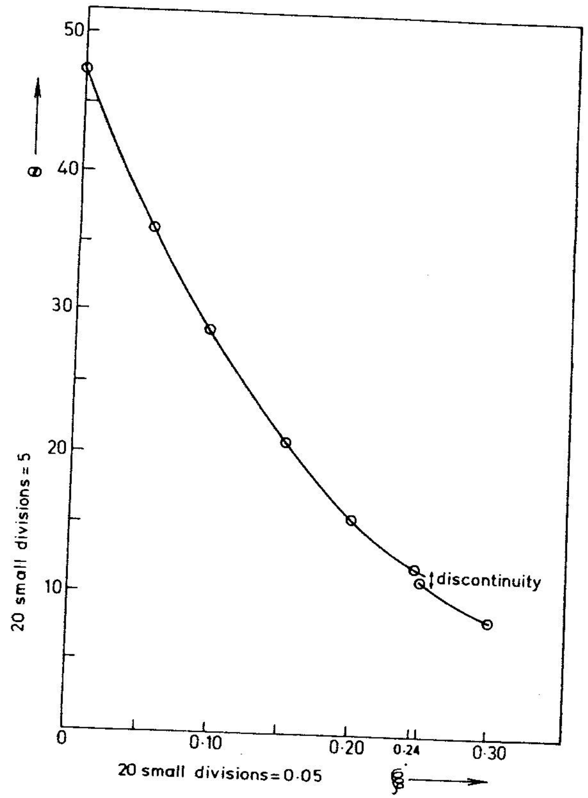

Fig. 3. The same description as in Fig. 1.

In the second case

$$[\Theta(\xi, \tau)]_{\xi=\tau/a_0} = \frac{\Theta_0'(a_0^2 - 1)}{a_0(b_0^2 - a_0^2)} e^{-a_1 \xi}$$

$$[\Theta(\xi, \tau)]_{\xi=\tau/b_0} = \frac{\Theta_0'(b_0^2 - 1)}{b_0(b_0^2 - a_0^2)} e^{-b_1 \xi}. \tag{4.2}$$

Since the expressions for the stress $\sigma(\xi, \tau)$ contain the temperature term, the two above mentioned points of discontinuity will appear in them also.

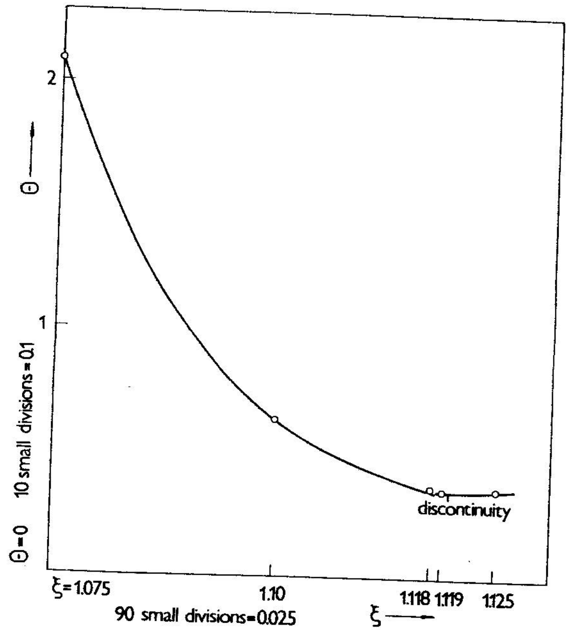

Fig. 4. The same description as in Fig. 1.

5. NUMERICAL CALCULATIONS

We have already seen in the last section that the deformation field is free of the points of discontinuity and only the temperature field contains sich discontinuities at the two points $\xi = \tau/a_0$ and $\xi = \tau/b_0$. This has been illustrated graphically in the adjoining figures 1 to 4 with the following values for the various material constants, vide, [6]. Non-dimensional time $\tau = 0.25$, thermal relaxation parameter $\tau_1 = 0.05$, $\Theta_0' = 1$, $\varepsilon_1 = 0.0003$.

6. DISCUSSION

Lord and Shulman [1] and other researchers in generalized thermoelasticity theories have shown that since the parabolic Fourier law of heat in the case of the classical thermal coupling is modified to a hyperbolic equation due to the introduction of a thermal relaxation parameter, points of discontinuity appear in the solution of the heat equation, though the solution of the mechanical motion remains free of such discontinuities.

In the present paper it can be seen from equation (3.15) that in the absence of the relaxation time τ_1 the material constants a_0 and b_0 reduce to one and zero, respectively. Substituting these values for a_0 and b_0 in the equation (4.2) we find that no jump in temperature exists at these two points, which agree with the results of the classical thermal coupling.

ACKONWLEDGEMENT

I am grateful to Dr. A. K. Pal of Jadavpur University for his kind help in the preparation of this paper.

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Received January 18th, 1985
Revised version received April 16th, 1986