SPHERICALLY SYMMETRIC CONFORMALLY FLAT STATIC SOLUTIONS IN SCHWINGER'S SCALAR-TENSOR THEORY

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We have obtained exact solutions for the conformally flat spherical symmetric static space-time in the scalar-tensor theory of gravitation proposed by Nordtvedt and Schwinger for vacuum and in the presence of a source-free electromagnetic field. We have also shown that there do not exist spherically symmetric static conformally flat solutions of Nordtvedt-Schwinger field equations representing disordered radiation except for the trivial empty flat space-time of Einstein's theory. In the vacuum case also the solution is only the flat space-time of Einstein's theory.

СФЕРИЧЕСКИ-СИММЕТРИЧНЫЕ И КОМФОРМНО-ПЛОСКИЕ СТАТИЧЕСКИЕ РЕШЕНИЯ В СКАЛЯРНО-ТЕНЗОРНОЙ ТЕОРИИ ШВИНГЕРА

В работе приведены точные решения для комформно-плоского и сферически-симметричного статического пространства-времени в скалярно-тензорной теории гравитации, предложенной Нордтведтом и Швингером для вакуума в присутствии электромагнитного поля без источников. Показано, что не существует сферически-симметричных и комформно-плоских статических решений полевых уравнений Нордтведта-Швингера, представляющих неупорядоченные излучения (удовлетворяющие уравнению состояния $= 3p$), кроме известного пустого плоского пространства-времени эйнштейновской теории гравитации.

I. INTRODUCTION

Within the framework of the general scalar-tensor theory of gravitation proposed by Nordtvedt [1] one can allow the parameter ω to be an arbitrary function of the scalar field Φ. This general class of scalar-tensor theories includes the Jordan [2] and Brans-Dicke [3] theories as special cases. Recently Barker [4] has proposed a special case of this general class of scalar-tensor theories where the

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Newtonian gravitational constant G does not vary with time. Also Schwinger [5] and Milton and Yee [6] have formulated a scalar-tensor theory as a mass-varying theory but it can be put in the form of a standard scalar-tensor theory with a suitable choice of the function $\omega(\Phi)$ and after a transformation to "*particle units*" has been carried out.

Reddy and Rao [7] have obtained spherically symmetric static conformally flat exact solutions in the Brans-Dicke theory in the presence of an electromagnetic field. Duttachoudhury and Bhattacharya [8] have shown that Birkhoff's theorem holds for Nordtvedt's general scalar-tensor theory both in vacuum and in the presence of electromagnetic fields when the scalar field is time-independent while Banerjee and Duttachaudhury [9] discussed static gravitational and Maxwell fields in the general scalar-tensor theory with ω given in the Barker form [4]. Recently the present authors [10] have discussed axisymmetric stationary gravitational and Maxwell fields in the scalar-tensor theory with a general $\omega(\Phi)$ for Nordtvedt's class and also for the form of $\omega(\Phi)$ given by Barker [4] and Schwinger [5]. Rao and Reddy [11] have considered conformally flat static spherically symmetric solutions in Barker's form of the general scalar-tensor theory. This paper presents some exact static spherically symmetric conformally flat solutions of the Nordtvedt-Schwinger scalar-tensor theory of gravitation.

II. GENERAL SCALAR-TENSOR THEORY AND FIELD EQUATIONS

The Nordtvedt field equations [1] are

$$R_{ij}-\frac{1}{2}g_{ij}R=-8\pi\Phi^{-1}T_{ij}-\omega\Phi^{-2}\left(\Phi_{,i}\Phi_{,j}-\frac{1}{2}g_{ij}\Phi_{,k}\Phi^{,k}\right)-$$
$$-\Phi^{-1}(\Phi_{;ij}-g_{ij}\Box\Phi) \tag{1}$$

$$(2\omega+3)\Box\Phi=8\pi T-\frac{d\omega}{d\Phi}(\Phi_{,k}\Phi^{,k}) \tag{2}$$

where the function $\omega(\Phi)$ is an arbitrary (positive definite) function of the scalar field Φ, T_{ij} is the stress energy tensor of the matter, $T=T^k_k$, and comma and semicolon denote partial and covariant derivatives, respectively. Special choice for $\omega(\Phi)$ yield the theories of Jordan [2], Brans-Dicke [3], Barker [4] and Schwinger [5].

We consider the static spherically symmetric conformally flat metric in the form

$$ds^2=e^{\alpha}(-dr^2-r^2\,d\Theta^2-r^2\sin^2\Theta\,d\Phi^2+dt^2) \tag{3}$$

where α is the function of r alone. We assume that the scalar field Φ also has spherical symmetry.

Taking Φ as a function of r only and using (3) in (1) and (2), the explicit field equations for the general scalar-tensor theory can be written as

$$8\pi\Phi^{-1}e^{\alpha}T^1_1=-\frac{3}{4}\alpha'^2-\frac{2\alpha'}{r}+\frac{\omega}{2}\Phi^{-2}\Phi'^2-\Phi^{-1}\left(\frac{3}{2}\alpha'\Phi'+\frac{2\Phi'}{r}\right) \tag{4}$$

$$8\pi\Phi^{-1}e^{\alpha}T^2_2=8\pi\Phi^{-1}e^{\alpha}T^3_3=-\alpha''-\frac{\alpha'^2}{4}-\frac{\alpha'}{r}-$$
$$-\frac{\omega}{2}\Phi^{-2}\Phi'^2-\Phi^{-1}\left(\Phi''+\frac{\Phi'\alpha'}{2}+\frac{\Phi'}{r}\right) \tag{5}$$

$$8\pi\Phi^{-1}e^{\alpha}T^4_4=-\alpha''-\frac{\alpha'^2}{4}-\frac{2\alpha'}{r}-\frac{\omega}{2}\Phi^{-2}\Phi'^2-$$
$$-\Phi^{-1}\left(\Phi''+\frac{\Phi'\alpha'}{2}+\frac{2\Phi'}{r}\right) \tag{6}$$

$$(2\omega+3)\left[\Phi''+\Phi'\left(\alpha'+\frac{2}{r}\right)\right]=-\frac{d\omega}{d\Phi}\Phi'^2=8\pi T\,e^{\alpha}.$$

Here a prime indicates differentiation with respect to r.

III. SOLUTION OF THE FIELD EQUATIONS

We now discuss the static solutions of the field equations (4)—(7) in the special case proposed by Schwinger [5] with ω in the form

$$2\omega+3=\frac{1}{\lambda\Phi},\quad \lambda \text{ is constant.} \tag{8}$$

As the field equations are highly nonlinear, solutions in some special case are presented. The question of overdeterminacy is settled by satisfing all the field equations by the actual substitution of the solution obtained.

III.1 Vacuum solution

From (5) and (6) on subtraction we can easily obtain

$$\Phi=k_1\,e^{-\alpha} \tag{9}$$

where k_1 is a constant of integration. Putting this value of Φ into (6) and using (8) we get

$$\alpha'=0,\quad \text{i.e.}\quad \alpha=\text{constant}.$$

Thus the solution of equations (4)—(7) is

$$\alpha=\text{const.},\quad \Phi=\text{const.} \tag{10}$$

Hence the only spherically symmetric static conformally flat vacuum solution of the Nordtvedt-Schwinger scalar-tensor theory is simply the empty flat space-time of Einstein's theory.

III.2 Electrovac solution

Here we consider the energy momentum tensor for a trace-free electromagnetic field in the form

$$T_{ij} = F_{il}F^l_j - \frac{1}{4} g_{ij}F_{lm}F^{lm} \tag{11}$$

where F_{ij} is the electromagnetic field tensor satisfying

$$F^{ij}_{;j} = 0 \tag{12}$$

and

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0. \tag{13}$$

For a static charged particle the only nonzero component of the electromagnetic field F_{ij} is F_{14}. The equations (12) and (13) now lead to $F_{14} = q/r^2$, where q is a constant which can be identified with the electric charge of the particle.

With the metric (3) the nonvanishing components of the energy-momentum tensor are

$$-T^1_1 = T^2_2 = T^3_3 = -T^4_4 = -\frac{1}{2}\frac{q^2}{r^4} e^{-2\alpha}. \tag{14}$$

Using (8) and (14) the field equations (4)—(7) reduce to the Nordtvedt--Schwinger-Maxwell field equations for a static charged point mass which admit the exact solution

$$e^{\alpha} = \left[C_2 + \frac{C_3}{r^2}\right]\Phi^{-1} \tag{15}$$

$$1/\Phi = (\lambda/4)\left[\frac{C_1}{(-C_2C_3)^{1/2}} \tanh^{-1}\left\{\frac{r(-C_2C_3)^{1/2}}{C_3}\right\} + \Phi_0\right]^2$$

where Φ_0, C_1 and C_2 are constants of integration and C_3 is set equal to $4\pi q^2$. Also C_2 and C_3 are non-zero. The solution (15) satisfies each of the Nordtvedt--Schwinger-Maxwell field equations provided the constants C_1, C_2 and C_3 are related by

$$C_1^2 + 12\, C_2C_3 = 0. \tag{16}$$

It can be seen that equation (16) with $C_1^2 > 0$ and $C_3 = 4\pi q^2 > 0$, implies $C_2 < 0$. Therefore $(-C_2C_3)^{1/2}$ in (15) is a real quantity. Thus equations (3) and (15) along with (16) constitute an exact static spherically symmetric conformally flat solution of the Nordtvedt-Maxwell equations in the special case proposed by Schwinger [5]. Physically, the solution (15) may be interpreted as describing the field of a charged particle at the origin surrounded by a scalar field in a conformally flat space-time. This solution is asymptotically flat after some adjustment of constants are made. Also eliminating C_3 from (15) and (16) and then putting $C_1 = 0$, (15) reduces to the flat space-time of Einstein's theory.

III.3 Disordered radiation

Here we consider the energy momentum tensor due to that of a perfect fluid distribution in the form

$$T_{ij} = (\varrho + p)u_iu_j - pg_{ij} \tag{17}$$

with equation of state

$$\varrho = 3p \tag{18}$$

where ϱ is energy density and p is the pressure of the fluid. From (3), (17) and (18) the components of T^i_j in comoving coordinates are

$$T^i_j = \mathrm{diag}\,(-p, -p, -p, 3p). \tag{19}$$

The conservation equation $T^i_{j;i} = 0$ leads to

$$\frac{dp}{dr} + (\varrho + p)\frac{\alpha'}{2} = 0. \tag{20}$$

Using (8), (17), (18) and (19) in the field equations (4)—(7) one gets the Nordtvedt-Schwinger field equations with disordered radiation.

Now using (20) in the difference of equations (5) and (6) and using (19) in the sum of equations (4) and (6) one easily gets $p = 0$ and so $\varrho = 0$ also. This leads to the vacuum field equations in which case, as shown in Section III.1, the only solution is the flat space-time of Einstein's theory.

Thus we have shown that there are no spherically symmetric conformally flat solutions of the Nordtvedt-Schwinger field equations representing disordered radiation.

IV. CONCLUSION

In order to understand fully the scalar-tensor theories of gravitation it is useful to have a knowledge of some exact solutions of these equations. The search for an analytic solution is important due to the fact that once such a solution is obtained one can study all of its physical properties. Exact static spherical symmetric conformally flat solutions in vacuum, in the presence of an electromagnetic field and for disordered radiation, are considered in the Nordtvedt-Schwinger scalar-tensor theory of gravitation. It is observed that unlike the Brans-Dicke theory, the only spherically symmetric conformally flat solution in the Nordtvedt--Schwinger theory is the flat space-time of Einsten's theory. The electrovac solution (15) in the Nordtvedt-Schwinger theory represents the field of a charged particle in a conformally flat space-time and may be useful in the study of interaction of electromagnetic and scalar-tensor gravitational fields. Also it may be noted that there are no static spherically symmetric conformally flat solutions for a disordered radiation in the Nordtvedt-Schwinger scalar-tensor theory.

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