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PROPAGATION OF THERMO-ELASTIC WAVES IN A HALF-SPACE WITH THERMAL RELAXATION

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The generalized dynamical theory of thermo-elasticity has been used to solve the problem of axisymmetric deformation of a half-space under a point load of temperature, step wise in time. The displacement potentials have been introduced and the Laplace-Hankel transform followed by the Cagniard-De Hoop technique has been made use of. The expressions for temperature, dilatation and displacements are obtained in integral forms. Wave geometry for temperature, dilatation and displacements in the half-space have been shown. It is seen that temperature and dilatation consists of conical and hemi-spherical wave-fronts and displacements are also of a similar type.

РАСПРОСТРАНЕНИЕ ТЕРМОУПРУГИХ ВОЛН В ПОЛУПРОСТРАНСТВЕ С ТЕПЛОВОЙ РЕЛАКСАЦИЕЙ

В работе для решения проблемы осесимметричного деформирования полупространства относительно точек термонагрузки использована обобщенная динамическая теория термоупругости. Введены потенциалы смещения и использовано преобразование Лапласа-Ханкеля с последующим применением метода Каньяра-Де Хуна. В интегральной форме получены выражения для температуры, растяжения и смещений. Продемонстрирована геометрия волн температуры, растяжения и смещений в полупространстве. Обнаружено, что температура и растяжение состоит из конических и полусферических волновых фронтов и смещения имеют также аналогичный характер.

I. INTRODUCTION

A. H. Nayfeh and S. Nemat-Nasser [1] studied the transient behaviour of a thermo-elastic wave in a solid half-space on the basis of the generalized dynamical theory of thermo-elasticity proposed by Lord and Shulman [2]. They used the Laplace–Fourier transform and the Cagniard–De Hoop technique to solve the problem of a two-dimensional thermo-elastic disturbance in which an instantaneous heat source is applied.

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In the present work the problem of an axisymmetric thermo-elastic disturbance in a solid half-space is considered by using the same theory. The stressfree boundary surface of the half-space is subjected to a point load of temperature, stepwise in time. The Laplace–Hankel transform followed by the Cagniard–De Hoop technique are used assuming the same relation between integral transform parameters as that of earlier workers [1].

The short time solutions are obtained in integral forms. Finally, wave geometry is used to show that thermal and dilatational waves consist of two types of wave-fronts, conical and hemispherical. The displacement are also of the same type.

II. NOTATIONS USED

τ_{rr}, τ_{rz}, $\tau_{\Theta\Theta}$ = stress components, T = temperature, e_{rr}, e_{rz}, $e_{\Theta\Theta}$ = strain components, T_0 = initial temperature, e = dilatation, ϱ = density, $\nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2}$, c_v = specific heat and constant deformation, $\beta = \alpha_t(3\lambda - 2\mu)$, α_t = coefficient of linear expansion, λ, μ = Lame's constant, τ_0 = thermal relaxation, τ = dimensionless thermal relaxation, ε = coupling parameter, k = coefficient of thermal conductivity, $\gamma^2 = \frac{\lambda + 2\mu}{\mu}$, u_r, u_z = displacement components.

III. STATEMENT OF THE PROBLEM AND BASIC EQUATIONS

We consider an axisymmetric thermo-elastic disturbance in a homogeneous, isotropic thermo-elastic medium occupying a half-space $z > 0$, the origin of the z-axis being taken on the boundary of the half-space and the positive direction of the z-axis into the medium. The motion is caused by a point load of temperature applied at the origin. The thermo-elastic half-space is initially undisturbed and stressfree.

The displacements, strains and stresses referred to the cylindrical coordinates (r, Θ, z) are

$$u_r = u_r(r, z), \quad u_\Theta = 0, \quad u_z = u_z(r, z)$$

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\Theta\Theta} = \frac{u_r}{r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \tag{1}$$

$$\tau_{rr} = 2\mu e_{rr} + \lambda e - \beta\tau$$

$$\tau_{zz} = 2\mu e_{zz} + \lambda e - \beta\tau$$

$$\tau_{\Theta\Theta} = 2\mu e_{\Theta\Theta} + \lambda e - \beta\tau$$

$$\tau_{rz} = \mu e_{rz}, \quad \tau_{r\Theta} = \tau_{\Theta z} = 0. \tag{2}$$

The equations of motion in the absence of body forces and the modified heat conduction equation with thermal relaxation are

$$\mu\left(\nabla^2 - \frac{1}{r^2}\right) u_r + (\lambda + \mu)\frac{\partial e}{\partial r} = \beta\frac{\partial \tau}{\partial r} + \varrho \ddot{u}_r, \tag{3a}$$

$$\mu\nabla^2 u_z + (\lambda + \mu)\frac{\partial e}{\partial z} = \beta\frac{\partial \tau}{\partial z} + \varrho \ddot{u}_z, \tag{3b}$$

$$(\varrho c_v \ddot{\tau} + \beta\tau_0 \ddot{e})\tau_0 + (\varrho c_v \dot{\tau} + \beta\tau_0 \dot{e}) = k\nabla^2\tau. \tag{4}$$

To obtain dimensionless equations we use the notation

$$\omega^* = \frac{\varrho c_v c^2}{k}, \quad c_1^2 = \frac{\lambda + 2\mu}{\varrho}, \quad \varepsilon = \frac{\beta^2 \tau_0}{\gamma^2 \mu \varrho c_v}, \quad \tau = \tau_0 \omega^*$$

and introduce $\frac{1}{\omega^*}$, $\frac{c_1}{\omega^*}$, $\frac{\varrho c_v c_1}{\beta\omega^*}$, $\tau_0 \frac{\varrho c_v c_1}{\beta}$ and $\frac{\varrho c_v \mu}{\beta}$ as the units of time, length, displacement, temperature, velocity and stress by Nayfeh and Nemat-Nasser [1].

The dimensionless equatins from (2), (3) and (4) are

$$\dot{\tau} + \tau\ddot{\tau} - \nabla^2\tau + \dot{e} + \tau\ddot{e} = 0 \tag{5}$$

$$\gamma^2 \tau_{zz} = \gamma^2 \frac{\partial u_z}{\partial z} + (\gamma^2 - 2)\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) - \gamma^2 \varepsilon\tau \tag{6a}$$

$$\gamma^2 \tau_{rr} = \gamma^2 \frac{\partial u_r}{\partial r} + 2(\gamma^2 - 2)\frac{u_r}{r} - \gamma^2 \varepsilon\tau \tag{6b}$$

$$\gamma^2 \tau_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \tag{6c}$$

$$\gamma^2 \tau_{\Theta\Theta} = (\gamma^2 - 2)\frac{\partial u_r}{\partial r} + 2(\gamma^2 - 1)\frac{u_r}{r} - \gamma^2 \varepsilon\tau \tag{6d}$$

$$\gamma^2 \ddot{u}_r = (\gamma^2 - 1)\frac{\partial e}{\partial r} + \left(\nabla^2 - \frac{1}{r^2}\right) u_r - \gamma^2 \varepsilon \frac{\partial \tau}{\partial r} \tag{7a}$$

$$\gamma^2 \ddot{u}_z = (\gamma^2 - 1)\frac{\partial e}{\partial z} + \nabla^2 u_z - \gamma^2 \varepsilon \frac{\partial \tau}{\partial z}. \tag{7b}$$

The boundary conditions for the problem are

$$\tau_{zz} = \tau_{rz} = 0 \quad \text{on } z=0,\ r \geqq 0 \tag{8a, 8b}$$

$$\tau = \frac{\tau_1 H(t)\,\delta(r)}{2\pi r}, \quad t>0 \quad \text{on } z=0 \tag{8c}$$

where τ_1 = constant.

The initial conditions are

$$u_z = u_r = \tau = 0, \quad z>0, \quad r>0 \quad \text{when } t=0. \tag{8d}$$

The ususal regularity conditions on the displacements and temperature require that u_r, u_z, τ tend to zero as both r and z tend to infinity.

IV. DISPLACEMENT POTENTIALS AND THE LAPLACE–HANKEL TRANSFORM

We introduce the displacement potentials

$$u_r = \partial\Phi/\partial r - \partial\Psi/\partial z \tag{9a}$$

$$u_z = \partial\Phi/\partial z + (1/r)\partial/\partial r\,(r\Psi). \tag{9b}$$

Substituting in (7a) and (7b) we get

$$\tau = \frac{1}{\varepsilon}[\nabla^2\Phi - \ddot{\Phi}] \tag{10a}$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)\Psi - \gamma^2\ddot{\Psi} = 0. \tag{10b}$$

From (9a) and (9b), (10) and (5) we get a coupled differential equation

$$\varepsilon(\tau\nabla^2\ddot{\Phi} + \nabla^2\dot{\Phi}) = \nabla^4\Phi - \tau\nabla^2\ddot{\Phi} - \nabla^2\ddot{\Phi} + \tau\dddot{\Phi} + \ddot{\Phi} - \nabla^2\dot{\Phi}) = 0. \tag{11}$$

From (9a), (9b) and (6) the stresses are

$$\gamma^2\tau_{zz} = \gamma^2\nabla^2\Phi - 2\left[\left\{\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\right\} - \left\{\frac{\partial^2\Psi}{\partial r\partial z} + \frac{1}{r}\frac{\partial\Psi}{\partial z} - \gamma^2\varepsilon\tau\right\}\right] \tag{12a}$$

$$\gamma^2\tau_{rz} = 2\frac{\partial^2\Phi}{\partial r\partial z} + \left(\nabla^2 - \frac{1}{r^2}\right)\Psi - 2\frac{\partial^2\Psi}{\partial z^2}. \tag{12b}$$

The dilatation

$$e = \nabla^2\Phi. \tag{13}$$

As usual the Laplace–Hankel transform is used with the parameter p and ξ, respectively in equation (9a)—(13) subject to an initial condition (8d); we use $J_0(\xi r)$ and $J_1(\xi r)$ for Φ, τ_{zz}, u_z, τ and Ψ, τ_{rz}, u_r, respectively.

$$\bar{u}'_r = -\xi\bar{\Phi}' - \frac{\partial\bar{\Psi}'}{\partial z} \tag{14a}$$

$$\bar{u}'_z = \frac{\partial\bar{\Phi}'}{\partial z} + \xi\bar{\Psi}' \tag{14b}$$

$$\bar{\tau}' = \frac{1}{\varepsilon}[(D^2-\xi^2)\bar{\Phi}' - p^2\bar{\Phi}'] \tag{15a}$$

$$\bar{e}' = (D^2-\xi^2)\bar{\Phi}' \tag{15b}$$

$$(D^2-\xi^2)\bar{\Psi}' - \gamma^2p^2\bar{\Psi}' = 0 \tag{16a}$$

$$\varepsilon(\tau p^2+p)(D^2-\xi^2)\bar{\Phi}' = [(D^2-\xi^2) - \tau p^2 - p]\times[(D^2-\xi^2) - p^2]\bar{\Phi}' \tag{16b}$$

$$\gamma^2\bar{\tau}'_{zz} = \gamma^2(D^2-\xi^2)\bar{\Phi}' + 2\left[\xi^2\bar{\Phi}' + \xi\frac{\partial\bar{\Psi}'}{\partial z}\right] - \gamma^2\varepsilon\bar{\tau}' \tag{17a}$$

$$\gamma^2\bar{\tau}'_{rz} = -\left[2\xi\frac{\partial\bar{\Phi}'}{\partial z} + (D^2+\xi^2)\bar{\Psi}'\right]. \tag{17b}$$

The boundary conditions (8a), (8b) and (8c) are transformed into

$$\bar{\tau}'_{zz} = \bar{\tau}'_{rz} = 0 \quad \text{on } z=0 \tag{18a}$$

$$\tau' = \frac{\tau_1}{p} \quad \text{on } z=0 \tag{18b}$$

V. SOLUTIONS

We assume the solutions of (16a) and (16b) in the form

$$\bar{\Psi}' = A_1 e^{-m_1 z} \tag{19a}$$

$$\bar{\Phi}' = A_2 e^{-m_2 z} + A_3 e^{-m_3 z}. \tag{19b}$$

Since $z\to\infty$, $\bar{\Phi}'$, $\bar{\Psi}'$ are finite, where A_1, A_2 and A_3 are independent of z but may be functions of ξ and p.

$$m_1^2 = \xi^2 + \gamma^2p^2 \tag{20a}$$

$$m_2^2 = \xi^2 + p^2 - \frac{p^2(1+\tau p)}{1+p(\tau-1)}\varepsilon \tag{20b}$$

$$m_3^2 = \xi^2 + \tau p^2 + p + \frac{p(1+\tau p)^2}{1+p(\tau-1)}\varepsilon. \tag{20c}$$

Using (19a), (19b) we get from (14a), (14b), (15a), (15b), (17a) and (17b)

$$\bar{u}'_r = A_1 m_1 \, e^{-m_1 z} - \xi [A_2 \, e^{-m_2 z} + A_3 \, e^{-m_3 z}] \quad (21a)$$

$$\bar{u}'_z = \xi A_1 \, e^{-m_1 z} - m_2 A_2 \, e^{-m_2 z} - m_3 A_3 \, e^{-m_3 z} \quad (21b)$$

$$\bar{\tau}' = \frac{1}{\varepsilon} [(m_2^2 - \xi^2 - p^2) A_2 \, e^{-m_2 z} + (m_3^2 - \xi^2 - p^2) A_3 \, e^{-m_3 z}] \quad (22a)$$

$$\bar{e}' = (m_2^2 - \xi^2) A_2 \, e^{-m_2 z} + (m_3^2 - \xi^2) A_3 \, e^{-m_3 z} \quad (22b)$$

$$\gamma^2 \bar{\tau}'_{zz} = -2\xi m_1 A_1 \, e^{-m_1 z} + [\gamma^2 (m_2^2 - \xi^2) + 2\xi^2] A_2 \, e^{-m_2 z} + [\gamma^2 (m_3^2 - \xi^2) + 2\xi^2] A_3 \, e^{-m_3 z} - \gamma^2 \varepsilon \bar{\tau}' \quad (23a)$$

$$\gamma^2 \bar{\tau}'_{rz} = -(m_1^2 + \xi^2) A_1 \, e^{-m_1 z} + 2\xi [m_2 A_2 \, e^{-m_2 z} + m_3 A_3 \, e^{-m_3 z}]. \quad (23b)$$

Using the boundary conditions (18a) and (18b) in equation (21)—(23), we obtain A_1, A_2 and A_3 as

$$A_1 = \frac{2\tau_1 \varepsilon \xi}{\Delta} [m_3 (\gamma^2 p^2 + 2\xi^2) - m_2 (\gamma^2 p^2 + 2\xi^2)] \quad (24a)$$

$$A_2 = \frac{\tau_1 \varepsilon}{\Delta} [4\xi^2 m_1 m_3 - (m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2)] \quad (24b)$$

$$A_3 = \frac{-\tau_1 \varepsilon}{\Delta} [4\xi^2 m_1 m_2 - (m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2)] \quad (24c)$$

$$|\Delta| = p (m_3 - m_2) [(m_3 + m_2)(2\xi^2 + \gamma^2 p^2)^2 - 4\xi^2 m_1 (m_2 m_3 + \xi^2 + p^2)] \quad (25)$$

$$[\bar{u}'_r, \bar{u}'_z, \bar{e}', \bar{\tau}'] = \sum_{k=1}^{3} [a_{1k}, a_{2k}, a_{3k}, a_{4k}] \, e^{-m_k z} \quad (26a, b, c, d)$$

$$a_{11} = 2\xi m_1 [(\gamma^2 p^2 + 2\xi^2)(m_3 - m_2)] \frac{\tau_1 \varepsilon}{\Delta} \quad (27a)$$

$$a_{12} = [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_3] \frac{\xi \tau_1 \varepsilon}{\Delta} \quad (27b)$$

$$a_{13} = [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_2] \frac{\xi \tau_1 \varepsilon}{\Delta} \quad (27c)$$

$$a_{21} = 2\xi^2 [(m_3 - m_2)(2\xi^2 + \gamma^2 p^2) \frac{\tau_1 \varepsilon}{\Delta} \quad (28a)$$

$$a_{22} = [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_3] \frac{m_2 \tau_1 \varepsilon}{\Delta} \quad (28b)$$

$$a_{23} = -[(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_2] \frac{m_3 \tau_1 \varepsilon}{\Delta} \quad (28c)$$

$$a_{31} = 0 \quad (29a)$$

$$a_{32} = -(m_2^2 - \xi^2) [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_3] \frac{\tau_1 \varepsilon}{\Delta} \quad (29b)$$

$$a_{33} = (m_3^2 - \xi^2) [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_2] \frac{\tau_1 \varepsilon}{\Delta} \quad (29c)$$

$$a_{41} = 0 \quad (30a)$$

$$a_{42} = -(m_2^2 - \xi^2 - p^2) [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_3] \frac{\tau_1}{\Delta} \quad (30b)$$

$$a_{43} = (m_3^2 - \xi^2 - p^2) [(m_1^2 + \xi^2)(2\xi^2 + \gamma^2 p^2) - 4\xi^2 m_1 m_2] \frac{\tau_1}{\Delta}. \quad (30c)$$

The stress components may also be found in a similar manner.

We shall now obtain the short time solution for the displacement, temperature, etc., using the Cagniard–De Hoop technique; for this we expand m_1, m_2 and m_3.

Short time expansion: To expand m_1, m_2 and m_3 for large values of p, we assume a relation between the Hankel parameter ξ and the Laplace parameter p and put $\xi = p\eta$ as in [1]. From (20a), (20b) and (20c) we get

$$m_1^2 = p^2 s_1^{*2} \quad (31a)$$

$$m_2^2 = p^2 s_2^{*2} - \frac{p^2 \varepsilon}{\tau - 1} \left[\tau - \frac{1}{p(\tau - 1)} \right] \quad (31b)$$

$$m_3^2 = p^2 s_3^{*2} + p + \frac{\tau p^2 \varepsilon}{\tau - 1} \left[\tau + \frac{\tau - 2}{p(\tau - 1)} \right] \quad (31c)$$

$$s_k^* = (\eta^2 - s k^2)^{1/2}, \quad k = 1, 2, 3. \quad s_1 = \gamma, \quad s_2 = 1, \quad s_3 = \tau^{1/2}$$

$$m_1 = p s_1^* \quad (32a)$$

$$m_2 = p s_2^* - \frac{\varepsilon p}{2(\tau - 1) s_2^*} \left[\tau - \frac{1}{p(\tau - 1)} \right] \quad (32b)$$

$$m_3 = p s_3^* + \frac{1}{2 s_3^*} + \frac{\tau \varepsilon p}{2(\tau - 1) s_3^*} \left[\tau + \frac{\tau - 2}{p(\tau - 1)} \right]. \quad (32c)$$

Equations (32) are same as equations (29) in Nemat-Nasser [1], using (31) and (32) in (27)—(30) and then in (26) we get

$$[\bar{u}'_r, \bar{u}'_z, \bar{e}', \bar{\tau}'] = \sum_{k=1}^{3} \left[\frac{A_{1k}}{p^2} + \frac{B_{1k}}{p^3}, \frac{A_{2k}}{p^2} + \frac{B_{2k}}{p^3}, \frac{A_{3k}}{p} + \frac{B_{3k}}{p^2}, \frac{A_{4k}}{p} + \frac{B_{4k}}{p^2} \right] e^{-m_k z} \quad (33a, b, c, d)$$

$$A_{11}=\frac{2\tau_1 s_1^*\eta(\gamma^2+2\eta^2)}{D^*(s_3^*+s_2^*)}\,\varepsilon,\quad B_{11}=\frac{s_1^*\eta(\gamma^2+2\eta^2)\tau_1}{(\tau-1)s_3^*D^*}\,\varepsilon \tag{34a, b}$$

$$A_{12}=\frac{\tau_1 M^*\eta}{(\tau-1)D^*}\,\varepsilon,\quad B_{12}=\frac{2\tau_1 s_1^*\eta^3}{(\tau-1)s_3^*D^*}\,\varepsilon \tag{34c, d}$$

$$A_{13}=\frac{\tau_1\eta}{\tau-1}\,\varepsilon,\quad B_{13}=0 \tag{34e, f}$$

$$A_{21}=\frac{2\tau_1\eta^2(\gamma^2+2\eta^2)}{D^*(s_3^*+s_2^*)}\,\varepsilon,\quad B_{21}=\frac{\tau_1\eta^2(\gamma^2+2\eta^2)}{(\tau-1)D^*s_3^*}\,\varepsilon \tag{35a, b}$$

$$A_{22}=\frac{\tau_1 M^* s_2^*}{(\tau-1)D^*}\,\varepsilon,\quad B_{22}=\frac{2\tau_1\eta^2 s_1^* s_2^*}{D^*(\tau-1)s_3^*}\,\varepsilon \tag{35c, d}$$

$$A_{23}=\frac{-\tau_1 s_3^*}{\tau-1}\,\varepsilon,\quad B_{23}=\frac{-\tau_1}{2s_3^*(\tau-1)}\,\varepsilon \tag{35e, f}$$

$$A_{31}=0,\quad B_{31}=0 \tag{36a, b}$$

$$A_{32}=\frac{\tau_1 M^*}{(\tau-1)D^*}\,\varepsilon,\quad B_{32}=\frac{\tau_1}{s_3^*D^*(\tau-1)}\,\varepsilon \tag{36c, d}$$

$$A_{33}=\frac{\tau_1\tau}{\tau-1}\,\varepsilon,\quad B_{33}=\tau_1\left[\frac{1}{\tau}+\frac{\tau-2}{\tau(\tau-1)}\right]\frac{\tau\varepsilon}{\tau-1} \tag{36e, f}$$

$$A_{41}=0,\quad B_{41}=0 \tag{37a, b}$$

$$A_{42}=\frac{\tau_1\tau M^*}{D^*(\tau-1)}\,\varepsilon,\quad B_{42}=\left[\frac{2\eta^2 s_1^*}{s_3^*M^*}-\frac{1}{\tau(\tau-1)}\right]A_{42} \tag{37c, d}$$

$$A_{43}=\frac{\tau_1[1+\tau\{2\eta^2 s_1^*(\tau-1)+\tau s_2^* D^*\}\varepsilon]}{(\tau-1)^2 s_2^* D^*}, \tag{37e}$$

$$B_{43}=\frac{2\eta^2 s_1^*(\tau-1)^2\varepsilon - s_2^*D^*\{\tau\varepsilon(\tau-2)-(\tau-1)^3\}}{(\tau-1)^3 s_2^* D^*} \tag{37f}$$

$$\Delta=p^7D^*(s_3^{*2}-s_2^{*2}) \tag{38}$$

$$D^*=(2\eta^2+\gamma^2)^2-4\eta^2 s_1^* s_2^* \tag{39a}$$

$$M^*=4\eta^2 s_1^* s_3-(\gamma^2+2\eta^2)^2. \tag{39b}$$

We now make use of the result

$$J_0(r\xi)=\frac{1}{2\pi}\int_0^{2\pi} e^{ir\xi\sin x}\,dx \tag{40a}$$

and then take

$$\eta=(q^2+\omega^2)^{1/2} \tag{40b}$$

$$\sin x=\frac{q}{(q^2+\omega^2)^{1/2}}. \tag{40c}$$

Taking the Hankel–Laplace inversion we get from (33a), (33b), (33c), (33d)

$$[e,\tau]=\mathscr{L}^{-1}\left[\frac{1}{\pi}\left\{\int_0^{\infty}\int_{-\infty}^{+\infty} p\sum_{k=2}^{3}\left\{\left(A_{3k}+\frac{B_{3k}}{p}\right),\right.\right.\right. \tag{41a}$$

$$\left.\left.\left.\left(A_{4k}+\frac{B_{4k}}{p}\right)\right\} e^{-p(s_k^*z-iqr)}\,dq\,d\omega\right] \tag{41b}$$

$$[u_r,u_z]=\mathscr{L}^{-1}\left[\frac{1}{\pi}\left\{\int_0^{\infty}\int_{-\infty}^{+\infty}\sum_{k=1}^{3}\left\{\left(A_{1k}+\frac{B_{1k}}{p}\right),\right.\right.\right. \tag{42a}$$

$$\left.\left.\left.\left(A_{2k}+\frac{B_{2k}}{p}\right)\right\} e^{-p(s_kz-iqr)}\,dq\,d\omega\right]. \tag{42b}$$

We assume $\tau\neq 1$, the integrands of (41a), (41b) and (42a) and (42b) have no singularities except at the zeros of D^*. We take $\tau=3$. The singularities of the integrands in (41) and (42) are the branch points in the q-plane at

$$q_1=\pm i(\omega^2+\gamma^2)^{1/2} \tag{43a}$$

$$q_2=\pm i(\omega^2+1)^{1/2} \tag{43b}$$

$$q_3=\pm i(\omega^2+\tau)^{1/2} \tag{43c}$$

and the simple poles at $q_R=\pm i(\omega^2+\gamma_R^2)^{1/2}$.

The poles correspond to the zeros of the Rayleigh function D^* where $\gamma_R=c_1/c_R$. c_R is the Rayleigh surface wave speed. The positive roots of these singularities lie in the upper half of the q-plane.

The Cagniard–De Hoop technique:

The integrands of (41a), (41b) have branch points

$$q_k=\pm i(\omega^2+s_k^2)^{1/2},\qquad k=2,3.$$

The necessary condition for the convergence of these integrals are that s_k^*, $k=2,3$, be single-valued functions with positive real parts on the path of integration. If we introduce a branch cut along the imaginary q-axis from each branch point to infinity, the condition on s_k^*, $k=2,3$ holds everywhere in this plane.

Again the integrands of (41a), (41b) have poles where $D^*=0$, the contour $q=q_k^{\pm}$ is found by solving

$$t=s_k^*z-iqr,\qquad k=2,3 \tag{44}$$

$$q_k^{\pm} = \pm \left\{\frac{t^2}{\varrho^2} - (\omega^2 + s_k^2)\right\}^{1/2} \cos \Phi - i\frac{t}{\varrho} \sin \Phi \tag{45}$$

$$k = 2, 3 \qquad t^2/\varrho^2 > (\omega^2 + s_k^2).$$

Each of the integrands (41a), (41b) tends to zero exponentially as q tends to infinity. The contribution from the arc of q_k at infinity vanishes. Moreover as t varies from $\varrho(\omega^2 + s_k^2)^{1/2}$ to infinity, q_k traces a hyperbola with a vertex at $q = i(\omega^2 + s_k^2)^{1/2} \sin \Phi$. The integration contour has two possible configurations in the q-plane depending on the values of Φ and ω.

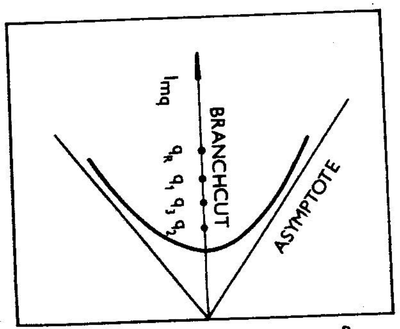

Fig. 1.

Case I: when $(\omega^2 + \tau)^{1/2} \sin \Phi < (\omega^2 + 1)^{1/2}$

$$\Phi < \Phi_{c1} \quad \text{when} \quad 0 < \omega < \infty$$

$$\text{or} \quad \Phi > \Phi_{c1} \quad \text{when} \quad \omega_{c1} < \omega < \infty$$

where

$$\Phi_{c1} = \sin^{-1} \frac{1}{\tau^{1/2}} \tag{46a}$$

$$\omega_{c1}' = (\tau \sin^2 \Phi - 1)^{1/2} \sec \Phi. \tag{46b}$$

The integration contour is simply given by q_k^{+} in Fig. 2a. In this case the vertex of $q_k^{\pm}$ does not lie on the branch cuts.

Case II: when $(\omega^2 + \pi)^{1/2} \sin \Phi > (\omega^2 + 1)^{1/2}$

$$\Phi > \Phi_{c1} \quad \text{when} \quad 0 < \omega < \omega_{c1}$$

the vertex of $q_3^{\prime}$ lies on the branch cut between the branch points at q_2 and q_3, Fig. 2b, and it is given by q_3^{+} for $t/\varrho > (\omega^2 + s_3^2)^{1/2}$ plus

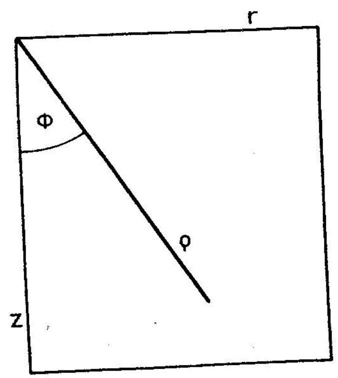

Fig. 2a.

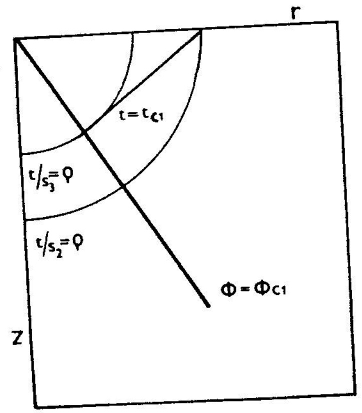

Fig. 2b.

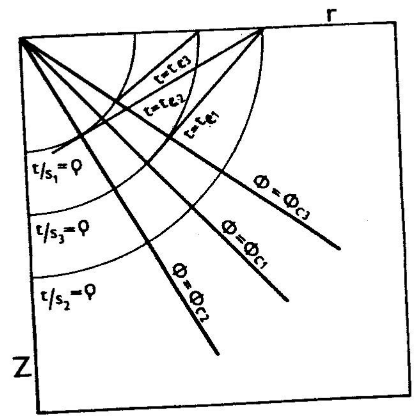

Fig. 2c.

$$q_3^{+} = -i\left[\left\{(\omega^2 + s_3^2) - \frac{t^2}{\varrho^2}\right\}^{1/2} \cos \Phi\right] + i\frac{t}{\varrho} \sin \Phi \pm \gamma_1, \tag{47a}$$

with ranges of t given by $\gamma_1 \to 0$

$$t_{32}^{(1)} = \varrho[(\omega^2 + 1)^{1/2} \sin \Phi + (s_3^2 - s_2^2)^{1/2} \cos \Phi] \leqq \tag{47b}$$

$$\leqq t \leqq \varrho(\omega^2 + s_3^2)^{1/2} = t_{32}^{(2)}.$$

From (41a), (41b) we get

$$[e, \tau] = \frac{1}{\pi} \mathscr{L}^{-1} \left[\sum_{k=2}^{3} \left\{\int_0^{\infty} \int_{\varrho q_k}^{\infty} p \operatorname{Re}\left[A_{3k} + \frac{B_{3k}}{p}, A_{4k} + \frac{B_{4k}}{p}\right] \times \right.\right. \tag{48a}$$

$$\times e^{-pt}\frac{dq_k}{dt}\,dt\,d\omega\Big\} + H(\Phi-\Phi_{c1})\int_0^{\omega_{c1}}\int_{t_{32}^{(1)}}^{t_{32}^{(2)}} p\,\mathrm{Re}\Big[A_{33}+\frac{B_{33}}{p},$$

$$A_{43}+\frac{B_{43}}{p}\Big]\,e^{-pt}\frac{dq_3^+}{dt}\,dt\,d\omega +$$

$$+H(\Phi-\Phi_{c1})\int_0^{\omega_{c1}}\int_{t_{32}^{(2)}}^{\infty} p\,\mathrm{Re}\Big[A_{33}+\frac{B_{33}}{p},\ A_{43}+\frac{B_{43}}{p}\Big]\times \tag{48b}$$

$$\times e^{-pt}\frac{dq_3^+}{dt}\,dt\,d\omega\Big].$$

The first two terms in (48a), (48b) are the contribution from q_k^+, $k=2, 3$, when the vertex of the hyperbola (45) lies below the branch point q_2. The last two terms are the contribution from q_3^+ when the vertex of (45) lies on the branch cut between the branch points q_2 and q_3. Equations (48a), (48b) are valid only for $0\leqq\Phi<\pi/2$, Fig. 1, which corresponds to the interior of the half-space.

Interchanging the order of integration and taking the Laplace inversion of (48a), (48b) we get

$$[e,\tau]=2\sum_{k=2}^{3}\Big[H(t-\varrho s_k)\int_0^{(t^2/\varrho-s_k^2)^{1/2}}\mathrm{Re}\Big[A_{3k}q_k(t,\omega)\frac{d^2q_k}{dt^2}+ \tag{49a, b}$$

$$+B_{3k}q_k(t,\omega)\frac{dq_k}{dt},\ A_{4k}q_k(t,\omega)\frac{d^2q_k}{dt^2}+B_{4k}q_k(t,\omega)\frac{dq_k}{dt}\Big]d\omega\Big]+$$

$$+H_{c1}\int_{A_{c1}}^{\tau_{c1}}\mathrm{Re}\Big[A_{33}q_3^+(t,\omega)\frac{d^2q_3^+}{dt^2}+B_{33}q_3^+(t,\omega)\frac{dq_3^+}{dt},$$

$$A_{43}q_3^+(t,\omega)\frac{d^2q_3^+}{dt^2}+B_{43}q_3^+(t,\omega)\frac{dq_3^+}{dt}\Big]d\omega\Big]$$

where

$$A_{c1}=\Big(\frac{t^2}{\varrho^2}-1\Big)^{1/2}$$

$$t_{c1}=\varrho\{\sin\Phi+(\tau-1)^{1/2}\cos\Phi\}$$

$$t'_{c1}=\varrho(\tau-1)^{1/2}\sec\Phi$$

$$\tau_{c1}=\Big[\Big\{\frac{t}{\varrho}-(\tau-1)^{1/2}\cos\Phi\Big\}^2\operatorname{cosec}^2\Phi-1\Big]^{1/2}$$

$$H_{c1}=H(\Phi-\Phi_{c1})\,H(t-t_{c1})\,H(t'_{c1}-t).$$

The first two terms in (49a), (49b) represent the dilatational and thermal wave motion behind the hemi-spherical wave-front at t/s_2 and t/s_3, Fig. 2c.

The third terms in (49a), (49b) represent the conical wavefront at $t=t_c$, for $\Phi>\Phi_{c1}$. This conical wave-front is the surface of a truncated cone given by $t=t_{c1}$ for $\Phi>\Phi_{c1}$.

The third term in (49a), (49b) alone contributes in front of the surface $t=t'_{c1}$ for $\Phi>\Phi_{c1}$ which is the equation of sphere $t=t'_{c1}$, $\Phi>\Phi_{c1}$.

VI. DISPLACEMENTS

To find out the displacements u_r, u_z we shall invert (42a), (42b) as in the case of ε and τ. The inversion is very difficult due to the nonvanishing term containing s_1^* in the exponential which introduces an extra branch point $i(\omega^2+\gamma^2)^{1/2}$ in the q-plane. The relative position of $i(\omega^2+\gamma^2)^{1/2}$ w.r.t. $i(\omega^2+1)^{1/2}$ and $i(\omega^2+\tau)^{1/2}$ depends on the values of τ and γ^2. As we have assumed $\tau=3$ we take $\gamma^2=4$ for the distinct positions of $i(\omega^2+\gamma^2)^{1/2}$ and $i(\omega^2+\tau)^{1/2}$. For $k=1, 2, 3$, the integration path of q_k^+ has the following possibilities in the q-plane depending on Φ and ω.

Case I

when (a) $(\omega^2+\tau)^{1/2}\sin\Phi<(\omega^2+1)^{1/2}$

i.e. $\Phi<\Phi_{c1}$ when $0<\omega<\infty$

or $\Phi>\Phi_{c1}$ when $\omega_{c1}<\omega<\infty$

and (b) $(\omega^2+\gamma^2)^{1/2}\sin\Phi<(\omega^1+1)^{1/2}$

$\Phi<\Phi_{c2}$ when $0<\omega<\infty$

or $\Phi>\Phi_{c2}$ when $\omega_{c2}<\omega<\infty$

$$\Phi_{c2}=\sin^{-1}\frac{1}{\gamma} \tag{50a}$$

$$\omega_{c2}=(\gamma^2\sin^2\Phi-1)^{1/2}\sec\Phi. \tag{50b}$$

The integration contour is simply given by q_k^+ in Fig. 3a. In this case the vertex of $q_k^\pm$ does not lie on the branch cuts.

Case II

when (a) $(\omega^2+\gamma^2)^{1/2}\sin\Phi>(\omega^2+1)^{1/2}$

$\Phi>\Phi_{c1}$ when $0<\omega<\omega_{c1}$

and (b) $(\omega^2+\gamma^2)^{1/2}\sin\Phi<(\omega^2+\tau)^{1/2}$

$\Phi<\Phi_{c3}$ when $0<\omega<\infty$

or $\Phi>\Phi_{c3}$ when $\omega_{c3}<\omega<\infty$

$$\Phi_{C3}=\sin^{-1}\frac{\tau^{1/2}}{\gamma} \tag{51a}$$

$$\omega_{c3} = (\gamma^2 \sin^2 \Phi - \tau)^{1/2} \sec \Phi. \tag{51b}$$

The vertex of q_1^+ and q_3^+ lies on the branch cut between the branch points q_2 and q_3, Fig. 3b.

$$q_3^+ = \mathrm{i}\left[\frac{t}{\varrho} \sin \Phi - \left\{(\omega^2 + s_3^2) - \frac{t^2}{\varrho^2}\right\} \cos \Phi\right] \pm \gamma_2 \ldots \tag{52}$$

Case III

when (a) $(\omega^2 + \tau)^{1/2} \sin \Phi > (\omega^1 + 1)^{1/2}$

$$\Phi > \Phi_{c2} \quad \text{when} \quad 0 < \omega < \omega_{c2}$$

and (b) $(\omega^2 + \gamma^2)^{1/2} \sin \Phi > (\omega^2 + \tau)^{1/2}$

$$\Phi > \Phi_{c3} \quad \text{when} \quad 0 < \omega < \omega_{c3}.$$

The vertex of q_3^+ lies on the branch cut between the branch points q_3 and q_1, Fig. 3c.

$$q_1^+ = \mathrm{i}\left[\frac{t}{\varrho} \sin \Phi - \left\{(\omega^2 + s_1^2) - \frac{t^2}{\varrho^2}\right\}^{1/2} \cos \Phi\right] \pm \gamma_3 \pm \gamma_2 \tag{53}$$

$$\gamma_2 \to 0, \quad \gamma_3 \to 0$$

$$t_{13}^{(1)} = \varrho\,[q_3 \sin \Phi + (s_1^2 - s_3^2)^{1/2} \cos \Phi] \leqslant t \leqslant \varrho q_1 = t_{13}^{(2)}$$

$$t_{12}^{(1)} = \varrho\,[q_2 \sin \Phi + (s_1^2 - s_2^2)^{1/2} \cos \Phi] \leqslant t \leqslant \varrho q_1 = t_{12}^{(2)}$$

$$[u_r, u_z] = \frac{1}{\pi} \mathscr{L}^{-1}\left[\sum_{k=1}^{3}\left[\int_0^\infty \int_{\varrho q_k}^{\infty} \mathrm{Re}\left\{A_{1k} + \frac{B_{1k}}{p}, A_{2k} + \frac{B_{2k}}{p}\right\} \times \right.\right.$$
$$\left. \times \mathrm{e}^{-pt} \frac{\mathrm{d}q_k}{\mathrm{d}t} \cdot \mathrm{d}t\, \mathrm{d}\omega\right] + H(\Phi - \Phi_{c1}) \int_0^\infty \int_{t_{32}^{(1)}}^{t_{32}^{(2)}} \mathrm{Re}\left\{A_{13} + \frac{B_{13}}{p}\right.$$
$$\left. A_{23} + \frac{B_{23}}{p}\right\} \mathrm{e}^{-pt} \frac{\mathrm{d}q_3^+}{\mathrm{d}t}\, \mathrm{d}t\, \mathrm{d}\omega + H(\Phi - \Phi_{c1}) \int_0^\infty \int_{t_{32}^{(2)}}^{\infty} \mathrm{Re}\left\{A_{13} + \frac{B_{13}}{p},\right.$$
$$\left. A_{23} + \frac{B_{23}}{p}\right\} \mathrm{e}^{-pt} \frac{\mathrm{d}q_3^+}{\mathrm{d}t}\, \mathrm{d}t\, \mathrm{d}\omega + \sum_{k=2,3}\left[H(\Phi - \Phi_{ck}) \times \right.$$
$$\left. \times \int_0^\infty \int_{t_{1k}^{(1)}}^{t_{1k}^{(2)}} \mathrm{Re}\left\{A_{11} + \frac{B_{11}}{p}, A_{21} + \frac{B_{21}}{p}\right\} \mathrm{e}^{-pt} \frac{\mathrm{d}q_1}{\mathrm{d}t}\, \mathrm{d}t\, \mathrm{d}\omega\right] +$$
$$+ \sum_{k=2,3}\left[H(\Phi - \Phi_{ck}) \int_0^\infty \int_{t_{1k}^{(2)}}^{\infty} \mathrm{Re}\left\{A_{11} + \frac{B_{11}}{p}, A_{21} + \frac{B_{21}}{p}\right\} \times \right.$$
$$\left. \times \mathrm{e}^{-pt} \frac{\mathrm{d}q_1}{\mathrm{d}t}\, \mathrm{d}t\, \mathrm{d}\omega\right]. \tag{54a), (54b}$$

The first three terms in (54a), (54b) are the contributions from q_k^+, $k = 1, 2, 3$ when the vertex of the hyperbola lies below the branch point q_2. The fourth and

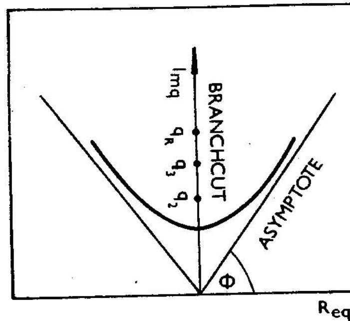

Fig. 3a.

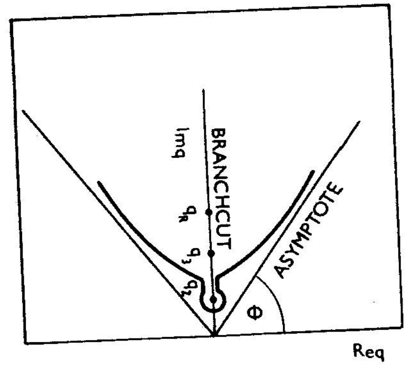

Fig. 3b.

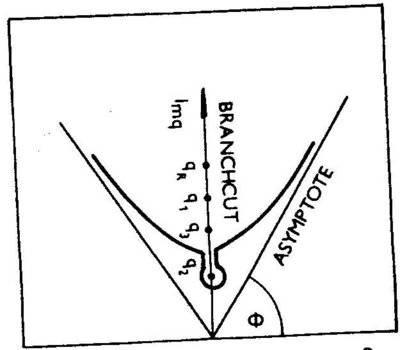

Fig. 3c.

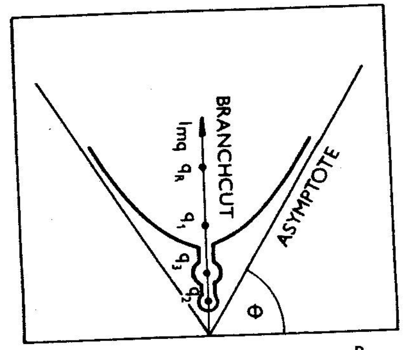

Fig. 3d.

fifth terms in (54a), (54b) are the contribution from q_3^+ when the vertex of (45) lies on the branch cut between the branch points q_2 and q_3. The last four terms in (54a), (54b) are the contributions from q_1^+ when the vertex lies on the branch cut between the branch points q_1, q_3. Equations (54a), (54b) are valid for $0 \leqslant \Phi < \pi/2$, which corresponds to the interior of the half-space. As $\Phi \to \pi/2$ the contour collapses on the imaginary q-axis, Fig. 1.

$$[u_r, u_z] = 2 \sum_{k=1}^{3}\left[\int_0^{(t^2 \varrho^{-2} - s_k^2)^{1/2}} \left[\mathrm{Re}\,\{A_{1k} q_k(t, \omega), A_{2k} q_k(t, \omega)\} \frac{\mathrm{d}q_k}{\mathrm{d}t} \times \right.\right.$$
$$\times H(t - \varrho s_k)\}\, \mathrm{d}t + \int_0^t \mathrm{Re}\,\{B_{1k} q_k(\bar{t}, \omega), B_{2k} q_k(\bar{t}, \omega)\} \times$$
$$\left.\left. \times \frac{\mathrm{d}q_k}{\mathrm{d}t} H(\bar{t} - \varrho s_k)\, \mathrm{d}\bar{t}\right] \mathrm{d}\omega\right] + \int_{A_{c1}}^{\tau_{c1}} \left[\mathrm{Re}\,\{A_{13} q_3^+(t, \omega),\right.$$

$$A_{23}q_3^+(t,\omega))\}\frac{dq_3^+}{dt}H_{c1}\,dt+\int_0^t \mathrm{Re}\,\{B_{13}q_3^+(\bar t,\omega),$$

$$\cdot\,B_{23}q_3^+(\bar t,\omega))\}\frac{dq_3^+}{d\bar t}\bar H_{c1}\,d\bar t\Big]\,d\omega+\sum_{k=2,3}\int_{A_{ck}}^{\tau_{ck}}\mathrm{Re}\,\{A_{11}q_1^+(t,\omega),$$

$$A_{21}q_1^+(t,\omega))\}\frac{dq_1^+}{dt}H_{ck}\,dt+\int_0^t \mathrm{Re}\,\{B_{11}q_1^+(\bar t,\omega),$$

$$B_{21}q_1^+(\bar t,\omega))\}\frac{dq_1^+}{d\bar t}H_{ck}\,d\bar t\Big|\,d\omega\ldots \qquad (55a, 55b)$$

$$A_{c2}=\left(\frac{t^2}{\varrho^2}-1\right)^{1/2}$$

$$t_{c2}=\varrho[\sin\Phi+(\gamma^2-1)^{1/2}\cos\Phi]$$

$$t'_{c2}=\varrho(\gamma^2-1)^{1/2}\sec\Phi$$

$$\tau_{c2}=\left[\left\{\frac{t}{\varrho}-(\gamma^2-1)^{1/2}\cos\Phi\right\}^2\operatorname{cosec}^2\Phi-1\right]^{1/2}$$

$$A_{c3}=\left(\frac{t^2}{\varrho^2}-\tau\right)^{1/2}$$

$$t_{c3}=\varrho\{\tau^{1/2}\sin\Phi+(\gamma^2-\tau)^{1/2}\cos\Phi\}$$

$$t'_{c3}=\varrho\{(\gamma^2-\tau)^{1/2}\sec\Phi\}$$

$$\tau_{c3}=\left[\left\{\frac{t}{\varrho}-(\gamma^2-\tau)^{1/2}\cos\Phi\right\}^2\operatorname{cosec}^2\Phi-\tau\right]^{1/2}.$$

$$H_{c1}=H(\Phi-\Phi_{c1})\,H(t-t_{c1})\,H(t'_{c1}-t)$$

$$\bar H_{c1}=H(\Phi-\Phi_{c1})\,H(\bar t-t_{c1})\,H(t'_{c1}-\bar t)$$

$$H_{ck}=H(\Phi-\Phi_{ck})\,H(t-t_{ck})\,H(t'_{ck}-t)$$

$$\bar H_{ck}=H(\Phi-\Phi_{ck})\,H(\bar t-t_{ck})\,H(t'_{ck}-\bar t),\qquad k=2,3.$$

The first three terms in equation (55a), (55b) represent the equivoluminal motion behind the hemi-spherical wave-fronts at t, $t/\tau^{1/2}$, t/γ, Fig. 3d. The fourth and fifth terms in (55a), (55b) represent the equivoluminal motion behind the conical wave-front at $t=t_{c1}$, for $\Phi>\Phi_{c1}$. This conical wave-front is the surface of a truncated cone given by $t=t_{c1}$ for $\Phi>\Phi_{c1}$. These terms also contribute to the head of the surfaces $t=t'_{c1}$, $\Phi>\Phi_{c1}$ which is the equation of a sphere.

Lastly, the remaining terms in (55a), (55b) represent the equivoluminal motion behind the conical wave-fronts at $t=t_{c2}$ for $\Phi>\Phi_{c2}$ and $t=t_{c3}$ for $\Phi>\Phi_{c3}$. These terms also contribute to the surface at $t=t'_{c2}$ for $\Phi>\Phi_{c2}$ and $t=t'_{c3}$ for $\Phi>\Phi_{c3}$ which represent the spherical surfaces.

VII. DISCUSSION

The analysis presented in the paper is applicable to materials conducting heat in which the classical theory of heat conduction represented by the parabolic type equation is replaced by a generalized law of heat conduction of the hyperbolic type. The observation of the second sound (thermal wave) in solid helium and sodium fluoride has led to the prediction of the second sound almost in all solids [1].

The plane-strain problem for the half-space was worked out by Nemat–Nasser who also carried out the numerical calculation of the problem. The corresponding axisymmetric half-space problem has been solved in the present work. The solutions for temperature, dilatation and displacements valid for short time have been obtained in integral forms and these can be calculated numerically for different values of Φ and taking the material parameters such as $\varepsilon=0.05$, $\tau=3$, $\gamma^2=4\ldots$ as in [1].

1. Temperature and dilatation

The first of two terms in (49) represent the dilatational motion behind the longitudinal wave-front for $k=2$ and the temperature motion behind the thermal wave-front for $k=3$. For a fixed value of ϱ, $t=s_2\varrho$ and $t=s_3\varrho$ indicate the time of arrival of the longitudinal and the thermal wave-fronts, respectively.

The last term in each of the expressions for the dilatation and temperature in (49) are the contributions of the conical wave-front (head wave-front). This is represented in the rz-plane, Fig. 2c, by a straight line. When ϱ is constant, $t=\varrho[\sin\Phi+2^{1/2}\cos\Phi]$ indicates the time of arrival of the conical wave-front.

In particular, when $\tau=0$, $\varepsilon=0$ the dilatation motion ceases and the temperature reduces to

$$\tau=\tau_1 H_{c1}\int_{A_{c1}}^{\tau_{c1}}\mathrm{Re}\left[\frac{q_3^+(t,\omega)}{D^*s_2^*}\frac{d^2q_3^+}{dt^2}+q_3^+(t,\omega)\frac{dq_3^+}{dt}\right]d\omega.$$

2. Displacements

The first three terms in (55) represent the shear motion for $k=1$, dilatational motion for $k=2$ and temperature motion for $k=3$. For a fixed value of ϱ, $t=s_1\varrho$, $t=s_2\varrho$, $t=s_3\varrho$ indicate the time of arrival of shear, longitudinal and thermal wave-fronts, respectively and $\varrho[\sin\Phi+2^{1/2}\cos\Phi]$, $\varrho[\sin\Phi+3^{1/2}\cos\Phi]$ and $\varrho[3^{1/2}\sin\Phi+\cos\Phi]$, represent the times of arrival of different conical wave-fronts. These conical wave-fronts are represented by straight lines in Fig. 3d.

REFERENCES

[1] Nayfeh, A. H., Nemat–Nasser, S.: J. of Appl. Math. and Phys. *23* (1972), 50.
[2] Lord, W. H., Shulman, Y. A.: J. Math. Solid *15* (1967), 299.
[3] Ewing, W. B., Jardetzky, W. S., Press, F.: *Elastic Waves in Layered Media.* McGraw Hill. New York 1957.

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