acta phys. slov. 33 (1983), No. 4

ON THE PROPAGATION OF A THERMO-ELASTIC PLANE WAVE IN A THIN INFINITE PLATE WITH THERMAL RELAXATION

S. G. MONDAL[1]), R. N. JANA[2]), Bengai

The period equation corresponding to a thermo-elastic plane wave in an infinite thermo-elastic plate with thermal relaxation has been obtained when the stressfree plane faces of the plate have been thermally insulated. This equation can be split up into symmetrical parts which in their turn can be expanded when the plate is very thin. In particular, the phase velocity-dispersion relation is obtained for such a plate in the case of a symmetrical vibration. Moreover, the phase velocity has been shown graphically and also the group velocity has been obtained in a tabular form.

О РАСПРОСТРАНЕНИИ ТЕРМОУПРУГОЙ ПЛОСКОЙ ВОЛНЫ В ТОНКОЙ БЕСКОНЕЧНОЙ ПЛАСТИНКЕ С ТЕПЛОВОЙ РЕЛАКСАЦИЕЙ

В работе получено периодическое уравнение, соответствующее температурной плоской волне в бесконечной термоупругой пластинке с термической релаксацией для случая, когда плоские грани пластинки, свободные от напряжений, термически изолированы. Это уравнение может быть разложено на симметрическую и антисимметрическую части, которые в их экстремальных точках могут быть разложены при условии, что пластинка очень тонкая. В частности, получено дисперсионное соотношение для фазовой скорости для пластинки, подвергающейся симметрическим вибрациям. Кроме того, приводится график фазовой скорости, а в форме таблицы также представлена групповая скорость.

I. INTRODUCTION

The propagation of body and surface waves in the elastic medium was extensively studied by Rayleigh, Lamb, Love and others on the basis of the classical theory of elasticity. By using the same theory Stoneley [1] predicted the existence of a wave which can propagate along the interface of two dissimilar elastic media in welded contact, which is known as the Stoneley wave. In [1] the propagation of waves in the infinite plate of a finite thickness in vacuum was studied from the standpoint of the classical theory of elasticity.

[1]) A. K. P. C. MAHAVIDYALAYA, P. O. BENGAI, Dt. Hooghly, West Bengal, India.
[2]) Department of Mathematics, Kalyani University, West Bengal, India.

In this work we wish to investigate the propagation of a thermo-elastic wave in an infinite plate in vacuum on the basis of the generalized dynamical theory of thermo-elasticity proposed by Lord and Shulman [2].

The period equation corresponding to the elastic plane wave in an infinite thermo-elastic plate with stressfree plane faces which are thermally insulated has been derived and separated in symmetric and antisymmetric parts. In particular, when the plate is very thin the phase velocity-dispersion relation is obtained for symmetric vibration and is shown graphically. The group velocities are also found out for some special values of the wave number and are shown in a tabular form.

II. NOTATIONS USED

τ_{ij} = stress components in Cartesian form; ϱ = constant mass density; u_i = displacement components; h_i = components of heat flux; c_v = specific heat at constant deformation; T_0 = initial temperature of the solid at which it is stressfree; T = temperature; $T_{,i}$ = temperature-gradient; α_t = coefficient of linear expansion; λ, μ = Lame's constant; $\gamma=\alpha_t(3\lambda+2\mu)$; $\beta^2=\dfrac{\lambda+2\mu}{\mu}$; e = dilatation; $e_{,i}$ = dilatation-gradient; $\nabla^2=\partial^2/\partial z^2+\partial^2/\partial y^2+\partial^2/\partial z^2$; τ_0 = thermal relaxation in time; τ = dimensionless thermal relaxation; K = coefficient of thermal conductivity; δ_{ij} = Kronecker delta; c = dimensionless apparent phase velocity on the surface; ε = coupling parameter; k = wave number.

III. BASIC EQUATIONS AND FORMULATION OF THE PROBLEM

We consider a plane thermo-elastic wave motion in a homogeneous, isotropic thermo-elastic infinite plate of thickness $2H$. The faces of the plate are bounded by planes the $x_3=\pm H$, which are stressfree and thermally insulated.

We introduce the orthogonal Cartesian frame of reference $0x_1x_2x_3$ and take the origin in the middle plane of the infinite plate which coincides with the plane $0x_1x_2$.

The basic equations of thermo-elasticity are

$$\tau_{ij,j}=\varrho\ddot{u}_i \tag{1}$$

$$-h_{i,i}=\varrho c_v\dot{T}+\gamma T_0\dot{e} \tag{2}$$

$$\tau_{ij}=\lambda e\delta_{ij}+\mu(u_{i,j}+u_{j,i})-\gamma T\delta_{ij} \tag{3}$$

$$\tau_0\dot{h}_i+h_i=-KT_{,i} \tag{4}$$

$$i,j=1,2,3.$$

To obtain the dimensionless equations we use the notations $\omega^*=\varrho c_v c_1^2/K$, $c_1^2=\lambda+2\mu/\varrho$, $\varepsilon=\gamma^2T_0/\beta^2\mu\varrho c_v$, $\tau=\tau_0\omega^*$ and introduce $1/\omega^*$, c_1/ω^*, $\varrho c_v c_1/\gamma\omega^*$,

T_0, $\varrho c_v c_1/\gamma$, $\varrho c_v\mu/\gamma$ as the unit of time, length, displacement, temperature, velocity and stress by Nayfeh and Nemat–Nasser [3].

Eliminating τ_{ij} from (1) and (3) and h_i from (2) and (4) we get

$$\beta^2\ddot{u}_i=(\beta^2-1)e_{,i}+\nabla^2u_i-\beta^2\varepsilon T_{,i} \tag{5}$$

$$\dot{T}+\tau\ddot{T}-\nabla^2T+\dot{e}+\tau\ddot{e}=0. \tag{6}$$

We consider the xz plane as the plane of motion of the thermo-elastic wave in the thermo-elastic continuum and take

$$x_1=x,\quad x_2=0,\ x_3=z,$$

$$u_1=u(x,z,t),\quad u_2=v=0,\quad u_3=w(x,z,t).$$

The dimensionless components in the x z coordinate are

$$T_{xz}=\partial u/\partial z+\partial w/\partial x \tag{7a}$$

$$T_{zz}=(\beta^2-2)\partial u/\partial z+\beta^2\partial w/\partial z-\beta^2\varepsilon T. \tag{7b}$$

The plane boundary faces $z=\pm H$ are stressfree and thermally insulated.

$$T_{zz}=T_{xz}=\partial T/\partial z=0. \tag{8}$$

IV. DISPLACEMENT POTENTIALS, PERIOD EQUATIONS OF SYMMETRICAL AND ANTISYMMETRICAL VIBRATIONS

We introduce the displacement potentials $\Phi(x,z,t)$ and $\Psi(x,z,t)$ defined by

$$u=\partial\Phi/\partial x-\partial\Psi/\partial z \tag{9a}$$

$$w=\partial\Phi/\partial z+\partial\Psi/\partial x \tag{9b}$$

and substituting into (5) we get

$$T=\tfrac{1}{\varepsilon}[\nabla^2\Phi-\ddot{\Phi}] \tag{10a}$$

$$\tfrac{1}{\beta^2}\nabla^2\Psi-\ddot{\Psi}=0. \tag{10b}$$

Eliminating T from equation (6) and (10a) a fourth order differential equation for Φ is obtained.

$$\nabla^4\Phi-\left[\left(1+\tau\frac{\partial}{\partial t}\right)(1+\varepsilon)+\frac{\partial}{\partial t}\right]\frac{\partial}{\partial t}\nabla^2\Phi+\left(1+\tau\frac{\partial}{\partial t}\right)\frac{\partial^3\Phi}{\partial t^3}=0. \tag{11}$$

We seek the solutions for Φ and Ψ in the form

$$\Phi=f(z)\exp\ ik(ct-x) \tag{12a}$$

$$\Psi = g(z) \exp i\,k(ct-x) \tag{12b}$$

$$i = (-1)^{1/2}.$$

Using equations (12a) in (11) we obtain

$$\frac{d^4 f(z)}{dz^4} + A\,\frac{d^2 f(z)}{dz^2} + Bf(z) = 0. \tag{13}$$

Where

$$A = k^2(c^2-2) - (i - \tau ck)(1+\varepsilon)ck \tag{14a}$$

$$B = k^4(1-c^2) + ck^3(i - \tau ck)(1+\varepsilon - c^2). \tag{14b}$$

We choose the solution for Φ and Ψ by (13) and (10b) respectively in the form

$$\Phi = [A_1 \cosh m_1 z + A_2 \sinh m_1 z + A_3 \cosh m_2 z + A_4 \sinh m_2 z] \exp ik(ct-x) \tag{15a}$$

$$\Psi = [A_5 \cosh m_3 z + A_6 \sinh m_3 z] \exp ik(ct-x) \tag{15b}$$

where A_1, A_2, A_3, A_4, A_5, A_6 are arbitrary constants.

$$m_1 = [\tfrac{1}{2}\{(A^2-4B)^{1/2} - A\}]^{1/2} \tag{16a}$$

$$m_2 = [-\tfrac{1}{2}\{(A^2-4B)^{1/2} + A\}]^{1/2} \tag{16b}$$

$$m_3 = ikb,\ b = (c^2\beta^2 - 1)^{1/2}. \tag{16c}$$

Using (15), (9) and the boundary conditions (8) in (7) and (10) we obtain

$$(2-c^2\beta^2)\{A_1 \cosh m_1 H + A_2 \sinh m_1 H + A_3 \cosh m_2 H + A_4 \sinh m_2 H\} + 2b\{A_5 \sinh m_3 H + A_6 \cosh m_3 H\} = 0 \tag{17a}$$

$$(2-c^2\beta^2)\{A_1 \cosh m_1 H - A_2 \sinh m_1 H + A_3 \cosh m_2 H - A_4 \sinh m_2 H\} + 2b\{-A_5 \sinh m_3 H + A_6 \cosh m_3 H\} = 0 \tag{17b}$$

$$(-2im_1)A_1 \sinh m_1 H + (-2im_1)A_2 \cosh m_1 H + (-2im_2)A_3 \sinh m_2 H + (-2im_2)A_4 \cosh m_2 H - k(2-c^2\beta^2)\{A_5 \cosh m_3 H + A_6 \sinh m_3 H\} = 0 \tag{18a}$$

$$(2im_1)A_1 \sinh m_1 H + (-2im_1)A_2 \cosh m_1 H + (2im_2)A_3 \sinh m_2 H + (-2im_2)A_4 \cosh m_2 H - K(2-c^2\beta^2)\{A_5 \cosh m_3 H - A_6 \sinh m_3 H\} = 0 \tag{18b}$$

$$b_1 m_1\{A_1 \sinh m_1 H + A_2 \cosh m_1 H\} + b_2 m_2\{A_3 \sinh m_2 H + A_4 \cosh m_2 H\} = 0 \tag{19a}$$

$$b_1 m_1\{-A_1 \sinh m_1 H + A_2 \cosh m_1 H\} + b_2 m_2\{-A_3 \sinh n_2 H + A_4 \cosh m_2 H\} = 0 \tag{19b}$$

where

$$b_1 = -k^2 + m_1^2 + c^2k^2 \tag{20a}$$

$$b_2 = -k^2 + m_2^2 + c^2k^2. \tag{20b}$$

Eliminating A_1, A_2, A_3, A_4, A_5, A_6 from (17)—(19) we get

$$[k(2-c^2\beta^2)^2 \sinh m^3 H(m_2 b_2 \cosh m_1 \sinh m_2 H - m_1 b_1 \sinh m_1 H \cosh m_2 H) + 2b \cosh m_3 H\{(-2im_1)m_2 b_2 + (2im_2)m_1 b_1\} \sinh m_1 H \sinh m_2 H] \times [k(2-c^2\beta^2)^2(m_2 b_2 \sinh m_1 H \times \cosh m_2 H - m_1 b_1 \cosh m_1 H \sinh m_2 H) \cosh m_3 H + 2b \sinh m_3 H\{(-2im_1)m_2 b_2 + (2im_2)m_1 b_1\} \cosh m_1 H \cosh m_2 H] = 0. \tag{21}$$

Now splitting up equation (21) into two separate systems we may write: Either

$$k(2-c^2\beta^2)^2 \sinh m_3 H(m_2 b_2 \cosh m_1 H \sinh m_2 H - m_1 b_1 \sinh m_1 H \cosh m_2 H) + 2b \cosh m_3 H\{(-2im_1)m_2 b_2 + (2im_2)m_1 b_1\} \sinh m_1 H \sinh m_2 H = 0 \tag{22a}$$

or

$$k(2-c^2\beta^2)^2 \cosh m_3 H(m_2 b_2 \sinh m_1 H \cosh m_2 H - m_1 b_1 \cosh m_1 H \sinh m_2 H) + 2b \sinh m_3 H\{(-2im_1)m_2 b_2 + (2im_2)m_1 b_1\} \cosh m_1 H \cosh m_2 H = 0. \tag{22b}$$

It can easily be verified that the coefficients of A_1, A_3, A_6 and A_2, A_4, A_5 are represented by equations (22a) and (22b), respectively.

Thus we consider the motion symmetrical about the z axis, assuming the solution in the form

$$\Phi_1 = [A_1 \cosh m_1 z + A_3 \cosh m_2 z] \exp ik(ct-x) \tag{23a}$$

$$\Psi_1 = [A_6 \sinh m_3 z] \exp ik(ct-x) \tag{23b}$$

and the antisymmetrical motion about the same axis, choosing the solution as

$$\Phi_2 = [A_2 \sinh m_1 z + A_4 \sinh m_2 z] \exp ik(ct-x) \tag{24a}$$

$$\Psi_2 = [A_5 \cosh m_3 z] \exp ik(ct-x), \tag{24b}$$

we arrive at equations (22a) and (22b) for the symmetrical and the antisymmetrical parts, respectively.

Equation (21) is the period equation of the thermoelastic plane wave in an infinite thermo-elastic plate of thickness $2H$. Equations (22a) and (22b) are its period equations for symmetrical and antisymmetrical vibrations about the z axis.

V. EXPANSION FOR THE CASE OF A VERY THIN PLATE: SYMMETRICAL VIBRATION

Expanding the hyperbolic functions of sine and cosine for small values of H and retaining terms up to second order we get from equation (22a)

$$\begin{aligned}6[k^2(2-c^2\beta^2)^2\{k^2+(i-\tau ck)(1+\varepsilon)ck\}-4\{k^4(1-c^2)+\\+ck^3(i-\tau ck)(1+\varepsilon-c^2)\}]+[(2-c^2\beta^2)^2\{[k^2(c^2-2)-\\-(i-\tau ck)(1+\varepsilon)ck]c^2k^2(\beta^2-1)+k^2(1-c^2)c^2\beta^2+\\+k^4(c^2-2)^2+(i-\tau ck)^2(1+\varepsilon)^2c^2k^2-2k^2(c^2-2)\times\\\times(i-\tau ck)(1+\varepsilon)ck+ck^3(i-\tau ck)(1+\varepsilon-c^2)\}-\\-4\{k^2(c^2-2)-(i-\tau ck)(1+\varepsilon)ck\}\{-5k^2+\\+c^2k^2(1+3\beta^2)-(i-\tau ck)(1+\varepsilon)ck\}]k^2H^2=0.\end{aligned} \quad (25)$$

Equation (25) represents the period equation of the thermo-elastic plane wave in the thermo-elastic thin plate for the symmetrical vibration about the z axis.

If the thickness of the plate is very thin, limit H tends to zero, and equation (25) reduces to

$$\begin{aligned}k^2(2-c^2\beta^2)^2\{k^2+(i-\tau ck)(1+\varepsilon)ck\}-4\{k^4(1-c^2)+\\+ck^3(i-\tau ck)(1+\varepsilon-c^2)\}=0.\end{aligned} \quad (26)$$

Neglecting the terms containing ε^2 but retaining the terms containing $\tau^2\varepsilon^2$ we get from (26)

$$\begin{aligned}(8-9c^2)^2c^2+6c^2\varepsilon(32-60c^2+27c^4)+(8-9c^2)^2k^2-\\-2(8-9c^2)^2c^2k^2\tau-6c^2k^2(32-60c^2+27c^4)\tau\varepsilon+\\+(8-9c^2)^2c^2k^2\tau^2+6c^2k^2(32-60c^2+27c^4)\tau^2\varepsilon+\\+(12-9c^2)^2c^2k^2\tau^2\varepsilon^2=0.\end{aligned} \quad (27)$$

Equation (27) is the phase velocity — dispersion relation in the dimensionless form of the thermo-elastic plane wave (for Poisson's ration $\sigma=0.25$) in a very thin thermo-elastic infinite plate when both the effects of coupling and of thermal relaxation are taken into account. Equation (27) shows that the wave in a thin plate is dispersive in nature.

We put $\tau=3$, $\varepsilon=0.05$ [3] in (27) and obtain from it for different values of k the sixth order algebraic equations with real coefficients involving c. The positive root of it always lies between 0 and 1.

Table 1

k	0	1	2	2.25	2.75	3	3.25	4	5
c	0.94	0.866	0.948	0.92	0.948	0.974	0.92	0.984	0.984

For different values of k the values of c are computed in Table 1.
As k tends to infinity equation (27) reduces to give

$$c^6-1.58c^4+0.38c^2+0.21=0. \quad (28)$$

Equation (28) shows that as $k\to\infty$, c approaches unity.

Particular case: When the relaxation parameter is absent and only the coupling parameter is taken into account, we get from (27) for different values of k a set of algebraic equations in c with real coefficients of the same order. The positive root of c lies between 0 and 1. The values of c are shown in Table 2.

Table 2

k	0	1	2	3
c	0.94	0.97	0.97	0.94

From (27) it is found that as $k\to\infty$, c agrees with the classical result.
When $\tau=0$, $\varepsilon=0$, as k tends to infinity, equation (27) is further reduced to give

$$(9c^2-8)^2=0. \quad (29)$$

The results of Table 1, Table 2 are represented graphically in Fig. 1; it is seen that as $k\to\infty$, c approaches 1 and c reduces to the classical result, $c=0.94$. Thus we can say that by the introduction of the thermal relaxation parameter the motion of the thermo-elastic wave is modified.

VI. GROUP VELOCITY

Since the equation (27) shows that the phase velocity c depends on the frequency, we are interested to find out the group velocity which plays an important role in the propagation of energy of wave motion. The rate of change of frequency with respect to wave number determines the group velocity,

$$d\omega/dk=c+k\,dc/dk.$$

Table 3 shows the group velocity corresponding to the values of c and k of the thermo-elastic plane wave at a very thin infinite plate from the phase velocity — dispersion relation (27), for $\tau=3$, $\varepsilon=0.05$.

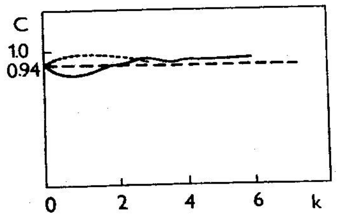

Fig. 1. Relations between phase-velocity c and wave number k in different cases.
The dashed line parallel to k-axis represents wave velocity in a thin plate in the absence of temperature, coupling and thermal relaxation terms in the equation (5) and (6) which correspond to the classical result, $c=0.94$.
The dotted curve represents the effect of the thermo-elastic wave in the thin infinite plate when the temperature and coupling terms in (5) and (6) are taken into account. It is seen that the phase-velocity has tendency to increase with k and then decrease until it approaches the classical results.
The continuous curve represents the phase-velocity as a function of the wave number in the most general case where temperature, coupling and relaxation parameters are taken into account. It is observed that the phase velocity decreases at the initial stage and then gradually increases; after that state it continues in a wave-like nature until it approaches the unity.

Table 3

$\frac{d\omega}{dk}$	0.94	0.955	0.959	0.958	1.029
k	0	1	2	3	4
c	0.94	0.866	0.948	0.974	0.984

VII. EXPANSION FOR THE CASE OF A VERY THIN PLATE: ANTISYMMETRICAL VIBRATION

As symmetric part expanding sinh m_1H, sinh m_2H, sinh m_3H, and cosh m_1H, cosh m_2H, cosh m_3H for small value of H and retaining up to H^2 terms, we obtain from (22b)

$$[6k^2c^4\beta^4+\{3(2-c^2\beta^2)^2[3k^2-k^2c^2(1+\beta^2)+ \\ +(i-\tau ck)(1+\varepsilon)ck]-4(1-c^2\beta^2)[7k^2-k^2c^2(3+\beta^2)+ \\ +3(i-\tau ck)(1+\varepsilon)ck]\}k^2H^2](A^2-4\beta)^{1/2}+ \\ +2k^2(2-c^2\beta^2)(1-c^2)\{k^2(c^2-2)-(i-\tau ck)(1+\varepsilon)ck\}k^2H^2=0. \tag{30}$$

The equation (30) shows that antisymmetric wave propagation is not possible for a very thin plate.

REFERENCES

[1] Ewing, W. B., Jardetzky, W. S., Press, F.: *Elastic waves in media*. Mc. Graw–Hill, New York 1957.
[2] Lord, W. H., Shulman, Y. A.: J. Mech. Phys. Solids *16* (1967), 299.
[3] Nayfeh, A. H., Nemat-Nasser, S.: J. Math. and Phys. *23* (1972), 50.

Received January 5th, 1982