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ALL VACUUM METRICS WITH SPACE-LIKE SYMMETRY AND SHEARING GEODESIC EIGENRAYS

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This paper contains the general solution of the Einstein equation for space-like symmetric vacuum having shearing geodesic eigenrays. The result is that the rotation of the eigenrays vanishes (except the Minkowski space-time), and the solutions belong to two groups. One of them contains solutions with functional dependence among some field quantities (the Papapetrou solutions and the Kasner solution). The other group contains new solutions.

ПЕРЕЧИСЛЕНИЕ ВАКУУМНЫХ МЕТРИК С ПРОСТРАНСТВЕННОПОДОБНОЙ СИММЕТРИЕЙ И СРЕЗАЮЩИЕ ГЕОДЕЗИЧЕСКИЕ СОБСТВЕННЫЕ ЛУЧИ

В работе приводятся все возможные вакуумные решения уравнений Эйнштейна для пространственноподобных симметричных гравитационных полей, имеющих временные срезяющие геодезические собственные лучи. Среди этих решений собственные лучи обладают вращательной симметрией только в случае поля Минковского. Решения разделяются на две группы. Первая из них содержит решения, для которых между пространственными величинами существует функциональная зависимость (решения Папапетру и решение Каснера). Вторую группу образуют новые решения.

I. INTRODUCTION

The spin coefficient technique has led to many new solutions of the Einstein equation of the general relativity. Without assuming any symmetry, the original Newman–Penrose equations can be solved for geodesic rays, and this class contains the Kerr solution, in which the shear of the rays vanishes too [1, 2, 3]. When the space-time has a non-null Killing vector, the problem can be reformulated in a three-dimensional space [4, 5], defined by the following decomposition of the line element:

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$$d\tilde{s}^2 = f(dy + \omega_r\, dx^r)^2 - f^{-1}\, ds^2$$
$$ds^2 = g_{rs}\, dx^r\, dx^s \qquad (1.1)$$
$$i = 1, 2, 3.$$

In the three-dimensional background space one can introduce a specially orthonormalized basic vector triad ($\boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}$) (see in Sect. II) whose structure is characterized by the invariant quantities (called spin coefficients)

$$\varkappa = -l_{i;k} m^i l^k \qquad \sigma = -l_{i;k} m^i m^k$$
$$\varrho = -l_{i;k} \bar{m}^i m^k \qquad \varepsilon = m_{i;k} \bar{m}^i l^k \qquad (1.2)$$
$$\tau = m_{i;k} \bar{m}^i m^k.$$

The Einstein equation becomes a system of partial differential equations for these spin coefficients and for the vector $\boldsymbol{G}$

$$\boldsymbol{G} = \frac{1}{2f}(\nabla f + i f^2 \nabla \times \boldsymbol{\omega}). \qquad (1.3)$$

For timelike symmetry the vacuum equations have been integrated for the class $\varkappa\sigma = 0$, [6, 7]. Since in the threedimensional formalism $\varkappa$ and σ belong to the eigenray congruence, and the eigenrays are the projections of the rays if and only if $\sigma = 0$, [4], these solutions have nongeodesic rays (except, of course, the subclass $\varkappa = \sigma = 0$). It is interesting that there is no generalization of the Kerr space-time among the $\varkappa\sigma = 0$, $\varkappa\bar{\varkappa} + \sigma\bar{\sigma} \neq 0$ solutions.

Since the structure of the spin coefficient equations does not become more complicated choosing the space-like Killing vector instead of the time-like one, it is obvious to try to solve the equations for the same class again. In this paper we will investigate the subcase of geodesic eigenrays for space-like symmetry. In several points the procedure of integration is very similar to the steps of the calculation in Ref. [6]. In these points we omit the details and refer to Ref. [6].

II. THE METHOD

For spatial symmetry the general formulation of the three-dimensional spin coefficient technique can be found in Ref. [5]. There exists a basic vector triad $\boldsymbol{z}_m = (\boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}})$, where $\boldsymbol{l}$ is real, and the index m takes the values 0, + and −, respectively. The triad is specially orthonormalized, namely

$$\boldsymbol{z}_m \boldsymbol{z}_n = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} \qquad (2.1)$$

If the eigenray equation has a solution, then it is a time-like curve, and $\boldsymbol{l}$ can be chosen as the tangent of the eigenray congruence when [6]

$$G_- = 0; \qquad G_m \equiv \boldsymbol{z}_m \boldsymbol{G}. \qquad (2.2)$$

By means of some rotation of $\boldsymbol{m}$

$$\varepsilon = 0 \qquad (2.3)$$

can always be achieved, [6] and there remains the freedom

$$\boldsymbol{m}' = \boldsymbol{m} e^{iC^0}; \qquad DC^0 = 0. \qquad (2.4)$$

Using the rotation a phase of vanishing D derivative can be removed from one of $\varkappa$, σ or G_+.

Now, imposing the condition that

$$\varkappa = 0; \qquad \sigma \neq 0, \qquad (2.5)$$

i.e. that the eigenrays by geodesic and shearing, we obtain the following field equations for the vacuum:

$$D\varrho = -\varrho^2 - \sigma\bar{\sigma} - G_0\bar{G}_0 \qquad (2.6a)$$
$$D\sigma = -(\varrho + \bar{\varrho})\sigma \qquad (2.6b)$$
$$D\tau = -\varrho\tau + \bar{\sigma}\bar{\tau} - G_0\bar{G}_- \qquad (2.6c)$$
$$\delta\varrho - \bar{\delta}\sigma = 2\sigma\tau - \bar{G}_0 G_+ \qquad (2.6d)$$
$$\delta\tau + \bar{\delta}\bar{\tau} = -2\tau\bar{\tau} - \sigma\bar{\sigma} + \varrho\bar{\varrho} - G_0\bar{G}_0 - G_+\bar{G}_- \qquad (2.6e)$$
$$DG_0 = (-2\bar{\varrho} + G_0 - \bar{G}_0)G_0 \qquad (2.6f)$$
$$\delta G_0 - DG_+ = (\bar{\varrho} + \bar{G}_0)G_+ \qquad (2.6g)$$
$$\bar{\delta}G_0 = \bar{\sigma}G_+ - \bar{G}_- G_0 \qquad (2.6h)$$
$$\bar{\delta}G_+ = -(\tau + \bar{G}_-)G_+ + (\varrho - \bar{\varrho})G_0, \qquad (2.6i)$$

where

$$D \equiv l^r \partial_r; \qquad \delta \equiv m^r \partial_r. \qquad (2.7)$$

The commutators of D, δ and $\bar{\delta}$ are as follows:

$$D\delta - \delta D = -\bar{\varrho}\delta - \sigma\bar{\delta} \qquad (2.8a)$$
$$\delta\bar{\delta} - \bar{\delta}\delta = \tau\delta - \bar{\tau}\bar{\delta} - (\varrho - \bar{\varrho})D. \qquad (2.8b)$$

A coordinate system can always be chosen so that

$$l^i = \delta^i_0 \qquad (2.9)$$

and there remains the freedom

$$t' = t + t^0(x^a); \qquad a = 1, 2 \tag{2.10a}$$

$$x^{a'} = x^{a'}(x^b). \tag{2.10b}$$

Rewriting $\boldsymbol{m}$ in the form

$$m^i = \omega\delta^i_0 + \xi^a\delta^i_a \tag{2.11}$$

the commutators (2.8) yield equations for ω and ξ^a.

III. THE INTEGRATION OF THE FIELD EQUATIONS

In order to obtain solutions, first we apply the commutators to the field quantities. Then, substituting the known derivatives, we may get further equations. When the system has become closed, proper coordinates are to be used when the equations become partial differential equations of the first order. Having integrated these equations, the three-dimensional quantities g_{ik} and $\boldsymbol{G}$ are obtained, and the four-dimensional metric can be reconstructed in the usual way [6].

As the first step one can observe that, according to eq. (2.6b),

$$D(\sigma/\bar{\sigma}) = 0; \tag{3.1}$$

thus, by means of the transformation (2.4), σ can be made real.

If $G_0 = 0$, eq. (2.6h) shows that G_+ also vanishes. But $\boldsymbol{G} = 0$ leads to flat space-time, thus here we assume that $G_0 \neq 0$. Now applying the commutator (2.8a) to $\ln G_0$ we get:

$$\delta(\ln(G_0\sigma)) = G_+ - 2\bar{\tau}. \tag{3.2}$$

Then, applying the operator D to the new equation, we get the equation

$$\gamma(3\bar{\delta}\varrho + \bar{\delta}\bar{\varrho} + 2\delta\sigma) + 2\sigma\delta\gamma = 0 \tag{3.3}$$

$$\gamma^2 \equiv G_0\bar{G}_0.$$

This equation has the same form as in Ref. [6], and from this point on we should repeat the steps of that paper, with the only difference that the operators $\delta_\pm$ are defined as

$$\delta_\pm \equiv R(\delta \pm i\bar{\delta}) \tag{3.4}$$

$$DR = \frac{1}{2}(\varrho + \bar{\varrho})R.$$

Finally we obtain the same consequences too, namely that

$$\varrho = \bar{\varrho} \tag{3.5}$$

$$\varrho^2 - \sigma^2 - \gamma^2 = 0 \tag{3.6}$$

$$\delta\varrho = 0 \tag{3.7a}$$

$$\delta\sigma = 0 \tag{3.7b}$$

$$\delta\gamma = 0. \tag{3.7c}$$

With these results one can start with the integration of the partial differential equations.

Now let us use a coordinate system of type (2.9). Combining eqs. (2.6a) and (3.6) one gets:

$$\frac{\partial}{\partial t}\varrho = -2\varrho^2, \tag{3.8}$$

whence

$$\varrho = \frac{1}{2}(t + t^0(x^a))^{-1}, \tag{3.9}$$

where t^0 is real. But such a t^0 can be removed by means of the transformation (2.10a), and then ϱ depends only on t. Comparing this form with eqs. (2.7), (2.11) and (3.7a) it can be seen that

$$\omega = 0. \tag{3.10}$$

Before integrating the remaining part of the system of equations we have to complete the system. First, the complex vector $\boldsymbol{G}$ comes from the Ernst potential $\varepsilon \equiv f + i\varphi$ as

$$\boldsymbol{G} = \frac{1}{2f}\nabla(f + i\varphi) \tag{3.11}$$

and, secondly, applying the commutators (2.8) to the expression (2.11) one gets:

$$D\xi^a = -\varrho\xi^a - \sigma\bar{\xi}^a \tag{3.12a}$$

$$\delta\bar{\xi}^a - \bar{\delta}\xi^a = \tau\bar{\xi}^a - \bar{\tau}\xi^a. \tag{3.12b}$$

Now one can immediately integrate the equations containing only t derivatives, namely eqs. (2.6a, b, f), the t component of (3.11) and (3.12a), together with the algebraic constraints (3.5, 3.6). The result is as follows:

$$\varrho = \frac{\sigma}{\sigma^0} = \frac{\gamma}{\gamma^0} = \frac{1}{2t}; \qquad \sigma^{02} + \gamma^{02} = 1 \tag{3.13a}$$

$$G_0 = -\frac{\gamma^0}{2t}\frac{t^{\gamma^0} - iQ}{t^{\gamma^0} + iQ} \tag{3.13b}$$

$$\xi^a = \frac{1}{\sqrt{2}t}(A^a t^{-\sigma^0/2} + iB^a t^{\sigma^0/2}) \qquad (3.13c)$$

$$\varepsilon \equiv f + i\varphi = i\varphi^0 + \frac{f^0}{t^{\gamma^0} + iQ}. \qquad (3.13d)$$

Now eqs. (3.7b, c) show that σ^0 and γ^0 are constant, while eqs. (2.6c, d) and (3.7) yield two algebraic equations:

$$2\sigma\tau = \bar{G}_0 G_+ \qquad (3.14a)$$

$$(\sigma^2 - \gamma^2)\tau = 0. \qquad (3.14b)$$

The remaining part of the system can be reduced to two equations:

$$\bar{\delta}(f + i\varphi) = 0 \qquad (3.15a)$$

$$\mathrm{Im}\,(\delta + \bar{\tau})\xi^a = 0. \qquad (3.15b)$$

Eq. (3.14b) has two solutions: either $\tau = 0$, or $\sigma^0 = \gamma^0 = 1/\sqrt{2}$. We must deal with these two cases separately.

If $\tau = 0$, eqs. (2.14a) and (2.15a) show that $\varphi^0 = 0$, while f^0 and Q are constant. According to eq. (3.15b) we can choose such coordinates that

$$A^a = \delta^a_2$$
$$B^a = \delta^a_3. \qquad (3.16)$$

Now the triad vectors (l^i, m^i, m^i) have been known and thus the three-dimensional metric tensor can be calculated. Namely, using the completeness of the triad

$$z^i_m z^m_k = \delta^i_k \qquad (3.17)$$

eq. (2.1) can be inverted as

$$g^{ik} = z^r_m z_{nr} z^{mi} z^{nk} = l^i l^k - m^i \bar{m}^k - \bar{m}^i m^k. \qquad (3.18)$$

(The actual values of the triad vectors can be taken from eqs. (2.9), (2.11), (3.10), (3.13c) and (3.16).)

Thus the three-metric has the form

$$ds^2 = dt^2 - t^{1+\sigma^0} dx^2 - t^{1-\sigma^0} dy^2. \qquad (3.19)$$

Since G is known, f and ω_i can be calculated in the same way as in Ref. [6].

Table 1

Metrics with $\sigma \neq 0$, $\varkappa = 0$

$\tau = 0$

$$d\hat{s}^2 = f(t)\,(dz - 2\gamma^0 Qx\,dy)^2 - f(t)^{-1}(dt^2 - t^{1+\sigma^0}dx^2 - t^{1-\sigma^0}dy^2)$$
$$f(t) = -t^{\gamma^0}(t^{2\gamma^0} + Q^2)^{-1}$$

σ^0 and Q are constants, $\gamma^0 = \sqrt{1-\sigma^{02}}$

$\tau \neq 0$, Q = constant

$$d\hat{s}^2 = -(x + Qy)\,t^{\sigma^0}(t^{2\sigma^0} + Q^2)^{-1}(dz + 2\sigma^0 Qy\,dx)^2 - 2dt\,(dz + 2\sigma^0 Qy\,dx) - (t^{2\sigma^0} + Q^2)\,(t^{1-2\sigma^0}dx^2 + t\,dy^2)$$

Q is a constant, $\sigma^0 = 1/\sqrt{2}$

$\tau \neq 0$, $Q \neq$ constant

$$d\hat{s}^2 = -x t^{\sigma^0}(t^{2\sigma^0} + y^2)^{-1}\left[dz + \frac{\sigma^0 y}{2x^4}(ay+b)^4\,dx\right]^2 - dt\left[2dz + \frac{\sigma^0 y}{x^4}(ay+b)^4\,dx\right] - x^{-6}t^{1-2\sigma^0}(t^{2\sigma^0} + y^2)\,(ay+b)^2 \times \{(ay+b)^2\,(t^{2\sigma^0} + y^2)\,dx^2 + 2x\,(ay+b)\,[at^{2\sigma^0} + (2ay+b)\,y]\,dx\,dy + x^2[a^2 t^{2\sigma^0} + (2ay+b)^2]\,dy^2\}$$

a and b are constants, $\sigma^0 = 1/\sqrt{2}$

If $\tau \neq 0$, the method of the integration of eqs. (3.14, 3.15) is analogous to the procedure written down in Ref. [6]. We introduce new differential operators by the following definitions:

$$\hat{\alpha} = \sqrt{-f^0} B^a \partial_a; \qquad \beta = \sqrt{-f^0}(A^a - QB^a)\partial_a \qquad (3.20)$$

and then eqs. (3.14, 3.15) get the form:

$$[\hat{\alpha}, \beta] = -2(\hat{\alpha}Q)\,\hat{\alpha} \qquad (3.21a)$$

$$\hat{\alpha}Q = \beta \ln(-f^0) \qquad (3.21b)$$

$$\beta Q = 0 \qquad (3.21c)$$

$$\hat{\alpha}\varphi^0 = 0 \qquad (3.21d)$$

$$\beta\varphi^0 = \hat{\alpha} f^0. \qquad (3.21e)$$

Applying the commutator $[\hat{\alpha}, \beta]$ to Q we find that

$$\beta(f^{0-3}\beta f^0) = 0. \qquad (3.22)$$

Comparing this with eq. (3.21c) it can be seen that there is an alternative: either Q is constant, or $(f^{0-3}\beta f^{0})$ is a functional of Q. These two cases can be treated similarly as in Ref. [6].

IV. THE RESULTS

The reconstruction of the 4-dimensional line element happens similarly as in Ref. [6] (in fact, the only difference is that here $f<0$). Three different line elements are obtained, and they are listed in Table 1. We note that a trivial homothetic factor (which can always be present in vacuum solutions) has been removed from the line elements.

V. CONCLUSIONS

These solutions are of the Petrov type I, similarly to the Kóta–Perjés solutions [8]. Only the first line element is a generalization of a $\varkappa=\sigma=0$ solution. In this solution both f and φ depend on t only, thus this metric belongs to a class analogous to the Papapetrou class for stationary space-times [8], unless $Q=0$. For $Q=0$ it is a Kasner solution [9]. Each solution has a singularity at $t=0$, and the third is singular at $y=-b/a$ too. For the physical meaning of the Kasner metric see [10].

The first metric has three Killing vectors as follows:

$$K_1^a=(0,1,0,2\gamma^0 Qy)$$
$$K_2^a=(0,0,1,0) \qquad (5.1)$$
$$K_3^a=(0,0,0,1)$$

with the commutators

$$[K_1,K_2]=-2\gamma^0 QK_3$$
$$[K_1,K_3]=0 \qquad (5.2)$$
$$[K_2,K_3]=0.$$

The second solution has two commuting Killing vectors:

$$K_1^a=(0,-Q,1,-2\gamma^0 Qx) \qquad (5.3)$$
$$K_2^a=(0,0,0,1)$$

while the only Killing vector of the third metric is

$$K^a=(0,0,0,1). \qquad (5.4)$$

Similarly to the stationary case, the class of shearing geodesic eigenrays contains less and more special solutions than the $\varkappa=\sigma=0$ class.

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