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NOTE OF POSITIONS OF PARTICLES IN CLASSICAL RELATIVISTIC MECHANICS

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The relation between world-lines and the position vector of a particle is studied from the gauge theoretical point of view. The expressions for the position vector of a free relativistic particle as well as of two interacting particles described by the Todorov–Komar model are derived under plausible assumptions. The relation between the physical meaning of basic canonical variables and the choice of a gauge is also discussed.

ЗАМЕЧАНИЕ О ПОЛОЖЕНИЯХ ЧАСТИЦ В КЛАССИЧЕСКОЙ РЕЛЯТИВИСТСКОЙ МЕХАНИКЕ

В работе в рамках калибровочной теории изучается соотношение между мировыми линиями и радиус-вектором частицы. При относительно реалистических предположениях выведены выражения для радиус-вектора свободной релятивистской частицы, а также двух взаимодействующих частиц, описываемых в рамках модели Тодорова–Комара. Обсуждается также связь между физическим значением основных динамических переменных и выбором калибровки.

I. INTRODUCTION

To outline the problem to be studied in the present paper let us consider the theory of a free relativistic particle. In the covariant formulation of this theory one starts from the action integral which is invariant with respect to reparametrizations $\tau \rightarrow \tau'$ of the world-lines $x^\mu(\tau)$. Hence by solving the corresponding equations of motion we obtain a classical world-line $x^\mu(\tau)$ the form of which depends upon used parametrization. Hence the world-line $x^\mu(\tau)$ has not a primary physical meaning.

In the paper presented the relation between $x^\mu(\tau)$ and the position of a particle at a time t will be studied from the gauge theoretical point of view. The paper is organized as follows. In this section we shall outline the concise classical theory of gauge systems. In Sec. II we shall treat a free relativistic particle as a gauge system. On the basis of some plausible requirements we shall derive the expression for the

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position vector of a considered particle. We shall also show that in quantum theory the Newton–Wigner position operator [3] corresponds to the initial value of the classical position vector. In Sec. III there are analogous considerations as regards the two interacting particles described by the Todorov–Komar model.

The dynamics of a gauge system is defined by a hamiltonian $H=H(p,q)$, $p=(p_1, p_2, \ldots, p_n)$, $q=(q_1, q_2, \ldots, q_n)$ and by the constraints $A_a(p,q)=0$, $a=1, 2, \ldots, m<n$ satisfying the relations

$$\{A_a, H\}=u_{ab}A_b \qquad \{A_a, A_b\}=v_{abc}A_c, \tag{1}$$

where the symbol $\{,\}$ denotes the Poisson bracket, u_{ab} and v_{abc} are certain functions and summation over repeated indices is understood throughout the paper.

The detailed theory of the system we consider can be found, e.g., in [1, 2]. Here we shall only list some necessary basic notions and equations.

The equations of motion for a considered system are of the form

$$\frac{df(q,p,t)}{dt}=\frac{\partial f}{\partial t}+\{f, H_D\}, \tag{2}$$

where H_D is the Dirac hamiltonian defined as $H_D=H+\lambda_a A_a$. Here λ_a are arbitrary functions depending upon the evolution parameter t. Hence solutions to (2) depend upon a choice of λ_a. The relations between the solutions q, p corresponding to different choices of λ_a are the following

$$q_k(t+dt,\lambda)-q_k(t+dt,\lambda')=dt(\lambda_a-\lambda_a')\{q_k(t), A_a\}+0((dt)^2). \tag{3}$$

(The same equations hold also for p_k.) The equations (1—3) mean that A_a are generators of certain transformations (gauge transformations) which do not change the physical state of a considered system.

The evolution of physical quantities cannot be influenced by the choice of λ_a. It means that physical quantities must have a zero Poisson bracket with any generator A_a of a corresponding gauge group.

At the end of this section let us consider m functions $B_a(p,q)$ for which $\det\{A_a, B_b\}\neq 0$. In this case we can choose λ_a in such a way that $dB_a/dt=0$. However, it means that we can put $B_a=\text{const}$ and the considered system can be fully described by $2(n-m)$ independent variables.

II. A FREE RELATIVISTIC PARTICLE AS A GAUGE SYSTEM

A free relativistic particle can be described by the lagrangian

$$L=-M(\dot{x}^\mu\dot{x}_\mu)^{1/2} \qquad \mu=0,1,2,3, \tag{4}$$

where $\dot{x}=dx/d\tau$, τ is some evolution parameter, $x^\mu=g^{\mu\nu}x_\nu$, $g^{\mu\nu}=(+,-,-,-)$ and M is the rest mass of a particle.

Going from L to hamiltonian H we obtain

$$H=0 \qquad A=p^\mu p_\mu-M^2=0, \tag{5}$$

where $p^\mu=\partial L/\partial\dot{x}_\mu$. Now $H_D=\lambda A$ and $\{x^\mu, x^\nu\}=\{p^\mu, p^\nu\}=0$, $\{x^\mu, p^\nu\}=g^{\mu\nu}$. It is not difficult to show that A is the generator of reparametrizations of the world-lines $x^\mu(\tau)$. Namely, the equations for x^μ are of the form

$$\frac{dx^\mu}{d\tau}=\lambda\{x^\mu, A\}. \tag{6}$$

By a reparametrization $\tau\to\tau'$ the Eqs. (6) take the form $dx^\mu/d\tau'=\lambda'\{x^\mu, A\}$, where $\lambda\, d\tau=\lambda'\, d\tau'$. Now

$$\begin{aligned} &x^\mu(\tau+d\tau,\lambda')-x^\mu(\tau+d\tau',\lambda')=\\ &=d\tau(\lambda'-\lambda)\{x^\mu(\tau), A\}+0((d\tau)^2)=\\ &=x^\mu(\tau+d\tau,\lambda')-x^\mu(\tau+d\tau,\lambda). \end{aligned}$$

However, the last equations mean that A is indeed the generator of the reparametrization of the world-lines $x^\mu(\tau)$.

As it was noted in the previous section, physical quantities F have to satisfy $\{F, A\}=0$. One can easily see that x^μ has no zero Poisson bracket with A. It means that x^i $(i=1,2,3)$ cannot be regarded as the components of the position vector of a particle. What quantities q^i $(i=1,2,3)$ can be identified as a position of a particle? To arrive at that identification we have to impose some requirements on q^i. We conjecture that the following requirements seem to be natural.

(i) $\{q^i, A\}=0$ for all i.

(ii) q^i have to be expressed by means of x^i, x^0, p^i, p^0, M and $c=1$ (velocity of light) only.

(iii) The transformation properties of q^i are the same as those of x^i. Here we assume x^0, p^0 to be invariant under space rotations and translations; $x^i\to x^i+a^i$, $p^i\to p^i$ under space translations and x^i, p^i are vectors under space rotations.

It follows from (ii) and (iii) that q^i is of the form

$$q^i=x^i+g(p^0, x^0, M, c)p^i.$$

By using (i) we obtain $g=b(p^0, M, c)-x^0/p^0$. However, we put $b=0$ because we cannot construct a quantity with a dimension of length by means of p^0, M, c and p^i only. Then

$$q^i=x^i-\frac{x^0}{p^0}p^i. \tag{7}$$

In quantum theory a wave function Ψ is independent of τ (owing to $H=0$) and physical states are projected by the equation

$$(\mathcal{P}^{\mu}\mathcal{P}_{\mu}-M^2)\Psi=0,$$

where the operators $\mathcal{X}^{\mu}$, $\mathcal{P}^{\mu}$ satisfy the relations

$$[\mathcal{X}^{\mu},\mathcal{X}^{\nu}]=[\mathcal{P}^{\mu},\mathcal{P}^{\nu}]=0, \qquad [\mathcal{X}^{\mu},\mathcal{P}^{\nu}]=i\hbar g^{\mu\nu}.$$

According to a general rule let us assign to q^i the operator

$$\mathcal{Q}^i=\mathcal{X}^i-\frac{1}{2}(\mathcal{X}^0(\mathcal{P}^0)^{-1}+(\mathcal{P}^0)^{-1}\mathcal{X}^0)\mathcal{P}^i=\mathcal{X}^i-(\mathcal{P}^0)^{-1}\mathcal{X}^0\mathcal{P}^i+\frac{i\hbar}{2}(\mathcal{P}^0)^{-2}\mathcal{P}^i.$$

If now we express q^i in terms of canonically conjugate variables ($p_i=-p^i$ is the canonically conjugate momentum to x^i) and drop the distinction between covariant and contravariant indices, we obtain

$$\mathcal{Q}_i=\mathcal{X}_i+(\mathcal{P}^0)^{-1}\mathcal{X}^0\mathcal{P}_i-\frac{i\hbar}{2}(\mathcal{P}^0)^{-2}\mathcal{P}_i,$$

where $[\mathcal{X}_i,\mathcal{P}_j]=i\hbar\delta_{ij}$ and $[\mathcal{X}^0,\mathcal{P}^0]=i\hbar$.

After putting $\mathcal{X}_i=i\hbar\partial/\partial p_i$, $\mathcal{X}^0=i\hbar\partial/\partial p^0$ and substituting $p^0=(p_kp_k+M^2)^{1/2}$ into the wave function Ψ, the operator $\mathcal{Q}_i$ can be written in the form

$$\mathcal{Q}_i=i\hbar\frac{\partial}{\partial p_i}-\frac{i\hbar p_i}{2(p_kp_k+M^2)}. \tag{8}$$

This operator is the position operator obtained first by Newton and Wigner [3] (see also [4]).

In the studied case, all gauge invariant (physical) quantities are integrals of motion. For this reason the q_i has to be interpreted as an "initial" value of the ith component of the position vector. Now we can ask: What is the dependence of the position upon the time? What is the relation between the time and the evolution parameter τ? In our simple example we know that the position x_i at the time t is simply given by $x_i(t)=q_i+tp_i/p^0$. This result can be obtained from the equations of motion if the gauge condition $B=x^0-\tau=0$ is chosen. Hence if $dx^0/d\tau=1$ then τ represents the time t. Is that choice of the gauge from some point of view important? It is not difficult to show that the energy $p^0=(p_kp_k+M^2)^{1/2}$ is the evolution generator of physical quantities only for the gauge of the type $B=x^0-\tau-\text{const}=0$. This assertion is also true for any system of non-interacting particles. Hence we conclude that for a system of non-interacting particles the quantities $x_{ai}(\tau)$ ($a=1,2,\ldots$, numerate particles) satisfying the equations of motion are the positions of particles at a time τ only if $x_a^0=\tau+\text{cons}$. Then the total energy $P^0=p_1^0+p_2^0+\ldots$ is the evolution generator of physical quantities.

III. THE TODOROV–KOMAR MODEL

Let us now consider two interacting particles described by the Todorov [5] — Komar [6] model. The dynamics in this model is defined by two first class constraints

$$A_a=p_a^2-m_a^2-f(X^2)=0 \qquad a=1,2, \tag{9}$$

where $p_a^2=p_a^{\mu}p_{a\mu}$, $X^2=x_{\mu}x_{\nu}(g^{\mu\nu}-P^{\mu}P^{\nu}/P^2)$, $P^{\mu}=p_1^{\mu}+p_2^{\mu}$, and $x^{\mu}=x_1^{\mu}-x_2^{\mu}$.

Here again the quantities x_a^{μ}, p_a^{μ} cannot be identified as positions and momenta of the considered particles because they have not zero Poisson brackets with the constraints (9). Let us denote by Q_a^i, P_a^i ($i=1,2,3$) the position and momentum of the ath particle and impose on Q_a^i, P_a^i the following requirements

(i) $\{Q_a^i, A_b\}=\{P_a^i, A_b\}=0$ for all a, b.

(ii) Q_a^i, P_a^i have to be expressed by means of the quantities occurring in the formalism.

(iii) The transformation properties of $Q_a^i(P_a^i)$ are the same as those of $x_a^i(p_a^i)$ (see (iii) in the previous section).

Let us now further assume that we know the functions

$$Q_a^i=Q_a^i(x_b^k, p_b^k, x_b^0, p_b^0) \qquad P_a^i=P_a^i(x_b^k, p_b^k, x_b^0, p_b^0) \tag{10}$$

satisfying (i)—(iii). Since Q_a^i, P_a^i are integrals of motion we again interpret them as "initial" data. One can intuitively expect that after putting $B_1=x_1^0-x_2^0=0$, $B_2=x_1^0-\tau=0$ and $A_a=0$ into (10) and solving (10) with respect to x_b^k, p_b^k, we obtain the positions and momenta of particles at a time τ, i.e.

$$x_a^k(\tau)=x_a^k(Q_b^i, P_b^i, \tau), \qquad p_a^k(\tau)=p_a^k(Q_b^i, P_b^i, \tau). \tag{11}$$

To make this assertion more plausible let us follow the next consideration. If we neglect and interaction, i.e. if we put $f=0$, then we have to obtain the results presented at the end of the previous section. It means that an evolution generator of the quantities (11) has to be expressed at $f=0$ as the sum of the energies of the considered free particles. In our case the gauge invariant quantity $P^0=p_1^0+p_2^0$ reduces at $f=0$ to $(p_{1k}p_{1k}+m_1^2)^{1/2}+(p_{2k}p_{2k}+m_2^2)^{1/2}$ and it is natural to regard P^0 as the total energy of the considered system. Moreover it is not difficult to show that P^0 is the evolution generator of the quantities (11). To prove this let us consider the equations of motion for $x_a^k(\tau)$, $p_a^k(\tau)$. They are

$$\frac{dx_a^k}{d\tau}=\lambda_b\{x_a^k, A_b\} \qquad \frac{dp_a^k}{d\tau}=\lambda_b\{p_a^k, A_b\}, \tag{12}$$

where λ_a are such that the gauge conditions $B_1=x_1^0-x_2^0=0$, $B_2=x_1^0-\tau=0$ are satisfied at any τ, i.e. $\lambda_1=2p_2^0/D$, $\lambda_2=2p_1^0/D$ where $D=4p_1^0p_2^0-2P^0F^0$ and

$F^0 = \partial f/\partial P^0$. Now the Eqs. (12) take the form (for brevity we write the equations for x_1^k, p_1^k only)

$$\frac{dx_1^k}{d\tau} = (4p_2^0 p_1^k - 2P^0 F_1^k)/D \qquad \frac{dp_1^k}{d\tau} = 2P^0 G_1^k/D, \tag{13}$$

where $F_1^k = \partial f/\partial p_{1k}$ and $G_1^k = \partial f/\partial x_{1k}$.

Let us now consider the quantity $(p_1^0 + p_2^0)$ where p_a^0 are solutions to the equations $A_a = B_a = 0$. There holds

$$\frac{\partial}{\partial p_1^k}(p_1^0 + p_2^0) = (4p_2^0 p_1^k - 2P^0 F_1^k)/D \qquad -\frac{\partial}{\partial x_1^k}(p_1^0 + p_2^0) = 2P^0 G_1^k/D. \tag{14}$$

By comparing Eqs. (13) and (14) one can see that $(p_1^0 + p_2^0)$ is indeed the evolution generator of the quantities $x_a^k(\tau)$, $p_a^k(\tau)$ given by (11).

Concluding this section we note that our calculations are consistent only if det $\{B_a, A_b\} \neq 0$, i.e. in this case only the gauge conditions $B_1 = x_1^0 - x_2^0 = 0$, $B_2 = x_1^0 - \tau = 0$ are allowed.

IV. CONCLUSION

Although from the formal point of view the theory of gauge systems (and constrained Hamiltonian systems in general) has been comparatively satisfactorily elaborated, we believe some of its problems still to be open. One of them is the identification of quantities (the determination of their physical meaning) having zero Poisson brackets with all generators of a corresponding gauge group. Another interesting problem is connected with the choice of a gauge. It seems to be no trivial question for which choice of gauge conditions the basic dynamical variables q_k, p_k represent physical quantities (e.g. positions and momenta of particles). We conjecture that the study of these and similar problems will enable us to understand better the theory of constrained systems.

The problem studied in this paper is closely related to the problem of the localizability of particles. Namely, in quantum theory the quantities q_i would be represented by the operators $\mathcal{Q}_i$ (the position operators). Eigenvalues and eigenfunctions of $\mathcal{Q}_i$ would represent possible values of the position and localized states of a considered particle. The existence of such $\mathcal{Q}_i$ (satisfying certain natural requirements) means, roughly speaking, that a considered particle is localizable. Various aspects of the localizability of particles and the extensive list of papers dealing with this problem can be found, e.g., in [7].

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