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FOKKER-PLANCK EQUATION IN A STRONG EXTERNAL UNIFORM MAGNETIC FIELD[1])

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The magnetized Fokker-Planck collision term for the electron-ion plasma is approximated. The collision term is evaluated explicitly in the case of unmagnetized Maxwellian ions for $1 < \Omega_e/\omega_{pi}\sqrt{2} < \lambda_0$ (λ_0 being the number of particles in a Debye sphere).

УРАВНЕНИЕ ФОККЕРА-ПЛАНКА ДЛЯ СИЛЬНОГО ВНЕШНЕГО ОДНОРОДНОГО МАГНИТНОГО ПОЛЯ

В работе сделана аппроксимация уравнения Фоккера-Планка с магнитным членом для электронно-ионной плазмы. Член, описывающий соударение, вычислен в явном виде для случая ненамагниченных максвелловских ионов с $1 < \Omega_e/\omega_{pi}\sqrt{2} < \lambda_0$ (λ_0 — представляет собой число частиц в сфере Дебая).

I. INTRODUCTION

The collision term of the kinetic equation for magnetized plasma has been studied by many authors (see [1] and references therein). The main problem is the fact that the calculated collision terms were so complex as to be unsuitable for practical use. Recently some good results have been obtained in this field [2]. The collision term of the magnetized kinetic equation was approximated in such a way that further mathematical treatment is possible.

In this paper we shall derive the collision term for the two-component plasma supposing the field particles plasma to be unmagnetized and Maxwellian. We also neglect the wave effects and the dependence of the interaction potential on the particle velocity [3]. We shall show that the dominant effects are given by the parameter $\ln(\lambda_0/\eta)$ (here $\eta = \Omega_e/\omega_{pi}\sqrt{2}$), which replaces the Coulomb logarithm $\ln \lambda_0$ in the terms describing the anisotropy of the collision term.

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II. THE FOKKER-PLANCK COEFFICIENT D

The magnetized Fokker-Planck collision term for the electron and Maxwellian ion plasma has the form [4]

$$\left(\frac{\partial f_e}{\partial t}\right)_{coll} = -\mathrm{div}\left[\mathbf{D}\cdot\left(\nabla + \frac{\mathbf{p}_e}{m_e k_B T_i}\right) f_e\right], \tag{1}$$

where $\mathrm{div} = \left(p_{e\perp}^{-1}\frac{\partial}{\partial p_{e\perp}} p_{e\perp}, \frac{\partial}{\partial p_{e\parallel}}\right)$, $\nabla = (\partial/\partial p_{e\perp}, \partial/\partial p_{e\parallel})$, $\mathbf{p}_e = (p_{e\perp}, p_{e\parallel})$, $f_e = f_e(p_{e\perp}^2, p_{e\parallel}^2)$ and $\mathbf{D}$ is the symmetrical tensor of the second order. Supposing that $\Omega_e \gg \Omega_i \to 0$ the components of $\mathbf{D}$ have the form

$$D_{\perp\perp} = -(2\pi)^{-3}\int d\mathbf{p}_i \int d\mathbf{k} \int_{-\infty}^{0} d\xi |\Phi(k)|^2 k_\perp^2 \cos(\varphi-\alpha)\times$$
$$\times \cos(\varphi-\alpha-\Omega_e\xi)\exp(i\mathbf{k}\cdot\Delta \mathbf{r}_{ei}) f_i^M(p_{i\perp}^2, p_{i\parallel}^2), \tag{2}$$

$$D_{\perp\parallel} = -(2\pi)^{-3}\int d\mathbf{p}_i \int d\mathbf{k} \int_{-\infty}^{0} d\xi |\Phi(k)|^2 k_\perp k_\parallel \cos(\varphi-\alpha)\times$$
$$\times \exp(i\mathbf{k}\cdot\Delta \mathbf{r}_{ei}) f_i^M(p_{i\perp}^2, p_{i\parallel}^2), \tag{3}$$

$$D_{\parallel\parallel} = -(2\pi)^{-3}\int d\mathbf{p}_i \int d\mathbf{k} \int_{-\infty}^{0} d\xi |\Phi(k)|^2 k_\parallel^2 \times$$
$$\times \exp(i\mathbf{k}\cdot\Delta \mathbf{r}_{ei}) f_i^M(p_{i\perp}^2, p_{i\parallel}^2), \tag{4}$$

where $\Phi(k)$ is the Fourier transform of the electron-ion interaction potential

$$\mathbf{k}\cdot\Delta\mathbf{r}_{ei} = \frac{k_\perp p_{e\perp}}{m_e\Omega_e}[\sin(\Omega_e\xi+\alpha-\varphi)+\sin(\varphi-\alpha)] - \tag{5}$$
$$-\frac{k_\perp p_{i\perp}}{m_i}\xi\cos(\varphi-\beta) + k_\parallel\xi\left(\frac{p_{e\parallel}}{m_e}-\frac{p_{i\parallel}}{m_i}\right)$$

and

$$f_i^M = N(2\pi m_i k_B T_i)^{-3/2}\exp\left(-\frac{p_{i\perp}^2+p_{i\parallel}^2}{2m_i k_B T_i}\right). \tag{6}$$

α, β and φ are the polar angles of $\mathbf{p}_e$, $\mathbf{p}_i$ and $\mathbf{k}$.

The components of $\mathbf{D}$ can be evaluated with help of the Bessel function identity

$$\exp(iz\sin\Psi) = \sum_{l=-\infty}^{+\infty} J_l(z)\exp(il\Psi).$$

After performing the $p_{i\perp}$, $p_{i\parallel}$ and ξ integrations and setting

$$k_\perp = k\sin\Theta,\quad k_\parallel = k\cos\Theta \tag{7}$$

one finds:

$$D_{\perp\perp} = -\mathscr{L}\left(\frac{m_e\Omega_e}{p_{e\perp}}\right)^2\int_0^\pi d\Theta\int_0^{k_0} dk\,k|\Phi(k)|^2\sin\Theta\sum_{n=-\infty}^{+\infty} n^2 A_n, \tag{8}$$

$$D_{\perp\parallel} = -\mathscr{L}\frac{m_e\Omega_e}{p_{e\perp}}\int_0^\pi d\Theta\int_0^{k_0} dk\,k^2|\Phi(k)|^2\sin\Theta\cos\Theta\sum_{n=-\infty}^{+\infty} nA_n, \tag{9}$$

$$D_{\parallel\parallel} = -\mathscr{L}\int_0^\pi d\Theta\int_0^{k_0} dk\,k^3|\Phi(k)|^2\sin\Theta\cos^2\Theta\sum_{n=-\infty}^{+\infty} A_n, \tag{10}$$

where

$$\mathscr{L} = \frac{N}{4\pi}\left(\frac{m_i}{2\pi k_B T_i}\right)^{1/2}, \tag{11}$$

$$A_n = J_n^2\left(\frac{kp_{e\perp}}{m_e\Omega_e}\sin\Theta\right)\exp\left[-\frac{m_i}{2k_B T_i}\left(n\frac{\Omega_e}{k}+\frac{p_{e\parallel}}{m_e}\cos\Theta\right)^2\right]. \tag{12}$$

Here we write $\int d\mathbf{k} \to \int_0^{k_0} dk\,k\int_0^\pi d\Theta$ to avoid a divergence of $\mathbf{D}$ for a large value of k (k_0 beig the inverse distance of the closest approach).

III. REDUCTION OF THE D COMPONETS

We restrict our attention to the strong magnetic field when $kp_{e\perp}/m_e\Omega_e < 1$. Using the asymptotic expansion for $J_n^2(x)$ [5] and neglecting the terms $0(x^2)$ we get after some manipulation:

$$D_{\perp\perp} = -\mathscr{L}\sum_{m=0}^{\infty}\frac{2^{2m-1}}{(2m)!}a^{-1}\left[\gamma\left(m+\frac{1}{2}, a^2\right) - a^{-2}\gamma\left(m+\frac{3}{2}, a^2\right)\right]B_m, \tag{13}$$

$$D_{\perp\parallel} = +\mathscr{L}\frac{p_{e\perp}}{m_e v_T}\sum_{m=0}^{\infty}\frac{2^{2m}}{(2m+1)!}a^{-2}\left[\gamma\left(m+\frac{3}{2}, a^2\right) - a^{-2}\gamma\left(m+\frac{5}{2}, a^2\right)\right]B_m, \tag{14}$$

$$D_{\parallel\parallel} = -\mathscr{L}a^{-3}\gamma\left(\frac{3}{2}, a^2\right)\int_0^{k_0} dk\,k^3|\Phi(k)|^2, \tag{15}$$

where $\gamma(\alpha, z)$ is the incomplete Gamma function [5], $v_T = (2k_B T_i/m_i)^{1/2}$

$$B_m = \int_0^{k_0} dk\,k^3|\Phi(k)|^2 b^{2m}\exp(-b^2), \tag{16}$$

and

$$a = \frac{p_{e\parallel}}{m_e v_T},\quad b = \frac{\Omega_e}{v_T k}. \tag{17}$$

To evaluate the integral B_m we shall further suppose that $\Phi(k)$ is of the Debye form (k_D being the inverse shielding length). The analytical evaluation of B_m shows that for the case of the strong magnetic field when

$$1 < \frac{\Omega_e}{\omega_{pi}\sqrt{2}} = \eta < \lambda_0 = \frac{k_0}{k_D}, \tag{18}$$

the dominant term is

$$B_0 \cong \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \ln \frac{\lambda_0}{\eta}, \tag{19}$$

while $B_m \cong \exp(-\eta^2)$, $m>0$. With logarithmical accuracy we can approximate the sum in Eqs. (13) and (14) with the first term ($m=0$) only. (The k integration in Eq. (15) gives with the same logarithmical accuracy $\ln \lambda_0$, i.e. the Coulomb logarithm). For the $\boldsymbol{D}$ components we then obtain:

$$D_{\perp\perp} = -\mathscr{L} \ln(\lambda_0/\eta) \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \frac{1}{2} a^{-1} \left[\gamma\left(\frac{1}{2}, a^2\right) - a^{-2}\gamma\left(\frac{3}{2}, a^2\right)\right] =$$

$$= -\mathscr{L} \ln(\lambda_0/\eta) \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \left[\frac{\pi^{1/2}}{2}\left(a^{-1} - \frac{a^{-3}}{2}\right) \operatorname{erf} a + \frac{a^{-2}}{2} e^{-a^2}\right], \tag{20}$$

$$D_{\perp\parallel} = +\mathscr{L} \ln(\lambda_0/\eta) \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \frac{p_{e\perp}}{2m_e v_T} a^{-2} \left[\gamma\left(\frac{3}{2}, a^2\right) - a^{-2}\gamma\left(\frac{5}{2}, a^2\right)\right] =$$

$$= -\mathscr{L} \ln(\lambda_0/\eta) \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \frac{p_{e\perp}}{2m_e v_T} \left[\frac{\pi^{1/2}}{2}\left(a^{-2} - \frac{3}{2} a^{-4}\right) \operatorname{erf} a + \frac{3}{2} a^{-3} e^{-a^2}\right], \tag{21}$$

$$D_{\parallel\parallel} = -\mathscr{L} \ln \lambda_0 \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 a^{-3} \gamma\left(\frac{3}{2}, a^2\right) =$$

$$= -\mathscr{L} \ln \lambda_0 \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \left[\frac{\pi^{1/2}}{2} a^{-3} \operatorname{erf} a - a^{-2} e^{-a^2}\right]. \tag{22}$$

Here the $\gamma(\alpha, z)$ function is expressed by the error function $\operatorname{erf}(z)$.

The $\boldsymbol{D}$ components can be further simplified using the assymptotic expansions of $\operatorname{erf}(z)$. We obtain:

1) for $p_{e\parallel}^2 < \left(\frac{m_e}{m_i}\right)^2 2k_B T_i m_i = (m_e v_T)^2$

$$D_{\perp\perp} \cong -\mathscr{L}^{e,i} \ln(\lambda_0/\eta) \pi^{-1/2} (m_e v_T)^{-1} \frac{2}{3}\left[1 + \frac{4}{5}\left(\frac{p_{e\parallel}}{m_e v_T}\right)^2\right], \tag{23}$$

$$D_{\perp\parallel} \cong +\mathscr{L}^{e,i} \ln(\lambda_0/\eta) \pi^{-1/2} (m_e v_T)^{-1} \frac{4}{15} \frac{p_{e\perp} p_{e\parallel}}{(m_e v_T)^2}, \tag{24}$$

$$D_{\parallel\parallel} \cong -\mathscr{L}^{e,i} \ln \lambda_0 \pi^{-1/2} (m_e v_T)^{-1} \frac{2}{3}\left[1 + \frac{2}{5}\left(\frac{p_{e\parallel}}{m_e v_T}\right)^2\right]; \tag{25}$$

2) for $p_{e\parallel}^2 > \left(\frac{m_e}{m_i}\right)^2 2k_B T_i m_i$

$$D_{\perp\perp} \cong -\mathscr{L}^{e,i} \ln(\lambda_0/\eta)(2p_{e\parallel})^{-1}\left[1 - \frac{1}{2}\left(\frac{m_e v_T}{p_{e\parallel}}\right)^2\right], \tag{26}$$

$$D_{\perp\parallel} \cong +\mathscr{L}^{e,i} \ln(\lambda_0/\eta) p_{e\perp}(2p_{e\parallel})^{-2}\left[1 - \frac{3}{2}\left(\frac{m_e v_T}{p_{e\parallel}}\right)^2\right], \tag{27}$$

$$D_{\parallel\parallel} \cong -\mathscr{L}^{e,i} \ln \lambda_0 \frac{(m_e v_T)^2}{2p_{e\parallel}^3}. \tag{28}$$

Here

$$\mathscr{L}^{e,i} = \left(\frac{Q_e Q_i}{\varepsilon_0}\right)^2 \frac{N m_e}{4\pi}. \tag{29}$$

From the above equations it is evident that the $\boldsymbol{D}$ tensor is anisotropical even if the plasma is homogeneous. The anisotropy describing the parameter is $\ln(\lambda_0/\eta)$ and because $1 < \eta < \lambda_0$ it strongly influences the $D_{\perp\perp}$ and $D_{\perp\parallel}$ components especially in the case of a high magnetic field.

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