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ON PATH INTEGRALS FOR SOME LAGRANGIANS WHICH ARE NOT QUADRATIC IN VELOCITIES

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Some speculations on the path integral formulation of quantum mechanics for some lagrangians which are not quadratic in velocities are presented.

ОБ ИНТЕГРАЛАХ ПО ТРАЕКТОРИЯМ ДЛЯ НЕКОТОРЫХ ЛАГРАНЖИАНОВ С НЕКВАДРАТИЧНОЙ ЗАВИСИМОСТЬЮ ОТ СКОРОСТЕЙ

В работе приведены некоторые соображения о формулировке квантовой механики в рамках интегралов по траекториям для лагранжианов, которые не зависят кваратично от скоростей.

I. INTRODUCTION

The probability amplitude $K(q, t; q_0, t_0)$ of the transition of the system from (q_0, t_0) to $(q, t>t_0)$ is one of the basic notions of Feynman's approach in quantum mechanics. In accordance with [1, 2] this amplitude can be formally written in the following form

$$K(q, t; q_0, t_0)=\int \mathrm{d}[q(t')] \exp \frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \mathrm{d}t' L(\dot{q}(t'), q(t'), t'), \tag{1}$$

where $\dot{q}(t)=\mathrm{d}q/\mathrm{d}t$ and L is the lagrangian of the system under consideration. The right-hand side of Eq. (1) is intuitively interpreted as the continual integral over all trajectories $q(t')$ satisfying $q(t_0)=q_0$, $q(t)=q$. The naive but very transparent definition of this integral is the following

$$K=\lim_{\varepsilon\to 0} \int \ldots \int \frac{1}{B_N}\left[\prod_{n=1}^{N-1} \frac{\mathrm{d}q_n}{B_n}\right] \times \\ \times \exp \frac{\mathrm{i}\varepsilon}{\hbar} \sum_{n=1}^{N} L\left(\frac{q_n-q_{n-1}}{\varepsilon}, \frac{q_n-q_{n-1}}{2}, t_0+n\varepsilon\right), \tag{2}$$

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where $\varepsilon=(t-t_0)/N$, $q_N=q$ and B_n's are suitable factors. If $L=m\dot{q}^2/2-V(q,t)$, then $B_n=(2\pi\, i\hbar\varepsilon/m)^{1/2}$ (we consider a onedimensional case only). As to the determination of B_n's we do not know yet recept how to solve this problem for lagrangians which are not quadratic in velocities. We think that it is not necessary to advocate the solution of this problem by means of some "deep physical motives". Besides, we believe that from the point of view of the completness of Feynman's approach and a wider outlook on the path integral formalism, the determination of B_n's represents an interesting problem.

In this paper we shall deal with the path integrals for lagrangians of the type

$$L=\frac{m\dot{q}^{\alpha+1}}{\alpha+1} \tag{3}$$

where $(\alpha+1)/\alpha=2k$, $k=1,2,3,\ldots$ and m is a constant. Though this class of lagrangians does not contain the physically interesting case $L=-m(1-\dot{q}^2)^{1/2}$, we think that the presented speculations are a good illustration of the possible solution of the above mentioned problem.

II. PATH INTEGRALS

According to Eq. (2) the amplitude

$$A=\frac{1}{B_1}\exp\frac{i\varepsilon}{\hbar}L\left(\frac{q_1-q_0}{\varepsilon},\ \frac{q_1+q_0}{2},\ t_0+\varepsilon\right) \tag{4}$$

is approximately equal to the short-time propagator $K(q_1,t_0+\varepsilon;q_0,t_0)$. Hence the determination of B_1 is equivalent to the determination of the short-time propagator. We shall write for the lagrangians (3)

$$A=F(\varepsilon,q_1-q_0)\exp\frac{im(q_1-q_0)^{\alpha+1}}{\hbar\varepsilon^{\alpha}(\alpha+1)}\cong K(q_1,t_0+\varepsilon;q_0,t_0), \tag{5}$$

where we assume that a suitable approximate F can be of the form

$$F(\varepsilon,q_1-q_0)=\sum_{j=0}^{N}f_j(\varepsilon)(q_1-q_0)^{j+\beta}. \tag{6}$$

The functions $f_j(\varepsilon)$ and the parameters β, N have to be determined by means of some natural requirements. As to the assumption (6) we shall show that this simple form of F allows us to obtain the equivalency between Feynman's and Schrödinger's formulation of quantum mechanics.

Let us now investigate the consequences for f_j, β, N following from the above mentioned equivalency. The wave function of the system can be defined by

$$\psi(q,t)=\int dq_0K(q,t;q_0,t_0)\psi(q_0,t_0). \tag{7}$$

The equivalency between Feynman's and Schrödinger's approach requires

$$\int dq_0K(q,t+\varepsilon;q_0,t)\psi(q_0,t)=\exp\left(-\frac{i\varepsilon H}{\hbar}\right)\psi(q,t), \tag{8}$$

where $H=(\alpha/\alpha+1)m^{-1/\alpha}(-i\hbar\partial/\partial q)^{(\alpha+1)/\alpha}$. Since we do not know the exact K it is sufficient to choose $F_j(\varepsilon)$, N and β in such a way as to have

$$\int dq_0K(q,t+\varepsilon;q_0,t)\psi(q_0,t)=\psi(q,t)-\frac{i\varepsilon H}{\hbar}\psi(q,t)+0(\varepsilon^{\gamma},\gamma>1). \tag{9}$$

Let us rearrange the left-hand side of Eq. (9) in the form of the right-hand side. We have

$$\psi(q,t+\varepsilon)=\int_{-\infty}^{+\infty}dy\exp\left[\frac{imy^{\alpha+1}}{\hbar(\alpha+1)\varepsilon^{\alpha}}\right]\left[\sum_{j=0}^{N}f_j(\varepsilon)y^{\beta+j}\right]\times$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}y^n(\partial/\partial q)^n\psi(q,t)=\sum_{j=0}^{N}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\times$$
$$\times f_j(\varepsilon)\varepsilon^{\alpha(j+n+1+\beta)/(\alpha+1)}I(j+n,\beta)(\partial/\partial q)^n\psi(q,t), \tag{10}$$

where

$$I(j+n,\beta)=\int_{-\infty}^{+\infty}dy\,y^{j+n+\beta}\exp\frac{imy^{\alpha+1}}{\hbar(\alpha+1)}.$$

Now we put $N=2k=(\alpha+1)/\alpha$ and

$$\sum_{j=0}^{N}g_jI(j,\beta)=1 \tag{11}$$

$$\sum_{j=0}^{N}g_jI(j+n,\beta)=0\quad\text{for}\quad n=1,2,\ldots,2k-1$$

$$\sum_{j=0}^{N}g_jI(j+2k,\beta)=\frac{(2k-1)!(-i)}{\hbar m^{1/\alpha}}(-i\hbar)^{2k}\frac{\alpha}{\alpha+1},$$

where we put $g_j=f_j(\varepsilon)\varepsilon^{\alpha(j+\beta+1)/(\alpha+1)}$.

If for a certain β the solution to Eqs. (11) exists, then $f_j\sim\varepsilon^{-\alpha(j+\beta+1)/(\alpha+1)}$ and the coefficients in $(\partial/\partial q)^n\psi$ are proportional to $\varepsilon^{n/2k}$. Hence, the relation (9) holds.

To illustrate this situation let us consider the case $\alpha=1$. If we look for the solution to Eqs. (11) for $N=2$ and $\beta=0$, we obtain $f_0=(m/2\pi\, i\hbar\varepsilon)^{1/2}$, $f_1=f_2=0$. If we put $\beta=1$, then we have no solution to f_j and if $\beta=2$, then $f_0=(m/2\pi\, i\hbar\varepsilon)^{1/2}(2m/i\hbar\varepsilon)$, $f_1=0$, $f_2=(m/2\pi\, i\hbar\varepsilon)^{1/2}(m^2/3\hbar^2\varepsilon^2)$. These results are not surprising. In the case of $\beta=0$ we obtain the exact propagator and for $\beta=2$ we have a good approximation (in the sense of Eq. (9)) to the exact propagator. Hence Eqs. (11) can have the solution for several values of the parameter β. In a general case it is difficult to decide what value of β is or is not appropriate from a certain point of view (e.g. the simplicity of the formalism).

III. CONCLUSION

Our speculations indicate that for lagrangians of the type (3) the "measure" $\mathrm{d}[q(t)]$ can be formally written in the form

$$\mathrm{d}[q(t)] = \lim_{\varepsilon \to 0} F(\varepsilon, q - q_{N-1}) \prod_{n=1}^{N-1} F(\varepsilon, q_n - q_{n-1})\, \mathrm{d}q_n,$$

where the function F can be determined from the requirement of the equivalency between the path integral approach in quantum mechanics and the Schrödinger one. The determination of F is not unambiguous by means of the procedure outlined in Sec. II because any shorttime propagator satisfying Eq. (9) is suitable for the path integral formulation. Our approach misses the proof that for given $\alpha = 1/2k - 1$, $k = 1, 2, \ldots$ there is such β for which the solution to Eqs. (11) exists.

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