acta phys. slov. 29. (1979), No. 4

THE NEW PATH INTEGRALS REPRESENTATION OF A POTENTIAL SCATTERING AMPLITUDE

VILIAM PAŽMA*, Bratislava

The new path integrals representation of a potential scattering amplitude of a particle with the spin $s = 1/2$ or $s = 0$ is found. The procedure is illustrated by simple examples.

НОВОЕ ПРЕДСТАВЛЕНИЕ АМПЛИТУДЫ ПОТЕНЦИАЛЬНОГО РАССЕЯНИЯ С ПОМОЩЮ ИНТЕГРАЛОВ ПО ТРАЕКТОРИИ

В предлагаеммой работе был найден новый способ изображения амплитуды потенциального рассеяния для частиц имеющих спин $c = 1/2$ и $c = 0$ с помощю набора интегралов по траектории. Использованная процедура показана для некоторых простых случаев.

I. INTRODUCTION AND METHOD

In our previous paper [1] we presented a formulation of relativistic quantum echanics of a particle with the spin $s = 1/2$ or $s = 0$ in terms of path integrals. In e above mentioned formulation the propagation function (a retarded Green nction) of a particle interacting with the potential $V(\boldsymbol{x},t)$ is given by $(c = 1)$

$$K(\boldsymbol{x}, T; \boldsymbol{x}_0, 0) = \Theta(T) \lim_{\tau \to 0} \prod_{k=1}^{N} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sqrt{m_k}}{C_k} \left| \prod_{k=1}^{N-1} \int \frac{\mathrm{d}^3 \boldsymbol{x}_k}{B_k} \right| \frac{1}{B_N} \lambda(N)\lambda(N-1)\ldots$$

$$\ldots\lambda(2)\lambda(1) \exp \frac{\mathrm{i}}{\hbar} \left| \frac{m_k \boldsymbol{a}_k^2}{2\tau} - \frac{\tau}{2}\left(m_k + \frac{M_0^2}{m_k}\right) - \tau V(\boldsymbol{x}_k, k\tau) \right|, \tag{1}$$

here $\tau = T/N$, $C_k = (\pi\hbar/2\mathrm{i}\tau)^{1/2}$, $B_k = (2\pi\mathrm{i}\hbar\tau/m_k)^{3/2}$, $\boldsymbol{a}_k = \boldsymbol{x}_k - \boldsymbol{x}_{k-1}$, $\boldsymbol{x}_N = \boldsymbol{x}$, $\Theta(T)$ the step function, M_0 is the rest mass of a particle and

$$\lambda(k) = \frac{1}{2}\left(1 + \frac{\beta M_0}{m_k} + \frac{\alpha \boldsymbol{a}_k}{\tau}\right) \quad \text{for the spin } s = 1/2 \tag{2}$$

* Ústav fyziky a biofyziky UK, Mlynská dolina, CS-816 31 BRATISLAVA.

$$\lambda(k)=\frac{1}{2}\left|\frac{\beta_0 M_0}{m_k}+\frac{\alpha_0}{2M_0}\left(\frac{m_k a_k^2}{\tau^2}-\frac{3i\hbar}{\tau}\right)\right| \quad \text{for } s=0, \tag{3}$$

where $\boldsymbol{\alpha}$, β are Dirac matrices and α_0, β_0 are (2×2) matrices occurring in the Feshbach—Villars representation of the Klein—Gordon equation.

The right-hand side of the Eq. (1) can be formally interpreted as the continual integral over all trajectories $(\boldsymbol{x}(t), m(t))$ $(\boldsymbol{x}_k=\boldsymbol{x}(k\tau),\ m_k=m(k\tau))$ which connect the points $(\boldsymbol{x}_0=\boldsymbol{x}(0),\ t=0)$ and $(\boldsymbol{x}=\boldsymbol{x}(T),\ t=T)$.

In what follows we put $\boldsymbol{x}(t)=\boldsymbol{x}_0+\int_0^t \mathrm{d}\xi \boldsymbol{v}(\xi)$ and we shall write K in the form

$$K=\Theta(T)\int \mathrm{d}[m(\xi),\boldsymbol{v}(\xi)]\delta\left(\boldsymbol{x}-\boldsymbol{x}_0-\int_0^T \mathrm{d}\xi \boldsymbol{v}(\xi)\right)\prod_\xi \lambda(\xi)\times \tag{4}$$
$$\times \exp\frac{i}{\hbar}\int_0^T \mathrm{d}\xi\left|\frac{mv^2}{2}-\frac{1}{2}\left(m+\frac{m_0^2}{m}\right)-V\left(\boldsymbol{x}_0+\int_0^\xi \mathrm{d}\xi' \boldsymbol{v}(\xi'),\xi\right)\right|,$$

where

$$\int \mathrm{d}[m(\xi),\boldsymbol{v}(\xi)] \rightarrow \prod_{k=1}^{N}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\sqrt{m_k}}{\left(\frac{\pi\hbar}{2i\tau}\right)^{1/2}}\int\frac{\mathrm{d}^3 v_k}{\left(\frac{2\pi i\hbar}{\tau m_k}\right)^{3/2}}.$$

II. SCATTERING AMPLITUDE

Let us now consider a particle with the spin $s=1/2$ which is scattered by the potential $V(\boldsymbol{x},t)$. The solutions of the Dirac equation corresponding to a free particle are chosen in the form (we do not write explicitly all quantum numbers)

$$\Psi_{\boldsymbol{p}}(\boldsymbol{x},t)=(2\pi\hbar)^{-3/2}u(\boldsymbol{p})\exp\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x}-Et),$$

where

$$Eu(\boldsymbol{p})=(\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta M_0)u(\boldsymbol{p}).$$

If before the scattering the particle was described by the wave function $\Psi_{\boldsymbol{p}_i}$, then at the time $T\rightarrow\infty$ the wave function of a particle can be formally written as (we assume $V(\boldsymbol{x},t)\rightarrow 0$ if $|\boldsymbol{x}|$, $|t|\rightarrow\infty$)

$$\Psi(\boldsymbol{x},T\rightarrow\infty)=\sum_{\boldsymbol{p}} C(\boldsymbol{p},\boldsymbol{p}_i;V)\Psi_{\boldsymbol{p}}(\boldsymbol{x},T\rightarrow\infty)=\lim_{T\rightarrow\infty}\int \mathrm{d}^3\boldsymbol{x}_0 K(\boldsymbol{x},T;\boldsymbol{x}_0,-T)\Psi_{\boldsymbol{p}_i}(\boldsymbol{x}_0,-T) \tag{5}$$

and the scattering amplitude M_{fi} is given by

$$M_{fi}=C(\boldsymbol{p}_f,\boldsymbol{p}_i;V)-C(\boldsymbol{p}_f,\boldsymbol{p}_i;V=0). \tag{6}$$

After putting (1) into the Eqs. (5. 6) and

i) performing integration over $\boldsymbol{x}_0$

ii) using the identity $e^V-1=V\int_0^1 \mathrm{d}\lambda e^{\lambda V}$

iii) performing the substitution $\boldsymbol{v}=\boldsymbol{\omega}+\boldsymbol{p}_i/m$

we obtain

$$M_{fi}=\left(-\frac{i}{\hbar}\right)\lim_{T\rightarrow\infty}\exp\frac{i}{\hbar}(E_f+E_i)T\cdot u^+(\boldsymbol{p}_f)\int\frac{\mathrm{d}^3x}{(2\pi\hbar)^3}\exp\left(-\frac{i}{\hbar}\boldsymbol{k}\cdot\boldsymbol{x}\right)\int_0^1 \mathrm{d}\lambda\times \tag{7}$$
$$\times\int \mathrm{d}[m,\boldsymbol{\omega}]\prod_\xi\frac{1}{2}\left(1+\frac{\hat{H}_i}{m}+\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\right)u(\boldsymbol{p}_i)\int_{-T}^{+T}\mathrm{d}\eta V\left(\boldsymbol{x}-\boldsymbol{p}_i\int_\eta^T \mathrm{d}\eta'/m-\right.$$
$$\left.-\int_\eta^T \mathrm{d}\eta'\boldsymbol{\omega},\eta\right)\exp\frac{i}{\hbar}\int_{-T}^{+T}\mathrm{d}\xi\left|\frac{m\omega^2}{2}-\frac{1}{2}\left(m+\frac{E_i}{m}\right)-\lambda V\left(\boldsymbol{x}-\boldsymbol{p}_i\int_\xi^T \mathrm{d}\xi'/m-\right.\right.$$
$$\left.\left.-\int_\xi^T \mathrm{d}\xi'\boldsymbol{\omega},\xi\right)\right|,$$

Where $\hat{H}_i=\boldsymbol{\alpha}\cdot\boldsymbol{p}_i+\beta M_0$, $\boldsymbol{k}=\boldsymbol{p}_f-\boldsymbol{p}_i$, $E_{f,i}=(\boldsymbol{p}_{f,i}^2+M_0^2)^{1/2}$ and u^+ means hermitian conjugated u.

If again we successively perform the substitutions

i) $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{p}_i\int_\eta^T \mathrm{d}\eta'/m-\int_\eta^T \mathrm{d}\eta'\boldsymbol{\omega}$

ii) $\boldsymbol{\omega}=\ \cdot+\Theta(\xi-\eta)\boldsymbol{k}/m$

iii) $\xi=s+\eta$,

then the expression (7) can be written in the following form

$$M_{fi}=\left(-\frac{i}{\hbar}\right)\lim_{T\rightarrow\infty}\exp\frac{i}{\hbar}(E_f+E_i)T\,u^+(\boldsymbol{p}_f)\int_{-T}^{+T}\mathrm{d}\eta\int\frac{\mathrm{d}^3y}{(2\pi\hbar)^3}V(\boldsymbol{y},\eta)\times \tag{8}$$
$$\times\exp\left(-\frac{i\boldsymbol{k}\boldsymbol{y}}{\hbar}\right)\int_0^1 \mathrm{d}\lambda M_f(\boldsymbol{y},\cdot,T,\eta)M_i(\boldsymbol{y},\ ,T,\eta)u(\boldsymbol{p}_i),$$

where

$$M_f=\int \mathrm{d}[m,\ \cdot]\prod_s\frac{1}{2}\left(1+\frac{\hat{H}_f}{m}+\boldsymbol{\alpha}\cdot\ \cdot\right)\exp\frac{i}{\hbar}\int_0^{T-\eta}\mathrm{d}s\left|\frac{m\ \ ^2}{2}-\right. \tag{9}$$
$$\left.-\frac{1}{2}\left(m+\frac{E_f^2}{m}\right)-\lambda V\left(\boldsymbol{y}+\boldsymbol{p}_f\int_0^s \mathrm{d}s'/m+\int_0^s \mathrm{d}s'\boldsymbol{\omega},s+\eta\right)\right|$$

and we obtain M_i from M_f by means of interchanges $f\rightarrow i$ and $\int_0^{T-\eta}\mathrm{d}s\rightarrow\int_{-T-\eta}^{0}\mathrm{d}s$.

In our opinion the expressions (8) and (9) offer a very simple and transparent picture of potential scattering and represent an unconventional approach to the scattering problems in relativistic quantum mechanics.

The above outlined procedure can be repeated without any difficulties for the case $s = 0$.

III. SIMPLE APPLICATIONS

If a particle moves almost as a free one, i.e. the potential is a sufficiently smooth function, then the part of action corresponding to a free particle is dominant. In this case only a small part of the vicinity of the "trajectory" $v = 0$, $m = E_f$ substantially contributes to M_f and we can write

$$M_f = \int d[m, v] \prod_s \frac{1}{2}\left(1+\frac{\hat{H}_f}{m}\right) \exp \frac{i}{\hbar}\int_0^{T-\eta} ds \left[\frac{mv^2}{2} - \frac{1}{2}\left(m+\frac{E_f^2}{m}\right) - \lambda V(y+sv_f, s+\eta)\right] = \exp\left(-\frac{i}{\hbar}(T-\eta)\hat{H}_f\right) \times \tag{10}$$

$$\times \exp\left(-\frac{i}{\hbar}\lambda \int_0^{T-\eta} ds V(y+sv_f, s+\eta)\right),$$

where $v_f = p_f/E_f$.

The character of this approximation can be better understood by means of the following consideration. Let us put $s = \xi/E_f$, $m = \mu E_f$. There holds

$$d|m, v| = d|\mu, v|$$

and

$$M_f = \int d[\mu, v] \prod_\xi \frac{1}{2}\left(1+\frac{\hat{H}_f}{\mu E_f}+\alpha \cdot \cdot\right) \exp \frac{i}{\hbar}\int_0^{(T-\eta)E_f} d\xi \left[\frac{\mu v^2}{2} - \frac{1}{2}\left(\mu+\frac{1}{\mu}\right)\right] \times$$

$$\times \exp\left(-\frac{i\lambda}{\hbar}\right)\int_0^{(T-\eta)E_f} \frac{d\xi}{E_f} V\left(y+v_f\frac{\xi}{E_f}+\frac{v_f}{E_f}d\xi'\left(\frac{1}{\mu}-1\right)+\right.$$

$$\left.+\frac{1}{E_f}\int_0^\xi d\xi' \cdot, \xi/E_f+\eta\right).$$

One can easily see that in the limit $E_f \to \infty$ the expression (10) is a good approximation of M_f.

The next example is a good illustration of the above mentioned approximation. If

$$V(x, t) = -x \cdot E(t)\Theta(T_0 - |t|),$$

then the integration in question can be easily performed and we have

$$M_f = \exp\left(\frac{i\lambda}{\hbar} y \cdot W^{(+)}(0, \eta)\right) T \exp\left(-\frac{i}{\hbar}\int_0^{T_0-\eta} ds[\hat{H}_f - \lambda \alpha W^{(+)}(s, \eta)]\right),$$

where T means the T-product of operators and

$$W^{(+)}(s, \eta) = \int_s^{T_0-\eta} ds' E(s'+\eta).$$

Using the formalism of disentangling operators elaborated by Feynman [2] we obtain

$$M_f = \exp\frac{i\lambda}{\hbar} y \cdot W^{(+)}(0, \eta) \exp\left(-\frac{i}{\hbar}(T_0-\eta)H_f\right) T \exp\frac{i\lambda}{\hbar}\int_0^{T_0-\eta} ds \times \tag{11}$$

$$\times \left[W^{(+)} p_f \hat{H}_f^{-1} + W^{(+)}\left(\alpha - p_f \hat{H}_f \exp\left(-\frac{2is\hat{H}_f}{\hbar}\right)\right)\right]$$

The second term in the exponent of the last term on the right-hand side of the Eq. (11) represents the contribution of the velocity of the so-called trembling motion. if the kinetic energy of a particle is sufficiently large, then this term is a quickly oscillating one and its contribution to M_f can be neglected. Thus we have the result following from the Eq. (10) directly.

IV. CONCLUSION

So far we have been able to calculate only some simple continual integrals and we do not know any effective method of their approximation. Despite this fact we conjecture that our results offer a simple and clear picture of the potential scattering and make clearer the procedure of calculation as well.

We are grateful to Dr. M. Petráš for his useful comments and advices.

REFERENCES

[1] Pažma, V.: *to be publ.* in Czech. J. Phys. (1979).
[2] Feynman, R. P.: Phys. Rev., *84* (1951), 108.

Received April 20th, 1979