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CAN THE LEFT-HAND CUT CONTRIBUTION OF THE SECOND RIEMANN SHEET TO THE PION FORM FACTOR BEHAVIOUR BE NEGLECTED?

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The influence of the left-hand cut on the second Riemann sheet on the pion form factor behaviour is investigated by means of the Padè approximations. By using the present experimental data in the region of the elastic threshold we came to the conclusion that the left-hand cut on the second Riemann sheet is almost of no importance for the explanation of the experimental behaviour of the pion form factor.

МОЖНО ЛИ ПРЕНЕБРЕЧЬ ВКЛАДОМ ОТ ЛЕВОГО РАЗРЕЗА НА ВТОРОМ ЛИСТЕ РИМАНОВОЙ ПОВЕРХНОСТИ В ПОВЕДЕНИЕ ПИОННОГО ФОРМФАКТОРА?

С помощью Паде приближений исследовано влияние левого разреза на втором листе римановой поверхности на поведение пионного формфактора. На основе использования современных экспериментальных данных для области упругого порога сделан вывод, что левый разрез на втором листе римановой поверхности не имеет почти никакого значения для объяснения экспериментального поведения пионного формфактора.

I. INTRODUCTION

There are many examples in particle physics in which the use of analytic functions has proved to be very useful. One could mention the dispersion relations [1], sum rules (see e.g. Ref. [2]), different bounds on the scattering amplitudes [3] and form factors [4], the exploitation of analyticity in the cos ϑ plane of binary reactions for the extraction of the corresponding coupling constants [5], etc.

The broad practical application of the analyticity in particle physics is caused by the fact that the only singularities which appear in concrete cases are the isolated poles and the branch points, which are rather simply manageable by standard mathematical methods. Moreover, in various approximation schemes only the

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nearby singularities are important and the distant singularities can be ignored. Then the question arises, which of the all singularities have to be considered as the nearby ones and which as the distant ones. Unfortunately, there is no exact criterion for such a choice and the distinction depends on the task under consideration.

In the case of the electromagnetic pion form factor $F_\pi(t)$ (t is a squared four momentum transfer) there are no poles on the physical sheet and the analytic properties of the $F_\pi(t)$ in the whole complex t plane are restricted only to a sequence of threshold branch points at $t=4m_\pi^2$, $16m_\pi^2$, $4m_K^2$, $4m_N^2$, ... on the positive real axis, where m_π, m_K and m_N is the mass of the pion, kaon and nucleon, respectively. The cuts associated with these branch points are chosen to extend to $+\infty$ along the real axis.

As a consequence of the elastic unitarity and the reality condition the singularity at $t=4m_\pi^2$ is a square root branch point. It generates two sheets of a Riemann surface in the complex t variable.

The singularities of the $F_\pi(t)$ on the second Riemann sheet can be reached by the analytic continuation of the $F_\pi(t)$ by means of the elastic unitarity condition through the elastic cut. They consist of the two complex conjugate ϱ meson poles, the left-hand cut for $-\infty<t\leq 0$ (this cut is a consequence of the presence of the $\pi\pi$ scattering amplitude in the unitarity condition) and the right-hand unitarity cut as on the first sheet.

If the left-hand cut on the second Riemann sheet plays some important role at all, then it contributes mainly to the behaviour of the pion form factor in the region of the elastic threshold, to which it is near.

In the construction of $F_\pi(t)$, which possesses all the fundamental properties and describes all the existing experimental data in space-like, as well as in time-like regions simultaneously [6, 7], the a priori assumption was used that the contribution of the left-hand cut on the second Riemann sheet was negligible. However, the ϱ meson parameters obtained are smaller than the values from the Particle Data Group. In our opinion, one could expect an improvement of the situation if the left-hand cut gave some non-negligible contribution to the pion form factor behaviour in the region of the elastic threshold.

In this paper we investigate by using the Padè approximations to what extent the omission of the left-hand cut in Refs. [6, 7] was justified.

II. SOME ARGUMENTS IN FAVOUR OF THE METHOD CHOSEN

In Ref. [8], starting with the phase representation (units $\hbar=c=m_\pi=1$ are used)

$$F_\pi(t)=P_n(t)\exp\left\{\frac{t}{\pi}\int_4^\infty \frac{\delta_1^1(t)}{t'(t'-t)}\,dt'\right\} \tag{1}$$

and using the following energy dependence of the $\pi\pi$ phase shift with a correct threshold and resonant behaviour

$$\mathrm{tg}\,\delta_1^1(t)=\frac{aq^3}{(1+q^2)(q_\varrho^2-q^2)} \tag{2}$$

or, equivalently,

$$\delta_1^1(t)=\frac{1}{2i}\ln\frac{(1+q^2)(q_\varrho^2-q^2)+iaq^3}{(1+q^2)(q_\varrho^2-q^2)-iaq^3}, \tag{3}$$

where q is the pion c.m. momentum, $q_\varrho=\left\{\frac{m_\varrho^2-4}{4}\right\}^{1/2}$ and a is a constant, the authors have found the following normalized pion form factor formula

$$F_\pi(t)=P_n(t)\frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)}\frac{(i+q_1)(i+q_2)(i+q_3)}{(i-q_1)} \tag{4}$$

with q_i ($i=1, \ldots, 4$) being the roots of the numerator of the logarithm in (3).

If the pion is regarded as a quark-antiquark bound state, then by using the dimensional counting [9, 10] one gets the following pion form factor asymptotic behaviour

$$F_\pi(t)\underset{t\to\pm\infty}{\sim}\frac{1}{t}, \tag{5}$$

which automatically requires $P_n(t)$ in (4) to be exactly equal to one. Then the expression (4) takes the form of the [1/3] Padè type approximation to the pion form factor and its analyticity structure consists of two ϱ meson poles at $q=-q_2$, $q=-q_4$, one pole at $q=-q_3$ and one zero at $q=q_1$, both situated on the negative imaginary axis of the q plane, onto which the pion form factor left-hand cut of the second Riemann sheet is mapped.

On the other hand, it is well known [11] that the poles and zeros of the Padè approximations, which are constructed from the coefficients of the Taylor series expansion, represent the singularities of the analytic function in consideration.

It is interesting to examine som simple examples. Consider for example the continued fraction expansion [11] (this is always [N/N] or the [$N\mp 1/N$] Padè approximant) for the function $\frac{1}{z}\ln(1+z)$

$$\frac{1}{z}\ln(1+z)=\cfrac{1}{1+\cfrac{z}{2+\cfrac{z}{3+\cfrac{4z}{4+\cfrac{4z}{5+\cfrac{9z}{\vdots}}}}}} \tag{6}$$

The [0/1] approximant has a pole at $z = -2$, the [1/1] approximant a pole at $z = -3/2$ and a zero at $z = -2$, etc. As the order increases, higher poles and zeros appear, which alternate on $(-\infty, -1)$ in order to represent the logarithmic cut.

In order to illustrate the same feature of the Padè approximants in case of the square rot cuts, let us consider the function $f(z) = \{4 - z^2\}^{1/2}/\{z^2 - 9/4\}$, which has two poles at $z = \pm 3/2$, the right-hand cut for $2 < z < +\infty$ and the left-hand cut for $-\infty < z < -2$. As the order of the diagonal Padè approximants increases, more accurate positions of the poles $z = \pm 3/2$ are obtained, while the other poles and zeros alternate on the intervals $(-\infty, -2)$, $(2, +\infty)$ in order to represent the corresponding cuts. The non-diagonal Padè approximants also reproduce the pole positions quite accurately; however, some of the zeros are placed symmetrically around the cut.

These examples prompt us to understand the pole at $q = -q_3$ and the zero at $q = q_1$ of (4) as an effective approximation of the left-hand cut of the pion form factor on the second Riemann sheet.

However, there is a question, to what extent the appearance of the pole at $q = -q_3$ and the zero at $q = q_1$ on the negative imaginary axis the reflection of the actual behaviour of the pion form factor are at the region of the elastic threshold and to what extent they are a consequence of a special choice of the energy dependence of the $\pi\pi$ phase shift (3), which by the way does not reproduce the experimental data on $\delta^1_1(t)$ too accurately.

To solve this problem, we propose the following method. We start with the Taylor series expansion around the elastic threshold in the q plane with all the fundamental properties of the pion form factor. Then, by the minimalization procedure, we find the optimal minimal number of terms, which describe the data inside the circle of the convergence, from which all the possible Padè approximants are constructed. The Padè approximant with the lowest value of χ^2 will be preferably chosen to give an information about the contribution of the pion form factor left-hand cut on the second Riemann sheeet.

The proposed method will be the subject of the next section.

III. AN ANALYSIS OF THE PION FORM FACTOR DATA IN THE REGION OF THE ELASTIC THRESHOLD

As we have mentioned in the introduction, one can prove the elastic threshold to be a square root branch point. It generates two sheets of the Riemann surface. The left-hand cut of the pion form factor is situated on the second sheet and the experimental data at the region of the elastic threshold are the most nearby pion form factor data, which could be affected by the existence of the cut.

By using the conformal mapping (again units $\hbar = c = m_\pi = 1$ are used)

$$q = \left\{\frac{t-4}{4}\right\}^{1/2} \tag{7}$$

we map the two sheets of the Riemann surface in the t variable onto the q plane and the elastic cut disappears.

Then $F_\pi(t)$ in the q plane can be expanded around the elastic threshold into the Taylor series

$$F_\pi(t) = \sum_{n=0}^{\infty} a_n q^n, \tag{8}$$

the convergence radius of which is determined by the distance from the origin to the first nearby singularity.

If we neglect the four pion cut (the $\pi\pi$ phase shift analysis [12] reveals, that the P-wave isovector inelasticity $\eta^1_1(t)$ starts to be different from one almost at the c.m. energy of 1 GeV), the convergence of (8) is extended nearly to the ϱ meson poles and we may use 57 experimental points from the range of momenta $-0.294\ \mathrm{GeV}^2 \leq t \leq 0.490\ \mathrm{GeV}^2$ for the analysis.

In order to comply with the reality condition $F^*_\pi(t) = F_\pi(t^*)$ the even and the odd coefficients of (8) have to be taken real and purely imaginary, respectively. With regard to this the pion form factor may be written in the following form

$$F_\pi(t) = \sum_{n=0}^{\infty} b_n (\mathrm{i})^n q^n \tag{9}$$

with b_n being real.

Then, imposing the threshold conditions [7]

$$\mathrm{Im}F_\pi(t)|_{q=0} = 0; \quad \left.\frac{\partial\, \mathrm{Im}F_\pi(t)}{\partial q}\right|_{q=0} = 0; \quad \left.\frac{\partial^2\, \mathrm{Im}F_\pi(t)}{\partial q^2}\right|_{q=0} = 0, \tag{10}$$

which are the consequence of the threshold behaviour of the $\pi\pi$ phase shift $\delta^1_1(t) \approx a^1_1 q^3$ (a^1_1 is the $\pi\pi$ scattering length), and imposing the elastic unitarity condition, one gets

$$b_1 = 0. \tag{11}$$

Finally, the normalization condition $F_\pi(0) = 1$ gives

$$b_0 = 1 - \sum_{n=2}^{\infty} b_n (-1)^n. \tag{12}$$

As a result one can write the following approximate parametrization

$$F_\pi(t) = 1 + \sum_{n=2}^{L} b_n [(\mathrm{i})^n q^n - (-1)^n] \tag{13}$$

consistent with the basic principles and suitable for the description of the pion form factor data from the range of momenta — $0.294\ \mathrm{GeV}^2 \leq t \leq 0.490\ \mathrm{GeV}^2$.

Further, by using the last-square method, the optimal degree L of the polynomial (13) will be found, which subsequently is used for the construction of the different Padè approximants.

The results of fitting the aforementioned 57 experimental points by expression (13) are summarized in Table 1.

Table 1

L	χ^2	χ^2/ndf
3	70.7	1.29
4	63.7	1.18
5	54.1	1.02
6	51.0	0.98
7	51.0	1.00

We have chosen the fit with $L=5$ as the optimal one, because in the case of the polynomial with $L=6$ the generated errors of the coefficients are larger than the coefficients themselves.

The values of the corresponding b_n are as follows:

$$\begin{aligned} b_2 &= -0.197860 \pm 0.016531 \\ b_3 &= -0.038254 \pm 0.003991 \\ b_4 &= 0.041677 \pm 0.008659 \\ b_5 &= 0.019305 \pm 0.004294 \end{aligned} \tag{14}$$

The [N/M] Padè approximant to the $F_\pi(t)$ will be constructed from the approximated power series expansion

$$F_\pi(t) = B_0 + B_1 q + B_2 q^2 + B_3 q^3 + B_4 q^4 + B_5 q^5 \tag{15}$$

$$\text{with } \begin{aligned} B_0 &= 1 - b_2 + b_3 - b_4 + b_5 \\ B_1 &= 0 \\ B_2 &= -b_2 \\ B_3 &= -ib_3 \\ B_4 &= b_4 \\ B_5 &= ib_5 \end{aligned} \tag{16}$$

represented as the ratio of two polynomials in q, $P_N(q)$ and $Q_M(q)$ of degree N and M, respectively $(N+M \leq L)$

$$F_\pi(t) = F_\pi^{[N/M]}(q) \equiv \frac{P_N(q)}{Q_M(q)}. \tag{17}$$

The restriction $N+M \leq 5$ allows generally to construct 21 different Padè approximants, each of which gives a different number of zeros and poles.

However, since the pion form factor is dominated by the ϱ meson, one can reduce the 21 Padè approximants to 10 with the polynomials in the denominator of at least the second degree.

The latter are compared with the experimental data from the range of the momenta $-0.294\ \mathrm{GeV}^2 \leq t \leq 0.490\ \mathrm{GeV}^2$ and always the corresponding value of χ^2 is calculated. The results are presented in Table 2, which reveals that the [1/2] Padè approximant gives the best reproduction of the considered data. It generates two complex conjugate ϱ meson poles at $\mathrm{Re}q_\varrho = \pm(2.33 \pm 0.39)$, $\mathrm{Im}q_\varrho = (-0.56 \pm 0.17)$ (to be compared with the Particle data [12] ϱ meson pole position at $q_\varrho = \pm 2.59 - i0.30$) and one zero for $q_0 = -i5.17$. This zero might be interpreted as the Padè approximant zero simulating the left-hand cut on the second Riemann sheet. However, since the zero is rather too distant from the origin and from the data region, one can conclude that the cut, which it simulates, is almost of no importance for the pion form factor behaviour near the elastic threshold, i.e. in the data region considered.

Essentially the same conclusion follows also from the Padè approximants with slightly higher χ^2. The common feature of those Padè approximants is that the generated poles and zeros are far away from the data region. An exception is the [1/4] Padè approximant, which still has not a too bad value of χ^2 and where on the cut position the approximant generates the pole and the zero rather very close to

Table 2

Approximant	χ^2	Positions of poles	Positions of zeros
[0,2]	1359.12	± 2.397	
[0,3]	68.18	$i5.998$ $\pm 2.188 - i0.413$	
[0,4]	97.69	$i4.000,\ -i8.664$ $\pm 2.105 - i0.305$	
[0,5]	84.22	$\pm 1.731 + i2.138$ $-i2.969$ $\pm 1.832 - i0.401$	
[1,2]	žž.83	$\pm 2.332 - i0.556$	$-i5.172$
[1,3]	122.36	$i3.193$ $\pm 2.149 - i0.291$	$i5.274$
[1,4]	77.97	$i5.260,\ -i1.198$ $\pm 2.112 - i0.421$	$-i1.210$
[2,2]	79.05	$\pm 1.970 - i0.390$	$\pm 3.435 - i1.306$
[2,3]	75.81	$-i12.043$ $\pm 1.939 - i0.457$	$\pm 2.771 - i1.610$
[3,2]	76.31	$\pm 1.929 - i0.449$	$i15.897$ $\pm 2.811 - i1.470$

the data region. If these two (the pole and the zero) were more distant from each other it would give a rather strong indication of importance of the examined cut, in contradiction to the conclusion drawn from the previous Padè approximants. However, the pole and the zero are so close to each other that their influences in the Padè approximant are mutually almost entirely canceled out and, again, one is left with the conclusion that the cut on the second Riemann sheet is almost entirely negligible.

Another interesting feature of all the Padè approximants considered is the fact that each of them feels the ϱ meson existence. The calculated positions are quite close (within the error bars) to the position determined from the Particle data table. This phenomenon is even more interesting if we take into account that the input information consisted only of the data from the energy region far below the ϱ meson.

Moreover, the position of the ϱ meson poles, calculated from the various Padè approximants is remarkably stable, when changing the order of the approximants, which gives a good reason to believe in the existence of such poles on the second Riemann sheet.

IV. CONCLUSION

Using the fundamental properties of the pion form factor as well as the regularization of the square root branch point at the elastic threshold by means of the conformal mapping and the successive Taylor expansion around the elastic threshold in the q plane, we have found an approximate parametrization suitable for the description of the pion form factor data in the region of the elastic threshold. By the least square method we have found the optimal minimal number of terms of the corresponding series, from which all the possible Padè approximants with the polynomials in the denominator of at least the second degree were constructed. The [1/2] Padè approximant, which generates two complex conjugate ϱ meson poles and one zero on the negative imaginary axis, gives the best reproduction of the data considered. As the zero is rather too far from the data region, we conclude that the contribution of the left-hand cut on the second Rieman sheet to the pion form factor behaviour is negligible. The same conclusion can be drawn also from the other Padè approximants in Table 2 with slightly higher χ^2.

However, as our conclusion rests on the presently available data, which are rather scarce and not very precise, a more accurate conclusion could be made only if more numerous and more precise data, mainly above the elastic threshold, were available.

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