A METHOD TO DETERMINE THE DIFFERENCE IN Δ^{++} AND Δ^0 POLE POSITIONS

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We propose a method for the determination of $\Delta^{++}-\Delta^0$ pole parameters. The method is free of the ambiguity due to pole-backgroud separation and takes into account all available information about the partial wave amplitudes. The method is tested by two model amplitudes with pole positions known beforehand. We present three different approaches based on the same method and calculate the corresponding pole shifts in model amplitudes.

МЕТОД ОПРЕДЕЛЕНИЯ РАЗНОСТИ МЕСТОПОЛОЖЕНИЯ ПОЛЮСОВ РЕЗОНАНСОВ Δ^{++} И Δ^0

Предлагается метод для определения разности параметров, определяющих полюса резонансов Δ^{++} и Δ^0. Этот метод не содержит неоднозначности, связанной с выделением полюса от фона, и учитывает всю доступную информацию об амплитудах парциальных волн. Метод проверен на двух модельных амплитудах с заранее заданными положениями полюсов. Представлены три различных подхода, основанных на одном и том же методе, и рассчитаны соответствующие сдвиги полюсов в модельных амплитудах.

I. INTRODUCTION

The explanation of mass differences of particles belonging to the same isospin multiplet is one of the most intricate questions in particle physics. The magnitude and the sign of this electromagnetic mass difference are apparently given by the (at present unknown) details of the internal structure of elementary particles.

Electromagnetic mass differences of stable particles are in most cases known quite accurately (typical cases are $m_P - m_N$, $m_{\pi^+} - m_{\pi^0}$ and $m_{K^+} - m_{K^0}$). The situation is much worse in what concerns the electromagnetic mass differences of resonances, like $m_{\Delta^{++}} - m_{\Delta^-}$, $m_{\varrho^+} - m_{\varrho^0}$ etc. Still, such cases may provide a valuable clue to the understanding of the origin of electromagnetic mass differences. Besides, in what concerns the resonance we have an additional piece of information, namely the e.m. differences of widths, e.g. $\Gamma_{\Delta^{++}} - \Gamma_{\Delta^0}$, etc.

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In order to determine the e.m. differences of resonance parameters one needs both accurate data (in fact an accuracy better than 1% is required) and a reliable method for determining the mass and the width of a resonance. The recent, very accurate data of Carter et al. [1] have made it possible to determine separately P_{33} phase shifts in π^+p and π^-p scattering and the errors of the data are so small that the e.m. differences of width and masses of Δ^{++} and Δ^0 can be determined. The simplest way of obtaining the resonance parameters from the data is to fit the amplitude near the point where the phase shift goes through 90° by the Breit-Wigner formula. The results, however, depend on the specific parametrization and in this sense are model dependent. It is thus quite natural that different authors obtain different results. In our opinion it is more advantageous to define the resonance parameters in an unambiguous and model independent way by relating the mass and the width directly to the position of the pole on the unphysical Riemann sheet of the partial wave amplitude (p.w.a.). Such a definition of the resonance parameters is at present becoming generally accepted. To determine the resonance pole on the unphysical sheet is a nontrivial problem since it requires (a tacit or explicit) analytic extrapolation of the p.w.a. onto the unphysical sheet.

A method which is being used for the determination of the mass differences of Δ^0, Δ^{++} resonances consists in fitting the data on each p.w.a. by an expression which has correct analytic properties and a resonance pole on the unphysical sheet. The fitting procedure then gives the parameters of the pole and thereby also the mass and the width of the resonance. This method is undoubted by superior to a simple fitting of the data by a BW formula. Still, even this method is not model independent since the pole position depends to some extent on the parametrization of the part of the p.w.a. which does not contain the pole. We shall refer to this ambiguity as to the problem of pole analytic background separation.

The results by Ball and Vasan [2, 3] in determining the parameters of Δ^{++} and Δ^0 obtained by methods of this type are summarized in Table 1.

It should also be mentioned that such methods make use only of the information about p.w.a. near the resonance and data about the p.w.a. at energies further away are practically neglected.

The aim of this paper is to present a method for the determination of e.m. differences of resonances which is free of the ambiguity due to the "pole" — "analytic background" separation and which takes into account the full available information about the p.w.a. The method will be tested by two model amplitudes with pole positions known beforehand. The two amplitudes may be considered as qualitative models of P_{33} π^+p and π^-p partial wave amplitudes. We present three different approaches based on the same method and calculate the corresponding differences in pole positions. The results should indicate the optimal procedure for calculating the $\Delta^{++}-\Delta^0$ pole parameters. In part II of this paper the method is briefly explained. In part III we construct the model amplitudes and describe three

Table 1

Difference in real and imaginary parts of pole positions of Δ^0 and Δ^{++} resonances

a) Results which were obtained by using resonance formulae.
The third column gives the values $\Gamma/2$.
b) SU(3) prediction

Author	$\mathrm{Re}\Delta^0 - \mathrm{Re}\Delta^{++}$ [MeV]	$\mathrm{Im}\Delta^0 - \mathrm{Im}\Delta^{++}$ [MeV]
[2]	0.	3.
[7][a]	1.	−2.6
[3][a]	0.5	3.1
[8][a]	1.7 ± 0.5	5. ± 1.
[9][b]	0.	3.

possible ways of testing the reliability and accuracy of the method. The concluding part IV contains a summary of the results and some comments.

II. METHOD AND MOTIVATION

The method is based on the statistical approach to the representation of data by analytic functions. So far it has been applied for resonance poles on πN and $\pi\pi$ partial waves. The procedure which has been used can be briefly described as follows.

Suppose that the analytic structure of the given partial wave amplitude $f^{\mathrm{I}}(s)$ in the s-plane is known. Further, we know from experiment the values and errors of

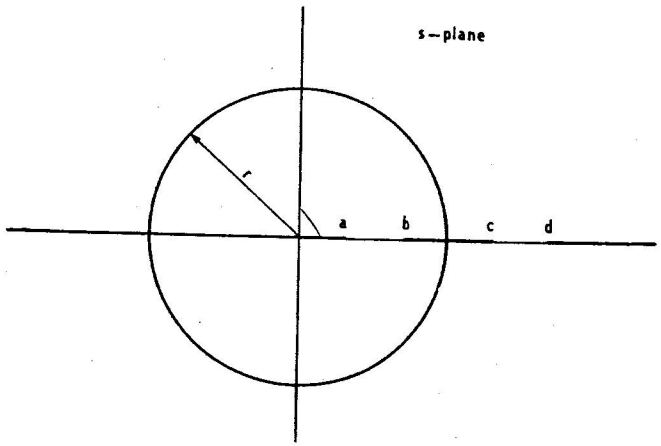

Fig. 1. The analyticity structure of the pion-nucleon p.w.a. on the second sheet. Resonance poles are not indicated. $a=(m_N-m_\pi)^2$, $b=(m_N-(m_\pi^2/m_N))^2$, $c=m_N^2+2m_\pi^2$, $d=(m_N+m_\pi)^2$, $r=m_N^2-m^2$.

p.w.a. along the part of the physical cut and from the theory we have some hypothesis about the behaviour of p.w.a. along the rest of the cuts. The same is true for amplitude $f^{\mathrm{II}}(s)$ on the second Riemann sheet. We use the conformal mapping which brings the part of the second sheet which is exterior to the circle in Fig. 1 into the unit disc. Naturally, the unit circle of this new plane (let us call it the z-plane) will correspond to cuts in the s-plane.

Now we can say that we have a function $f^{\mathrm{II}}(z)$ which is analytic (except for possible poles) inside a unit disc and we know its values and errors along the boundary. We want to test whether such a function has some singularities or not.

If there is a pair of complex conjugate poles, (which would correspond to the presence of a resonance) we can write

$$f^{\mathrm{II}}(z)=\frac{\alpha}{z-\lambda}+\frac{\alpha^*}{z-\lambda^*}+\sum_{n=0}^{\infty}a_n z^n. \tag{1}$$

Expanding the pole terms,

$$f^{\mathrm{II}}(z)=\sum_{n=1}^{\infty}\{\alpha\lambda^{n-1}+\alpha^*\lambda^{*n-1}\}z^{-n}+\sum_{n=0}^{\infty}a_n z^n. \tag{2}$$

Let us define

$$Q_n=\frac{1}{2\pi\,\mathrm{i}}\oint f^{\mathrm{II}}(z)z^{n-1}\,\mathrm{d}z. \tag{3}$$

If we know the function $f^{\mathrm{II}}(z)$ exactly, then for $n\geq 1$ $Q_n=\alpha\lambda^{n-1}+\alpha^*\lambda^{*n-1}$. Since we know the values $f^{\mathrm{II}}(z)$ only with a given accuracy, we can say according to the statistical approach that Q_n are random distributed variables with mean values $\alpha\lambda^{n-1}+\alpha^*\lambda^{*n-1}$ (For more details see e.g. [4, 5]). To make full use of the statistical character of data we use instead of (3) a slightly modified formula

$$Q_n=\frac{1}{2\pi\,\mathrm{i}}\oint\frac{f^{\mathrm{II}}(z)}{w(z)}z^{n-1}\,\mathrm{d}z, \tag{4}$$

where $w(z)$ is a weight function which is analytic and free of zeros inside the unit disc. It is constructed from errors in such a way that

$$|w(z_i)|\approx\varepsilon_i;\quad |z|=1 \tag{5}$$

in points z_i where the errors are given. In those parts of the circle where no data exist, we require that $|w(z)|$ is much higher than in the data region. The values of $w(z)$ in the parts of the boundary not covered by the data are interpreted as errors of the hypothesis about the behaviour of $f^{\mathrm{II}}(z)$ in these regions. Then Q_N are gaussian distributed with mean values the $\alpha\lambda^{n-1}+\alpha^*\lambda^{*n-1}$ and with the dispersion 1 (where $\alpha=\alpha/w(\lambda)$).

We construct the χ^2_N function

$$\chi^2_N=\sum_{1}^{N}|Q_n-\alpha\lambda^{n-1}-\alpha^*\lambda^{*n-1}|^2 \tag{6}$$

and by minimizing (6) we find the parameters α, λ. The method can be simply generalized for the case of a function with m pairs of poles on the unphysical Riemann sheet, i.e. the function

$$f(z)=\sum_1^m\left(\frac{\alpha_i}{z-\lambda_i}+\frac{\alpha_i^*}{z-\lambda_i^*}\right)+\sum_0^\infty a_n z^n.$$

Then, instead of (6) we minimize the expression

$$\chi_N^2=\sum_1^N\left|Q_n-\sum_1^m(\alpha_i\lambda_i^{n-1}+\alpha_i^*\lambda_i^{*n-1})\right|^2. \tag{7}$$

In our case, we have two p.w.a's, $F_1(z)$ and $F_2(z)$, each of which is expected to have the resonance pole corresponding to Δ^{++} and Δ^0, respectively.
The functions $F_1(z)$, $F_2(z)$ are of the form:

$$F_1(z)=\frac{\alpha_1}{z-\lambda_1}+\frac{\alpha_1^*}{z-\lambda_1^*}+\sum_0 a_n^{(1)}z^n$$

$$F_2(z)=\frac{\alpha_2}{z-\lambda_2}+\frac{\alpha_2^*}{z-\lambda_2^*}+\sum_0^\infty a_n^{(2)}z^n.$$

The final aim is to find $\Delta\lambda=\lambda_1-\lambda_2$ and $\Delta\alpha=\alpha_1-\alpha_2$ but in this paper we want first to answer the following question: Is the above method sensitive enough to determine $\Delta\lambda$? In other words: if we construct from the data on π^+p and π^-p the functions $F_1(z)$ and $F_2(z)$, how accurately can we determine $\Delta\lambda=\lambda_1-\lambda_2$? The difference $\Delta\lambda$ can be calculated in two ways: a) First we determine λ_1, λ_2 from the data on F_1 and F_2 separately and get $\Delta\lambda$ simply as $\lambda_2-\lambda_1$. This direct approach has one serious drawback. When computing Q_n's, we know $F_1(z)$ and $F_2(z)$ only along a part of the unit circle. On the remaining part we construct the amplitudes from some hypothesis about p.w.a. High errors in this part of the circle may result in high systematic errors of Q_n coefficients and consequently high errors of λ_1, and λ_2. Thus the final result $\Delta\lambda=\lambda_1-\lambda_2$ may be influenced by large systematic errors. b) We construct the analytic function, which has two pairs of resonance poles:

$$\varphi=F_1-F_2. \tag{8}$$

The functions F_1 and F_2 do not differ considerably on the left-hand cut. When we approximate the function φ on the left-hand cut by zero, then the systematic error of the Q_n coefficients from φ will be much lower compared to the case a) and therefore the result will be more reliable. On the other hand the data have to be very accurate, since the absolute value of φ is small.
Instead of $\varphi(z)$ we can also use the function

$$\tilde{\varphi}(z)=\varphi(z)\,\frac{z-\lambda_1}{1-z\lambda_1^*}\,\frac{z-\lambda_2}{1-z\lambda_2^*}\,\frac{z-\lambda_1^*}{1-z\lambda_1}\,\frac{z-\lambda_2^*}{1-z\lambda_2}, \tag{8a}$$

where the factors $(z-\lambda_i)/(1-\lambda_i^*z)$ are the so-called Blaschke factors. The function $\tilde{\varphi}(z)$ has no poles inside a unit disc. Therefore the corresponding Q_n coefficients are random variables with the mean value zero. We minimize the χ^2, which is of the form

$$\chi_N^2=\sum_1^N|Q_n|^2,$$

where Q_n are functions of λ_1, λ_2.

In the following parts we shall work only with the functions $\varphi(z)$ and $\tilde{\varphi}(z)$, since the separate pole determination as explained in a) is burdened by the large systematic error.

III. EXAMPLE OF A p.w.a. WITH THE RESONANCE POLE

In order to test the method we have worked first with the model p.w.a.'s. These amplitudes were contructed from the Jost functions, which have analytic properties similar (including the pair of complex conjugated poles) those of the p.w.a. on the first and second Riemann sheets. Our aim is to decide, which of the three approachos to be presented in this part is able to reproduce the poles correctly if the "data" are calculated from the model p.w.a.

The model amplitude is of the form

$$F^{\mathrm{I}}(s)=\frac{S(k)-1}{2ik} \tag{9}$$

where

$$S(k)=f(k)/f(-k) \tag{9a}$$

$$f(k)=(k-a-ib)(k+a-ib) \tag{9b}$$

with

$$k(s)=1/2(s-(m_N+m_\pi)^2)^{1/2}, \tag{9c}$$

m_N and m_π are masses of the nucleon and the pion.
The phase in (9c) are chosen so that for real $s>(m_N+m_\pi)^2$ we have $k(s)$ real positive and

$$k(s)=-k^*(s^*).$$

Then the amplitude defined by (9) is real analytic.

The function $F^{\mathrm{I}}(s)$ has the right-hand cut $((m_N+m_\pi)^2, \infty)$.

The second sheet is reached via the elastic unitarity condition

$$F^{\mathrm{II}}(s)=\frac{F^{\mathrm{I}}(s)}{1+2ikF^{\mathrm{I}}(s)}. \tag{10}$$

Here the model amplitude has the pair of complex conjugated poles

$$s_{1,2}=(m_N+m_\pi)^2+4(a^2-b^2)\pm 8iab. \tag{11}$$

In the real situation we use two sets of data constructed from the f_{π^-p} and f_{π^+p} amplitudes. Therefore we use two model functions F_1 and F_2 with the pole parameters λ_1 and $\lambda_1+\Delta\lambda$ corresponding to Δ^{++} and Δ^0, respectively.

The parameters a, b of the model amplitude F_1 were fixed at values corresponding to the pole position

$$\sqrt{s_{1,2}}=(1211\pm i50)\ \text{MeV}. \tag{12}$$

For the function F_2 we have chosen five different values of pole position, which differ from (12) by 1 to 5 MeV. We have tested three different approaches of determining the relative shift of the pole position of the function $\varphi=F_1-F_2$.

1) Let $\lambda_2=\lambda_1+\Delta\lambda$ be the pole position of F_2. The coefficients of the terms in the singular part of the Laurent expansion of φ are

$$a_n=(\alpha_1\lambda_1^{n-1}+\alpha_1^*\lambda_1^{*n-1})-[\alpha_2(\lambda_1+\Delta\lambda)^{n-1}+ \tag{13}$$
$$+\alpha_2^*(\lambda_1^*+\Delta\lambda^*)^{n-1}],$$

where α_1, α_2 are the residues corresponding to λ_1, λ_2. Hence, for χ^2_N we get

$$\chi^2_N=\sum_1^N|Q_n-a_n|^2\approx\sum_1^N|Q_n-2\mathrm{Re}(\Delta\alpha\lambda_1^{n-1})-2(n-1)\times \tag{14}$$
$$\times\mathrm{Re}(\alpha_1\lambda_1^{n-2}\Delta\lambda)|^2,$$

where $\Delta\alpha=\alpha_2-\alpha_1$ and Q_n for F_1 is defined by (4). Nonlinear terms in $\Delta\alpha$, $\Delta\lambda$ were neglected. λ_1, α_1 are fixed and χ^2 is minimized with respect to $\Delta\lambda$, $\Delta\alpha$.

This method requires a high accuracy of computing the Q_n coefficients since their absolute value is small.

When calculating the real physical case, also the error of the input value (12) must be taken into account.

2) First we calculate the Q_n coefficients only for the function F_1, which has one pole at λ_1 given by Eq. (12). Minimizing χ^2 we get the values λ_1 and α_1 which are used as an input in (7), where Q_n correspond to the function φ given by (8). Keeping λ_1 and α_1 fixed, the $\lambda_1+\Delta\lambda$ and $\alpha_1+\Delta\alpha$ are obtained by minimizing the two-pole formula (7). Next we minimize (7) with respect to all four complex parameters λ_1, α_1, λ_2, α_2 with initial values from the two preceding steeps. Since the number of free parameters is rather high in this case, we are looking for the most probable initial values to avoid the false local minima in the process of minimization.

3) In the last approach the functions F_1 and φ were multiplied by the Blaschke product — the factor which is equal to zero in points where the functions F_1 or φ

Table 2

Results of testing three different ways of calculating the $\Delta^{++}-\Delta^0$ pole parameters. The first column gives the difference in the pole positions of the model amplitudes F_1 and F_2. The column 2 to 4 shows how this pole difference is reproduced by the proposed methods.

	Input values [MeV]	The result of "one-pole fit" [MeV]	The result of "two-pole fit" [MeV]	The result of "Blaschke factor" approach [MeV]
	0. +1.i	−0.011 +0.964i	0.001 +0.999i	0.0004 +1.0000i
	0. +2.i	−0.043 +1.86i	0.001 +1.998	0.0004 +1.9997i
	0. +3.i	−0.098 +2.689	0.001 +2.997i	0.0003 +2.9995i
	0. +4.i	−0.161 +3.454	0.001 +3.9957i	0.0003 +3.9992i
	0. +5.i	−0.243 +4.159i	0.001 +4.9944i	0.0002 +4.9989i
Pole pos. of F_1	1211. +50i	——	1211.01 +49.986i	1211.00 +50.00i

have poles. (See Eq. (8a)). Thus instead of testing the hypothesis about the number of poles we are testing the analyticity of the function inside a given domain.

The main advantage of this approach is that the only free parameters which enter are the pole positions λ_1, λ_2. On the other hand the minimizing procedure is more time consuming compared to the previous methods.

For the same reason as in the case 2) we first found the pole position λ_1 for the function F_1 and then fixed the pole position λ_2 for the function φ at λ_1.

In the last step — the minimization with respect to both positions, we get the final values λ_1, λ_2. After the same calculation had been repeated with a higher accuracy, the output values λ_1, λ_2 started to deviate from the input pole position. This effect is due to a false minimum in χ^2, which may be so close to the true minimum, that it is difficult to avoid it even by choosing the appropriate starting values.

Much better results, i.e. convergence when accuracy had been increased, were achieved with one pole fixed and the second pole determined by minimization. In the next step the second pole was fixed and the first one was calculated with an increased accuracy.

All the results concerning the model p.w.a. are shown in Table 2.

The next part of this paper contains some comments and concluding remarks.

IV. DISCUSSION

There are a few points which should be stresses:

i) We need accurate data on the real axis in the region close to the poles. Other parts of the cuts are less important but they are considerably suppressed by the weight function.

ii) The first approach does not give satisfactory results since it reproduces the input parameters λ_1 and $\Delta\lambda$ only approximately (See Table 2).

iii) The methods 2) and 3) are almost equivalent and rather promising. The "Blaschke factor" method is not so influenced by the systematic error, since it reproduces not only the difference in the pole position but also the values of the input parameters.

iv) In Table 2 we do not give the statistical error. The problem is that the errors do not have a statistical character but depend also on the method. A more correct way of calculating the statistical errors is developed in [6] and is based on the following idea:

Suppose that the exp. data and errors are given. We generate randomly a few set of "data" so that all of them lie in the corridor given by exp. errors. For each such set we determine the parameters λ_1 and $\Delta\lambda$. The final error is then calculated from all the values λ_1 and $\Delta\lambda$ using the standard statistical procedure. In our model example this method was not used. In conclusion we can say that the methods 2) and 3) are able to reproduce the difference $\Delta\lambda$ in the pole positions with a sufficient accuracy.

Both approaches are model independent and do not use any special assumptions about the background or the left-hand cuts.

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