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RELATIVISTIC GRAVITATION FROM MASSLESS SYSTEMS OF SCALAR AND VECTOR FIELDS

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Within the laws of Einstein's gravitation theory, a massless system consisting of two fields is discussed. One field is scalar, of long range, the other is vector field of short range. Diffuse sources of these fields are admitted, to avoid singularities. A proportionality between the sources is assumed. Both fields are minimally coupled to gravitation, and contribute positive definitely to the time component of the energy momentum tensor. A class of static, spherically symmetric solutions of the equations is obtained, in the weak field limit. The solutions are regular everywhere, stable, and can represent large or small physical systems. The gravitational field presents a Schwarzschild-type asymptotic behaviour. The radius of the structure is determined unambiguously. The dependence of the energy on the various parameters characterizing the system is discussed in some detail.

РЕЛЯТИВИСТСКОЕ ПОЛЕ ТЯГОТЕНИЯ СИСТЕМЫ СКАЛЯРНОГО И ВЕКТОРНОГО ПОЛЕЙ С НУЛЕВОЙ МАССОЙ

В статье обсуждается на основе законов эйнштейновской теории гравитации система с нулевой массой, состоящая из двух полей, одно из которых является дальнодействующим скалярным полем, а второе короткодействующим векторным полем. Во избежание сингулярностей допущены неточные источники этих полей. Предполагается наличие пропорциональности между источниками. Между обоими полями существует минимальная связь через гравитацию, и эти поля вносят вклад в положительно определённую временную составляющую тензора энергии-импульса. В пределе слабого поля получен класс статических и сферически-симметричных решений уравнений. Полученные решения везде регулярны, стабильны и могут представлять большие и малые физические системы. Гравитационное поле имеет асимптотику шварцшильдовского типа. Радиус системы определяется однозначно. Довольно подробно обсуждается зависимость энергии от различных параметров, характеризующих систему.

I. INTRODUCTION

It is an old belief that general relativity occupies a foremost place in the description of elementary physical structures (Einstein and Rosen [1]). Nonsingular solutions of field equations are particularly looked for, in which the energy

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momentum tensor depends on a minimum number of simple physical quantities. Massive static systems are more usually studied, where the attractive effects of self-gravitation are balanced by some kind of repulsive interaction. This interaction may be in the form of pressure of electrostatic repulsion (Bonnor [2]). However, the scalar interactions are also introduced to obtain equilibrium (Das [3], Bekenstein [4], Wolk et al. [5], Teixeira et al. [6, 7]). Diffuse sources of fields are commonly admitted, if one wants solutions without singularities.

In the present paper a simple structure is studied, not containing matter explicitly. It consists of two fields, together with the corresponding diffuse sources. One is a repulsive vector field of short range, the other is an attractive scalar field of long range. Both fields contribute positive definitely to the time component of the energy momentum tensor, and are minimally coupled to gravitation. In Sect. II, the covariant equations governing the system are obtained from a Lagrangian density, and the static, spherically symmetric equations are written in the weak field limit. In Sect. III, exact solutions for the vector and the scalar fields are obtained. In sec. IV, expressions for the gravitational potentials are presented. Finally, three independent parameters which characterize the system are discussed in Sec. V, and the influence each of them exerts on the gravitational mass of the system is clearly explained. It is also shown that the solutions obtained may serve as a basis to describe large or small actual physical systems.

II. THE EQUATIONS

One starts from the lagrangian density

$$\mathscr{L}=\mathscr{L}_G+\mathscr{L}_V+\mathscr{L}_S. \tag{1}$$

$$K\mathscr{L}_G=\frac{1}{2}(-g)^{1/2}R, \qquad K=8\pi G/c^4, \tag{2}$$

$$K\mathscr{L}_V=(-g)^{1/2}[(V_{\mu,\nu}-V_{\nu,\mu})V_{\alpha,\beta}g^{\mu\beta}+\kappa^2V_\nu V_\alpha]g^{\nu\alpha}-8\pi J^\alpha_{\cdot}V_\alpha \tag{3}$$

$$K\mathscr{L}_S=(-g)^{1/2}S_{,\alpha}S_{,\beta}g^{\alpha\beta}-8\pi\sigma_{\cdot}S. \tag{4}$$

In these equations R is the scalar curvature, g is the determinant of the metric potential $g_{\mu\nu}$, V_μ is a repulsive vector field of short range (κ^{-1}) and S is an attractive scalar field of long range. A subscripted comma means an ordinary derivative. The vector quantity $J^\alpha_{\cdot}$ and the scalar quantity $\sigma_{\cdot}$ are introduced to avoid singularities; they are densities of weight +1, and represent the diffuse sources of V_α and S, respectively (Das [3]).

From the invariance of the action integral upon variations of the metric potentials one obtains the Einstein equations [8],

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-2(V_{\mu\alpha}V^\alpha_\nu+\kappa^2V_\mu V_\nu+S_{,\mu}S_{,\nu})+ \tag{5}$$

$$+\left(\frac{1}{2}V^\alpha_\beta V^\beta_\alpha+\kappa^2V^\alpha V_\alpha+S^{,\alpha}S_{,\alpha}\right)g_{\mu\nu},$$

$$V_{\mu\nu}\equiv V_{\nu,\mu}-V_{\mu,\nu}, \tag{6}$$

while variations of the vector and scalar potentials give

$$V^{\mu\nu}_{;\nu}-\kappa^2V^\mu=-4\pi J^\mu, \qquad J^\mu\equiv(-g)^{-1/2}J^\mu_{\cdot}, \tag{7}$$

$$S^{;\mu}_{;\mu}=-4\pi\sigma, \qquad \sigma\equiv(-g)^{-1/2}\sigma_{\cdot}. \tag{8}$$

The semicolons mean covariant derivatives, and the quantities J^μ and σ have the weight zero. From the Bianchi identities one obtains

$$J^\alpha V_{\alpha\nu}-\sigma S_{,\nu}=0. \tag{9}$$

We now adjust these equations for the case of static, spherically symmetric systems. We write

$$ds^2=e^{2\eta}(dx^0)^2-e^{2\lambda}dr^2-r^2\,d\Theta^2-r^2\sin^2\Theta\,d\Phi^2, \tag{10}$$

$$V_\mu=V\delta^0_\mu, \quad J^\mu=J\delta^\mu_0, \tag{11}$$

and consider all quantities (η, λ, V, J, S, σ) functions of r only. We then obtain, as independent equations,

$$\eta'+\lambda'=r(\kappa^2V^2e^{-2\eta+2\lambda}+S'^2), \tag{12}$$

$$[r(1-e^{-2\lambda})]'=r^2[(\kappa^2V^2+V'^2e^{-2\lambda})\,e^{-2\eta}+S'^2e^{-2\lambda}], \tag{13}$$

$$r^{-2}e^{-\eta-\lambda}(r^2V'e^{-\eta-\lambda})'-\kappa^2Ve^{-2\eta}=-4\pi J, \tag{14}$$

$$r^{-2}e^{-\eta-\lambda}(r^2S'e^{\eta-\lambda})'=4\pi\sigma, \tag{15}$$

$$JV'+\sigma S'=0, \tag{16}$$

where a prime means d/dr. Since in these five equations we have six functions, one constraint is necessary to obtain explicit solutions. We consider here the case where the sources $J(r)$ and $\sigma(r)$ bear a constant ratio,

$$J=\omega\sigma, \qquad \omega=\text{const.} \tag{17}$$

One finds difficulty in obtaining the exact integration of the field equations. We then try an approximate method: we expand the four fields (η, λ, V, S) and the two sources (J, σ) in integral powers of some dimensionless parameter ε. This parameter is identified later. We have been able to obtain the exact solution in the lowest order of approximation, in which J, σ, V, S are proportional to ε, while η

and λ are proportional to ε^2. In this order of approximation the field equations become

$$\eta' + \lambda' = r(\kappa^2 V^2 + S'^2), \tag{18}$$

$$(r\lambda)' = \frac{1}{2} r^2(\kappa^2 V^2 + V'^2 + S'^2), \tag{19}$$

$$V'' + 2V'/r - \kappa^2 V = -4\pi\omega\sigma, \tag{20}$$

$$S'' + 2S'/r = 4\pi\sigma, \tag{21}$$

$$(\omega V' + S')\,\sigma = 0, \tag{22}$$

where (17) has been used.

From the last three equations one obtains the field V, S, and the source σ, then from (19) and (18) one gets the gravitational potentials λ and η, consecutively.

III. VECTOR AND SCALAR FIELDS

One initially considers the region $r \leqslant a$, where the diffuse source σ exists. From (20) to (22) one then obtains the solutions, regular in the origin,

$$V_i(r) = \alpha j_0(\nu r), \qquad \nu \equiv \kappa(\omega^2 - 1)^{-1/2}, \tag{23}$$

$$4\pi\sigma(r) = \alpha\omega\nu^2 j_0(\nu r), \tag{24}$$

$$S_i(r) = -\alpha\omega[j_0(\nu r) + \beta], \tag{25}$$

where $j_0(x) = x^{-1}\sin x$ is the spherical Bessel function of order zero, and α, β are constants of integration. The subscript i means internal. One finds that the parameter ω necessarily satisfies $\omega^2 > 1$, otherwise the mathematical solutions obtained are physically unsatisfactory; this subject is further discussed in Sec. V.

In the region $r > a$, where the source $\sigma = 0$, one obtains from (20)

$$V_e(r) = \alpha j_0(\nu a)(a/r)\, e^{-\kappa(r-a)}, \tag{26}$$

where the continuity of the vector field through $r = a$ was imposed. The subscript e means external. One observes the rapid decay of the short range field, for a distance increasing from the origin. One also imposes the continuity of the radial derivative of the vector field, and obtains

$$\nu a j_1(\nu a) = (1 + \kappa a) j_0(\nu a), \tag{27}$$

where $j_1(x) = -\mathrm{d}j_0(x)/\mathrm{d}x$ is the spherical Bessel function of order one. This relation represents a constraint for the radius a, for a given set of parameters κ and ω. Since variations of sign in the diffuse source of fields induce instability in the system, one finds from (24) that only the smallest positive value of νa satisfying (27) is of physical interest, namely

$$\pi/2 < \nu a < \pi. \tag{28}$$

The external scalar field is obtained from (21), with $\sigma = 0$:

$$S_e(r) = -\alpha\omega[j_0(\nu a) + \beta](a/r), \qquad \beta = (1 - \omega^{-2})^{1/2}, \tag{29}$$

where the continuity of the field and of its radial derivative were again imposed. In order to obtain the value of β, use was made of the relation (27). One observes the hyperbolic behaviour (r^{-1}) of the scalar field in the regions outside the sources.

IV. GRAVITATIONAL FIELD

In the internal region ($r \leqslant a$) one obtains, using (19), (23) and (25),

$$\lambda_i(r) = \frac{1}{2}\alpha^2[\omega^2 + j_0(2\nu r) - (\omega^2 + 1) j_0^2(\nu r)], \tag{30}$$

while from (18) one obtains

$$\eta_i(r) = \eta(0) + \alpha^2\left[(2\omega^2 - 1)\Sigma(\nu r) + \left(\omega^2 - \frac{1}{2}\right) j_0(2\nu r) - \omega^2 + \frac{1}{2} j_0^2(\nu r)\right]. \tag{31}$$

For convenience, we have introduced the constant

$$\eta(0) = -\alpha^2\left[(2\omega^2 - 1)\sum(\nu a) - (1 - \omega^{-2})\, e^{2\kappa a} Ei(-2\kappa a) - \frac{1}{2}\right], \tag{32}$$

where the function $\Sigma(x)$ and the exponential integral $Ei(-x)$ are defined by

$$\Sigma(x) = \int_0^x t[j_0(t)]^2\, \mathrm{d}t, \quad Ei(-x) = -\int_x^\infty t^{-1} e^{-t}\, \mathrm{d}t, \quad x > 0. \tag{33}$$

An easy inspection of (30) shows that $\lambda(0) = 0$; less trivially, one finds that $\eta(0) < 0$, and that both η_i and λ_i increase monotonically outwards. All these general features are also encountered in the weak field limit of the internal Schwarzschild solution.

In the external region ($r > a$), one obtains from (19), (26) and (29),

$$\lambda_e(r) = Gm/rc^2 - \frac{1}{2}(1 + \kappa a)\, V_e^2(r) - \frac{1}{2} S_e^2(r), \tag{34}$$

where m is the mass parameter, given by

$$Gm/ac^2 = \alpha^2\left[\omega^2 - \frac{1}{2} - \omega^2 j_0(2\nu a)\right]. \tag{35}$$

Finally, from (18) one gets

$$\eta_e(r) = -Gm/rc^2 + \frac{1}{2}(1+\kappa r)\, V^2(r) + [\kappa a V(a)\, e^{\kappa a}]^2 Ei(-2\kappa r). \qquad (36)$$

One observes, in (34) and (36), the usual Schwarzschild gravitational behaviour in the asymptotic regions,

$$\eta(r) = -\lambda(r) = -Gm/rc^2, \qquad r \to \infty. \qquad (37)$$

The continuity properties of the gravitational potentials are easily seen from (18) and (19). As in the cases of massive spheres, one finds that η, λ and η' are continuous through $r = a$. In addition, one finds that also λ' and η'' are continuous, in our system.

V. DISCUSSION

Three independent parameters characterize our physical systems: κ, α and ω. The inverse length parameter κ is mainly responsible for the size of the system; indeed, one finds from (27) and (23) that the radius a is inversely proportional to κ.

The parameter α is dimesionless. In Sect. III, we have found that all vector and scalar field quantities are proportional to α, while in Sect. IV we have found that the gravitational potentials η and λ are proportional to α^2. This suggests to identify α as the small, dimensionless parameter in terms of which the series expansion of Sec. II were made. As it can be seen in (35), the smallness of α^2 implies $m/a \ll c^2/G$, a condition usually met both in large physical systems (stars, galaxies) and small ones (atomic nuclei).

Finally, we have found that the dimensionless parameter $\omega = J/\sigma$ must satisfy $\omega^2 > 1$. This has a simple physical interpretation. The collapse of the system is only prevented when there is a sufficient source J of a repulsive, short range vector field to balance the attractive effects of the long range scalar field on the corresponding source σ.

From (5), (10) and (11) one finds that the time component of the energy momentum tensor is

$$(8\pi G/c^4)\, T_{00} = \kappa^2 V^2 + (V'^2 + S'^2 e^{2\eta})\, e^{-2\lambda}; \qquad (38)$$

this is an exact result, and shows that both fields V and S contribute positive definitely to T_{00}.

An alternative expression for the mass m is obtained from (35) and (27):

$$Gm/c^2 = \alpha^2\kappa^{-1} W(\omega), \quad W(\omega) = (\omega^2-1)^{1/2}\left[\left(\omega^2 - \frac{1}{2}\right)\csc^{-1}|\omega| + (\omega^2-1)^{1/2}\right]. \qquad (39)$$

This expression exhibits more clearly the dependence of m on the parameters α, κ and ω. The inverse cosecant is taken between $\pi/2$ and π, in order to satisfy (28). A direct computation of the function $W(\omega)$ shows that the energy mc^2 of the system monotonically increases with $|\omega|$. The following two extreme behaviours are obtained, for small and large $|\omega|$:

$$Gm/c^2 \simeq \pi(\alpha^2/\kappa)(\delta/8)^{1/2} \quad \text{for} \quad |\omega| = 1+\delta, \quad 0<\delta\ll 1; \qquad (40)$$

$$Gm/c^2 \simeq \pi(\alpha^2/\kappa)\, |\omega|^3 \quad \text{for} \quad |\omega| \ll 1. \qquad (41)$$

We have not attempted to demonstrate rigorously the stability of our system. However, a nonrelativistic form of reasoning is appropriate [6] in the present case, where the Newtonian concept of force can be used: Starting from an equilibrium configuration, admit a small perturbation, which produces some local compression of the diffuse sources. Since $\omega^2 > 1$, the additional repulsive, short range forces will exceed the additional forces of the long range, attractive field. As a consequence, a tendency to local rarefaction is manifested. In the reverse situation of a local small expansion, the same final tendency to restore the equilibrium configuration is observed.

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