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STATISTICAL INTERPRETATION OF SOME PRODUCTION PARAMETERS BY MEANS OF THE GENERALIZED AND MODIFIED URN MODEL

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It is assumed that an intermediate state (urn) contains arbitrarily many sorts of constituents, each sort including arbitrarily many members. If one constituent is drawn at random, then a given number of constituents of each sort is created (or annihilated) in the state under consideration. This procedure is repeated.

An approach is developed which allows to obtain the normalized probability that in N draws ($N \geqslant 1$) the constituent of a given sort is drawn K times ($0 \leqslant K \leqslant N$); even the probability can be included that in the first α as well as in the last ω draws the constituent of the given sort is (or is not) drawn. It is shown that already the case of two sorts of constituents promises several interesting physical implications.

In the corresponding limits the binomial as well as the Poisson and the Gaussian distributions follow. Those asymptotic cases allow to interpret statistically several high energy parameters like the slope or the Feynman scaling variable. Conditions are stated which allow to deduce how many sorts of constituents are to be introduced in order to explain the asymptotics observed in experiments. The statistical expression of the confinement is obtained as well.

СТАТИСТИЧЕСКАЯ ИНТЕРПРЕТАЦИЯ НЕКОТОРЫХ ПАРАМЕТРОВ РОЖДЕНИЯ, ОСНОВАННАЯ НА ОБОБЩЁННОЙ И МОДИФИЦИРОВАННОЙ МОДЕЛИ ЯЩИКА

Работа исходит из предположения, что промежуточное состояние (ящик) содержит произвольное число различных сортов составляющих с произвольным количеством элементов. Если случайно вынуть из ящика одну из составляющих, то это означает, что рождается (или аннигилирует) соответствующее число элементов данного сорта составляющих. Эта процедура повторяется.

В данной работе развит подход, позволяющий получить нормированную вероятность того, что при N розыгрышах ($N \geqslant 1$) составляющая данного сорта вытянута K раз ($0 \leqslant K \leqslant N$); в полученное выражение можно также включить вероятность того, что во всех первых α розыгрыщах и во всех последних ω розыгрыщах составляющая данного сорта вытянута (или не вытянута). Показано, что уже случай двух сортов составляющих обещает интересные физические выводы.

В случае соответствующих предельных переходов получаются биномиальное распределение как и распределения Пуассона и Гаусса. Эти асимптотические случаи позволяют дать статистическую интерпретацию некоторых высо-

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коэнергетических параметров, таких как, например, наклоны сечений или масштабная переменная Фейнмана. Сформулированы также условия, позволяющие сделать заключение о числе составляющих, для объяснения асимптотик, наблюдаемых на опыте. Кроме того, получено также статистическое выражение для конфайнмента.

I. INTRODUCTION

A variety of experimental data obtained from collisions of elementary particles at very high energies presented in the form of convenient figures is often fitted by heuristic formulae involving several free parameters. For instance the formula

$$a(1-x_F)^c \exp(-bP_T) \tag{1}$$

is used for the parametrization of the Lorentz-invariant cross section in the inclusive muon pairs production at 150 GeV by π^+ mesons and protons on beryllium [1]. In expression (1), a, b, c are fitting parameters, x_F is the Feynman scaling variable and P_T is the transverse momentum. On the other hand there are well known cases when the cross section or the intensities of beams decrease exponentially, say, with the energy E,

$$\sim \exp\left(-\lambda \frac{E}{E_0}\right), \tag{2}$$

where E_0 fixes the scale.

Due to the fact that the cross sections themselves express a certain kind of probability distributions it is quite natural to look for some statistical approaches in order to get the distributions, e. g., of the form (1) or (2). In the case under discussion it turns out that the simple urn model presented by B. Friedman in 1949, [2], conveniently generalized and modified leads to expression (1) or (2), thereby allowing a statistical interpretation of the parameters involved. Moreover, that generalized and modified model might be considered also as the one which contains several fundamental statistical features of present-day approaches to high energy production phenomena, thereby fulfilling the necessary conditions to become eventually a "standard" statistical model [3] in that domain of phenomena.

Friedman formulated his problem as follows: An urn contains α white and β black balls (we change slightly his notation). One ball is drawn at random and then $1+\xi$ additional balls of the same colour as the drawn ball and η additional balls of the opposite colour are added to the urn. This procedure is repeated. What is the probability distribution of the number of white balls in the urn after N draws?

Solving his problem Friedman obtained a difference-differential equation for the characteristic function of the number of the white balls in the urn after N draws and he treated also the case when the N-th draw is a white ball. However, the solution of the aforementioned equation was obtained only in three special cases, namely, in the case of the Polya-Eggenberger distribution (when ξ is arbitrary and

$\eta=0$), in the case of the Ehrenfest model ($\xi=-1$, $\eta=1$) as well as in the case of the safety campaign ($\xi=0$, η arbitrary).

Polya and Eggenberger [4] derived their probability distribution of "contagion" for the number of infected cases during an epidemic. The corresponding distribution (e.g. also [5]) with its special cases is discussed in more detail, e.g. in [6].

Kac [7] has treated the case $\xi=-1$, $\eta=1$ in another formulation of the Ehrenfest model [8] of heat exchange between two isolated bodies of unequal temperatures. From the model Newton's law of cooling can be derived; more references to the connection of this approach with the Brownian motion can be found in [7].

Schrödinger and Kohlrausch [9] pointed out that the Ehrenfest model was also equivalent to a random walk problem (more about that as well as about the classical gambler's ruin problem can be found, e.g. in [10]; compair also with [11]).

The generalization of the aforementioned simple urn problem is formulated in the next Section of the present paper and in Sect. III a procedure is developed which leads to its solution in an explicit form. The solution of the Friedman problem then follows in Sect. IV as a special case of the generalized probability distribution. Already this special case leads to the curves which have several interpretations; let us mention at least some of them: i) the fast hadron distributions, ii) the distribution of the deep inelastic structure function, iii) the energy distribution of different muons for Drell-Yan trimuons and the invariant mass distributions, iv) the rapidity distributions of away side jets resulting from different quark-gluon scattering processes, etc; moreover, qualitatively also the shape of the probability distribution can be obtained, as it follows from the solution of the non-linear Fokker-Planck equation in the laser problem.

Section V contains the asymptotic cases of the generalized distribution when the numbers of the constituents and then the number of draws are large; in those cases the binomial and then the Poisson distributions are established. Section VI deals with a modification of the urn model which takes into account the case that there exists also a non-vanishing probability of drawing no ball at all from the urn: in this case the aforementioned asymptotics is slightly modified. This case allows to express statistically also the confinement of the constituents of the given sort. The last Section contains several concluding remarks.

II. GENERALIZATION OF THE SIMPLE URN MODEL

Let us consider in an intermediate state or in a box, in Ref. [2] called the "urn", a set of objects (like molecules, atoms, nucleons, quarks, partons or their clusters) with some property which gives rise to a distinction between their subsets (like a set of quantum numbers). Such objects wil be called in the present paper "balls", and their distinguishing property "colour"; each ball has one colour. The problem to be solved is formulated as follows:

i) There are s sorts of coloured balls (s is an arbitrary positive integer). There are β_r balls of the r-th colour; in the present paper we use always

$$r = 1, 2, \ldots, s;$$

all β_r are arbitrary.

ii) A ball, say of the f-th colour,

$$f = 1, 2, \ldots, s$$

is drawn at random and put back. Then just c_{fr} new balls are added there to the balls of the r-th colour (c_{fr} 's need not be positive, whereby creation as well as annihilation of the balls might be introduced).

All balls are mixed together and the next draw might be performed.

First of all we look for the (generalized) probability that in N draws ($N \geq 1$) the ball of the f-th colour is drawn altogether K-times ($0 \leq K \leq N$).

The procedure developed in the present paper does allow to distinguish the sequence of the draws in which the coloured ball under consideration is (or is not) drawn. This circumstance allows then to obtain the aforementioned probability, e.g., with the inclusion of the probability that the ball of the f-th colour is (or is not) drawn in a special set of draws. We specify this probability at least for two cases, namely for the case when the ball of the f-th colour a) is drawn in all first α_+ as well as in all last ω_+ draws, where $\alpha_+ + \omega_+ \leq K$; b) is not drawn at all in the first α_- and in the last ω_- draws, where $\alpha_- + \omega_- \leq N - K$.

As far as there is no restriction on the sequence of the draws (i.e. $\alpha_\pm = \omega_\pm = 0$) both cases a) and b) lead to the generalized probability mentioned above.

III. SOLUTION OF THE GENERALIZED URN PROBLEM

Let us start with the statement that the intermediate state contains altogether

$$\Sigma' \beta_r$$

coloured balls, where

$$\Sigma' \equiv \sum_{r=1}^{s}. \tag{3}$$

Elementary considerations lead us to the following conclusions:

1. The probability that in the first draw the ball of the f-th colour

a) is drawn is given by

$$P_{K=1}^{(N=1)} = \frac{\beta_f}{\Sigma' \beta_r}, \tag{4}$$

b) is not drawn is given by

$$P_{K=0}^{(N=1)} = \sum_{(j)} \frac{\beta_j}{\Sigma' \beta_r} \equiv 1 - \frac{\beta_f}{\Sigma' \beta_r}, \tag{5}$$

where

$$\sum_{(j)} \equiv \sum_{\substack{i=j_L \\ (j \neq z)}}^{s}; \tag{6}$$

in the present case $j_L = 1$ and $z \equiv f$. It will be seen later that in the limit when the starting numbers of the balls, β_r (as well as when the number of draws, N) are very big, it is convenient to interpret just the probability (5) as the Feynman scaling variable, x_F, (cf. rel. (35)).

2. Let us draw altogether two times.

A. Let the first draw be the ball of the f-th colour. The probability that in the second draw the ball of the f-th colour

a) is drawn again, is given by

$$P_{K=2}^{(N=2)} = \frac{\beta_f}{\Sigma' \beta_r} \frac{\beta_f + c_{ff}}{\Sigma' (\beta_r + c_{fr})}; \tag{7}$$

b) is not drawn, is given by

$$\frac{\beta_f}{\Sigma' \beta_r} \sum_{(j)} \frac{\beta_j + c_{fj}}{\Sigma' (\beta_r + c_{fr})}. \tag{8}$$

B. Let the first draw be not the ball of the f-th colour. The probability that in the second draw the ball of the f-th colour

a) is drawn, is given by

$$\sum_{(j)} \frac{\beta_j}{\Sigma' \beta_r} \frac{\beta_f + c_{jf}}{\Sigma' (\beta_r + c_{jr})}, \tag{9}$$

b) is again not drawn, is given by

$$P_{K=0}^{(N=2)} = \sum_{(j_1)} \frac{\beta_{j_1}}{\Sigma' \beta_r} \sum_{(j_2)} \frac{\beta_{j_2} + c_{j_1 j_2}}{\Sigma' (\beta_r + c_{j_1, r})}. \tag{10}$$

The sum of the expressions (8) and (9) gives the probability that in two draws the ball of the f-th colour is drawn only once (and the sequence of the draws is not distinguished).

3. The continuation of the aforementioned procedure leads to the following conclusions:

A. The probability that in N draws the ball of the f-th colour is drawn $K = N$ times is given by

$$P_N^{(N)} = \frac{\beta_f}{\Sigma'\beta_r}\frac{\beta_f + c_{ff}}{\Sigma'(\beta_r + c_{fr})}\cdots\frac{\beta_f + (N-1)c_{ff}}{\Sigma'[\beta_r + (N-1)c_{fr}]} \equiv \tag{11}$$

$$\equiv \left(\frac{c_{ff}}{\Sigma' c_{fr}}\right)^N \frac{\Gamma(\Sigma'\beta_r/\Sigma' c_{fr})}{\Gamma(\beta_f/c_{ff})}\frac{\Gamma[(\beta_f/c_{ff})+N]}{\Gamma[(\Sigma'\beta_r/\Sigma' c_{fr})+N]}.$$

B. The probability that in $N(\geq 2)$ draws the ball of the f-th colour is drawn $K = N-1$ times can be found as follows:

a) if that ball is not drawn

i) just in the first draw, the probability is given by

$$\sum_{(j)} \frac{\beta_j}{\Sigma'\beta_r}\frac{\beta_f + c_{jf}}{\Sigma'(\beta_r + c_{jr})}\frac{\beta_f + c_{jf} + c_{ff}}{\Sigma'(\beta_r + c_{jr} + c_{fr})}\cdots\frac{\beta_f + c_{jf} + (N-2)c_{ff}}{\Sigma'[\beta_r + c_{jr} + (N-2)c_{fr}]} \equiv \tag{12}$$

$$\equiv \sum_{(j)} \frac{\beta_j}{\Sigma'\beta_r}\left(\frac{c_{ff}}{\Sigma' c_{fr}}\right)^K \frac{\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}} + K\right)\Big/\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}}\right)}{\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}} + K\right)\Big/\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}}\right)};$$

ii) only in the second draw, the probability is given by

$$\frac{\beta_f}{\Sigma'\beta_r}\sum_{(j)} \frac{\beta_j + c_{fj}}{\Sigma'(\beta_r + c_{fr})}\frac{\beta_f + c_{jf} + c_{ff}}{\Sigma'(\beta_r + c_{fr} + c_{jr})}\cdots\frac{\beta_f + c_{jf} + (N-2)c_{ff}}{\Sigma'[\beta_r + c_{jr} + (N-2)c_{fr}]} \equiv \tag{13}$$

$$\equiv \sum_{(j)} \frac{c_{ff}\Gamma\left(\frac{\beta_f}{c_{ff}} + 1\right)\Big/\Gamma\left(\frac{\beta_f}{c_{ff}}\right)}{(\Sigma' c_{fr})\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + 1\right)\Big/\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}}\right)} \frac{c_{fj}\Gamma\left(\frac{\beta_j}{c_{fj}} + 2\right)\Big/\Gamma\left(\frac{\beta_j}{c_{fj}} + 1\right)}{(\Sigma' c_{fr})\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + 2\right)\Big/\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + 1\right)} \times$$

$$\times \frac{c_{ff}^{K-1}\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}} + K\right)\Big/\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}} + 1\right)}{(\Sigma' c_{fr})^{K-1}\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}} + K\right)\Big/\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}} + 1\right)};$$

iii) only in the v-th draw, where $v = \tau + 1$, $\tau = 0, 1, 2, \ldots, K$ (we recall that $K = N-1$), the probability is given by

$$P_{K=N-1}^{(N)}(\tau) = \sum_{(j)} \frac{(c_{ff})^K}{(\Sigma' c_{fr})^N} \frac{\Gamma\left(\frac{\beta_f}{c_{ff}} + \tau\right)\Big/\Gamma\left(\frac{\beta_f}{c_{ff}}\right)}{\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + \tau\right)\Big/\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}}\right)} \times \tag{14}$$

$$\times \frac{c_{fj}\Gamma\left(\frac{\beta_j}{c_{fj}} + \tau + 1\right)\Big/\Gamma\left(\frac{\beta_j}{c_{fj}} + \tau\right)}{\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + \tau + 1\right)\Big/\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{fr}} + \tau\right)} \frac{\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}} + K\right)\Big/\Gamma\left(\frac{\beta_f + c_{jf}}{c_{ff}} + \tau\right)}{\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}} + K\right)\Big/\Gamma\left(\frac{\Sigma'(\beta_r + c_{jr})}{\Sigma' c_{fr}} + \tau\right)}.$$

b) Using rel. (14), the summation

$$\sum_{\tau=0}^{K} P_{K=N-1}^{(N)}(\tau) \equiv P_{K=N-1}^{(N)} \tag{15}$$

gives the (total) probability that in N draws the ball of the f-th colour is drawn $K = N-1$ times.

c) If instead of the l.h.s. of rel. (15) the following summation is considered,

$$\sum_{\tau=\tau_{\min}}^{\tau_{\max}} P_{K=N-1}^{(N)}(\tau) \tag{16}$$

(with $0 \leq \tau_{\min} \leq \tau_{\max} \leq K$), the probability is obtained that in N draws the ball of the f-th colour is drawn $K = N-1$ times including the probability that it is drawn in all first $\tau_{\min}$ as well as in all last $K - \tau_{\max}$ draws.

d) On the other hand, using rel. (14), also the probability can be deduced that the ball of the f-th colour is drawn $K = N-1$ times including the probability that

i) it is not drawn once in the first $(\tau_u + 1)$ draws, $\tau_u = 0, 1, 2, \ldots, K$; it is given by

$$\sum_{\tau=0}^{\tau_u} P_{K=N-1}^{(N)}(\tau); \tag{17}$$

ii) it is not drawn once in the last $(N - \tau_l)$ draws $\tau_l = 0, 1, 2, \ldots, K$; it is given by

$$\sum_{\tau=\tau_l}^{K} P_{K=N-1}^{(N)}(\tau). \tag{18}$$

C. With respect to the procedure just briefly outlined the probability can be looked for that in N draws the ball of the f-th colour is drawn $K = N-2$ times with the condition that it is not drawn in the first as well as in the v_1-th draw ($v_1 = 2, 3, 4, \ldots, N$); then, in the next step with the condition that it is not drawn out in the second nor in the v_2-nd draw ($v_2 = 3, 4, \ldots, N$), etc. If we add together all those expressions, the sought probability follows. With the appropriate summation analogous to (16) and (17) again the probabilities can be obtained that the ball of the f-th colour is (or is not) drawn in some first as well as in some last draws.

Similarly we can continue in cases when the ball of the f-th colour is drawn K-times where $K = N-3, N-4, \ldots, 2, 1, 0$.

4. The aforementioned procedure leads to the probability $P_K^{(N)}$ that in N draws the ball of the f-th colour is drawn K times; we express it by means of the function $\mathcal{P}_K^{(N)}$, which is introduced as follows,

$$\mathscr{P}_K^{(N)}\{z, j_L; b_j \text{ for } j=j_L, j_L+1, \ldots, s; \quad (\tau_\sigma)_{\min}, (\tau_\sigma)_{\max} \text{ for } \sigma=1,2,\ldots,N-K\} \equiv \tag{19}$$

$$\equiv \frac{(b_z c_{zz})^K}{(\Sigma' c_{zr})^N} \frac{\Gamma(\Sigma'\beta_r/\Sigma' c_{zr})}{\Gamma(\beta_z/c_{zz})} \sum_{(j_1)}\sum_{(j_2)} \cdots \sum_{(j_{N-K})} \frac{\Gamma\left(\frac{\beta_z + V_{j_{N-K},z}}{c_{zz}} + K\right)}{\Gamma\left(\frac{\Sigma'(\beta_r + V_{j_{N-K},r})}{\Sigma' c_{zr}} + K\right)} \times$$

$$\times \prod_{\sigma=1}^{N-K} \sum_{\tau_\sigma=(\tau_\sigma)_{\min}}^{(\tau_\sigma)_{\max}} b_{j_\sigma} \frac{\Gamma\left(\frac{\beta_z + V_{j_\sigma,z} - c_{j_\sigma,z}}{c_{zz}} + \tau_\sigma\right)}{\Gamma\left(\frac{\beta_z + V_{j_\sigma,z}}{c_{zz}} + \tau_\sigma\right)} \times$$

$$\times \frac{c_{z,j_\sigma}\Gamma\left(\frac{\beta_{j_\sigma} + V_{j_\sigma,j_\sigma} - c_{j_\sigma,j_\sigma}}{c_{z,j_\sigma}} + \tau_\sigma + 1\right)}{\Gamma\left(\frac{\beta_{j_\sigma} + V_{j_\sigma,j_\sigma} - c_{j_\sigma,j_\sigma}}{c_{z,j_\sigma}} + \tau_\sigma\right)} \frac{\Gamma\left(\frac{\Sigma'(\beta_r + V_{j_\sigma,r})}{\Sigma' c_{zr}} + \tau_\sigma\right)}{\Gamma\left(\frac{\Sigma'(\beta_r + V_{j_\sigma,r} - c_{j_\sigma,r})}{\Sigma' c_{zr}} + \tau_\sigma + 1\right)},$$

where

$$V_{j_\sigma,t} \equiv c_{j_1,t} + c_{j_2,t} + \ldots + c_{j_\sigma,t} \tag{20}$$

and the summations over $(j)'$ s are to be understood in the sense of rel. (6).

Now, for $K<N$ the probability $P_K^{(N)}$ is given by

$$P_K^{(N)} = \mathscr{P}_K^{(N)}\{z=f, j_L=1; \text{ all } b_j \equiv 1; \quad (\tau_\sigma)_{\min} = \tau_{\sigma-1}, (\tau_\sigma)_{\max} = K \text{ for } \sigma=1,2,\ldots,N-K; \quad (\tau_1)_{\min} = \tau_0 = 0\}; \tag{21}$$

if $K=N$, the probability $P_{K=N}^{(N)}$ is given by rel. (11), which can be obtained also from rel. (21) with apt definitions.

The probability (21) together with (11) is normalized in the sense that

$$\sum_{K=0}^{N} P_K^{(N)} = 1. \tag{22}$$

We note that the number of summations over $(j)'$ s in rel. (19) says how many times the ball of the f-th colour is not drawn. Moreover, the structure of rel. (19) is such that the ball of the f-th colour is not drawn just the σ-th times ($\sigma = 1, 2, \ldots, N-K$) in the $(\tau_\sigma+\sigma)$-th draw.

5. Taking into account the aforementioned procedure, the following probabilities are obtained:

A. The probability that in N draws the ball of the f-th colour is drawn K times and including the probability that

a) it is drawn in all first α_+, ($\alpha_+ = 0, 1, 2, \ldots, K$) as well as in all last ω_+, ($\omega_+ = 0, 1, 2, \ldots, K$) draws, $\alpha_+ + \omega_+ \leq K$; it is given as follows,

$$\mathscr{P}_K^{(N)}\{z=f, j_L=1; \text{ all } b_j \equiv 1; \quad (\tau_\sigma)_{\min} = \tau_{\sigma-1}, (\tau_\sigma)_{\max} = K-\omega_+ \text{ for } \sigma=1,2,\ldots,N-K; \quad \tau_0 = \alpha_+\}, \tag{23}$$

where $\mathscr{P}_K^{(N)}$ is given by rel. (19); if $\alpha_+ = \omega_+ = 0$, the expression (23) gives the probability (21);

b) it is not drawn in all first α_-($\alpha_- = 0, 1, 2, \ldots, N-K$) nor in all last ω_- ($\omega_- = 0, 1, 2, \ldots, N-K$) draws, $0 \leq \alpha_- + \omega_- \leq N-K$; it is given as follows,

$$\mathscr{P}_K^{(N)}\{z=f, j_L=1; \text{ all } b_j \equiv 1; \tag{24}$$

(a) $(\tau_\sigma)_{\min} = 0, (\tau_\sigma)_{\max} = 0$ for $\sigma = 1, 2, \ldots, \alpha_-$;
(b) $(\tau_\sigma)_{\min} = \tau_{\sigma-1}, (\tau_\sigma)_{\max} = K$ for $\sigma' = \alpha_- + 1, \alpha_- + 2, \ldots, N-K-\omega_-$;
(c) $(\tau_\sigma)_{\min} = K, (\tau_\sigma)_{\max} = K$ for $\sigma = N-K-\omega_- + 1, \ldots, N-K$;
$\tau_0 = 0\}$,

where $\mathscr{P}_K^{(N)}$ is given by rel. (19).

As to the expression (24) it is worthwhile to note that
(i) if no first draws are specified ($\alpha_- = 0, \omega_- \neq 0$), then the line (a) is to be omitted,
(ii) if no last draws are specified ($\omega_- = 0, \alpha_- \neq 0$), then the line (c) is to be omitted,
(iii) if $\alpha_- = 0, \omega_- = 0$, then both lines (a) and (c) are to be omitted; in the latter case the probability (21) is again reproduced;
(iv) if $\alpha_- + \omega_- = N-K$, then the line (b) is to be omitted.

B. Similarly the inclusion of the probability might be considered that the ball of the given colour is (or is not) drawn in another sequence of draws; in the present Section we do not enter into those details nor in other generalizations.

IV. SPECIAL CASES

1. Let us discuss in some more details the case of two colours, say white and black; then $s=2$ and let $\beta_1 \equiv \alpha, \beta_2 \equiv \beta$. Moreover in any draw we add ξ balls of the colour which was drawn as well as η balls of the other colour, i.e. $c_{11} = c_{22} = \xi$, $c_{12} = c_{21} = \eta$. The probability to draw K times the white ball in N draws follows from rel. (21) in the form

$$P_K^{(N)}(\alpha, \beta; \xi, \eta) = \frac{\xi^K}{(\xi+\eta)^N} \frac{\Gamma\left(\frac{\alpha+\beta}{\xi+\eta}\right)}{\Gamma\left(\frac{\alpha}{\xi}\right)} \times \tag{25}$$

$$\times \frac{\Gamma\left(\frac{\alpha+(N-K)\eta}{\xi}+K\right)}{\Gamma\left(\frac{\alpha+\beta}{\xi+\eta}+N\right)} \prod_{\sigma=1}^{N-K} \sum_{\tau_\sigma=\tau_{\sigma-1}}^{K} [\beta + (\sigma-1)\xi + \tau_\sigma\eta] \frac{\Gamma\left(\frac{\alpha+(\sigma-1)\eta}{\xi}+\tau_\sigma\right)}{\Gamma\left(\frac{\alpha+\sigma\eta}{\xi}+\tau_\sigma\right)}$$

with $\tau_0=0$ and if $K=N$, then $\prod_{\sigma=1}^{N-K}\ldots\equiv 1$. Rel. (25) represents the solution of the Friedman problem; it allows also to express simply the reversibility property of the system: the probability of drawing a white ball K times in N draws is equal to the probability of drawing a black ball $(N-K)$ times, i.e.

$$P_K^{(N)}(\alpha,\beta;\xi,\eta)=P_{N-K}^{(N)}(\beta,\alpha;\xi,\eta). \tag{26}$$

For instance, the expression $P_N^{(N)}(\beta,\alpha;\xi,\eta)$ gives also the probability that in N draws the white ball is not drawn at all.

Rel. (25) implies the Polya distribution if $\eta=0$ and using the relation

$$\prod_{\sigma=1}^{N-K}\sum_{\tau_\sigma=\tau_{\sigma-1}}^{K}1=\binom{N-\tau_0}{N-K}\equiv\binom{N-\tau_0}{K-\tau_0}\underset{N\to\infty}{\longrightarrow}\frac{N^{K-\tau_0}}{(K-\tau_0)!} \tag{27}$$

(for the proof cf. ref. [12]); it has the following form (with $\tau_0=0$),

$$P_K^{(N)}(\alpha,\beta;\xi,0)=\binom{N}{K}\frac{\Gamma\left(\frac{\alpha}{\xi}+K\right)}{\Gamma\left(\frac{\alpha}{\xi}\right)}\frac{\Gamma\left(\frac{\beta}{\xi}+N-K\right)}{\Gamma\left(\frac{\beta}{\xi}\right)}\frac{\Gamma\left(\frac{\alpha+\beta}{\xi}\right)}{\Gamma\left(\frac{\alpha+\beta}{\xi}+N\right)}.$$

2. The normalized probability distribution (25) is presented in the form of several Figures for some values of the parameters involved; those are chosen only to see more clearly the typical behaviour of that probability distribution; the change of those parameters leads to the change of the height and of the shape of the extremes as well as to the shift of their location. Especially, Fig. 1a presents the distribution (25) for a "symmetrical" case $(\alpha=\beta)$: it is interesting to compare it with Fig. 1b taken from [13], where the rapidity distribution of away side jets resulting from different quark-gluon scattering processes is presented. Moreover, a qualitatively similar behaviour is seen also in Fig. 1c, where a class of solutions of the nonlinear Fokker-Planck equation for the laser model (with different values of the drift term) is presented (taken from [14]).

Furthermore, we see in Fig. 2a an "unsymmetrical" case $(\alpha\neq\beta)$ of the probability (25), while in Fig. 2b the fits to data on deep inelastic electroproduction

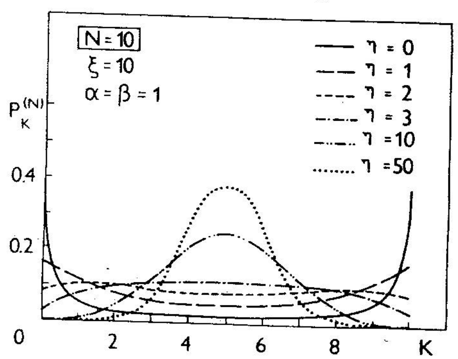

Fig. 1a. The normalized probability distribution $P_K^{(N)}(\alpha,\beta;\xi,\eta)$, rel. (25), for several values of the parameter η.

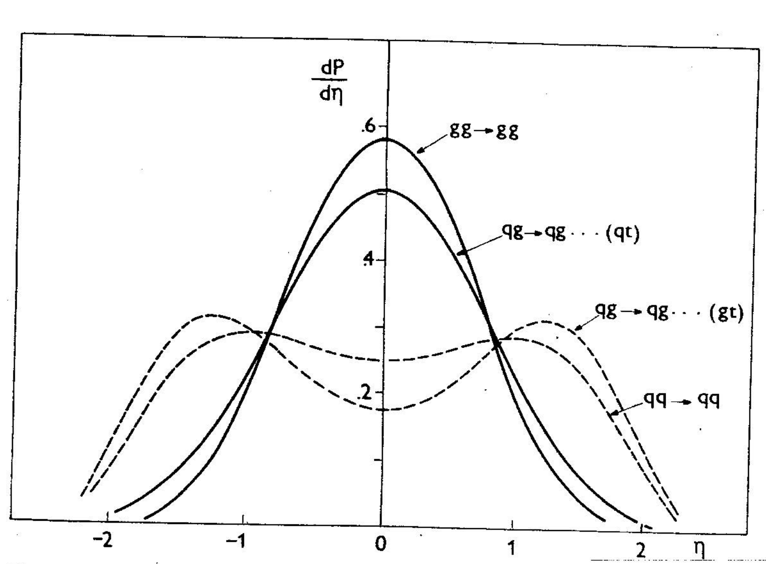

Fig. 1b. The rapidity distribution of away side jets resulting from different quark-gluon scattering processes; the curve (qt) corresponds to the quark trigger and (gt) to the gluon trigger (from [13]).

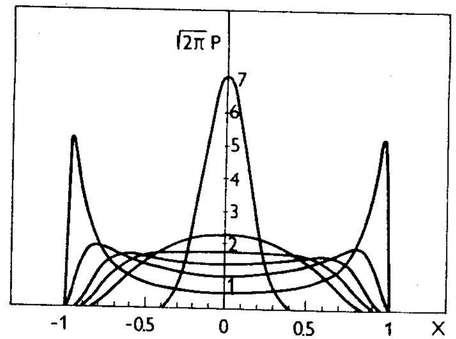

Fig. 1c. A distribution as it follows from the non-linear Fokker-Planck equation for the laser model with different values of the drift term (from [14]).

(taken from [15]), in Fig. 2c the energy distribution of fast (μ_F^-) and of slow (μ_S^-) negative muons as well as of positive muons (μ^+) for Drell-Yan trimuons averaged over the Harward-Pennsylvania-Wisconsin-Fermilab spectrum, and in Fig. 2d the invariant mass distributions for the same spectrum (the two last Figures are taken from [16]), in Fig. 2e the fast hadron distributions taken from Ref. [3], [17] (the full curves correspond to the experimental data). Moreover, other dependences of the probability (25) are seen in Figs. 3a, 3b, 3c. A lot of other high energy distributions, recently published in different papers might be quoted as well. The comparison of those Figures with the behaviour of the probability (25) suggests that the statistical approach developed in the present paper contains several common features of a wide class of production phenomena, where statistical approaches are to be considered.

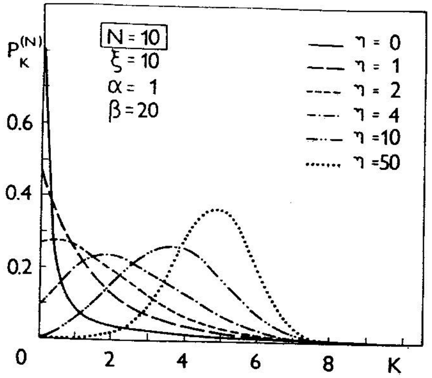

Fig. 2a. The probability $P_K^{(N)}$ $(\alpha, \beta; \xi, \eta)$, rel. (25), for an "unsymmetrical" case, $\alpha \neq \beta$.

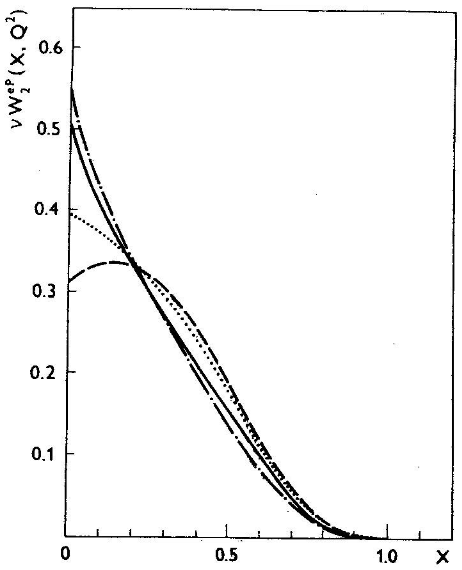

Fig. 2b. The fits to data on deep inelastic electro-production (from [15]).

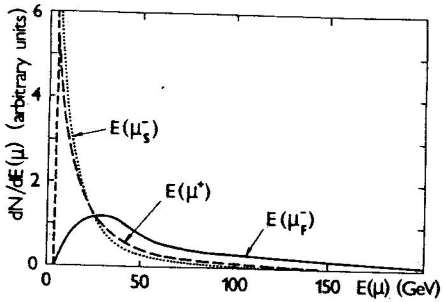

Fig. 2c. The energy distribution of fast (μ_F^-) and slow (μ_s^-) negative muons as well as of positive muons (μ^+) for Drell-Yan trimuons (from [16]).

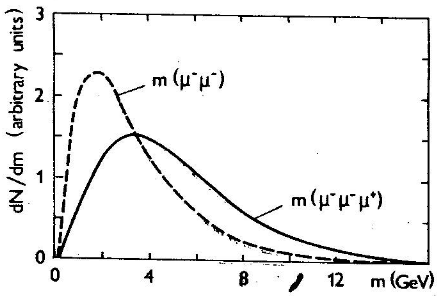

Fig. 2d. The invariant mass distributions (from [16]).

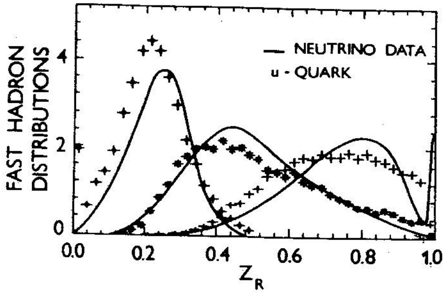

Fig. 2e. Fast hadron distributions (from [3], [17]).

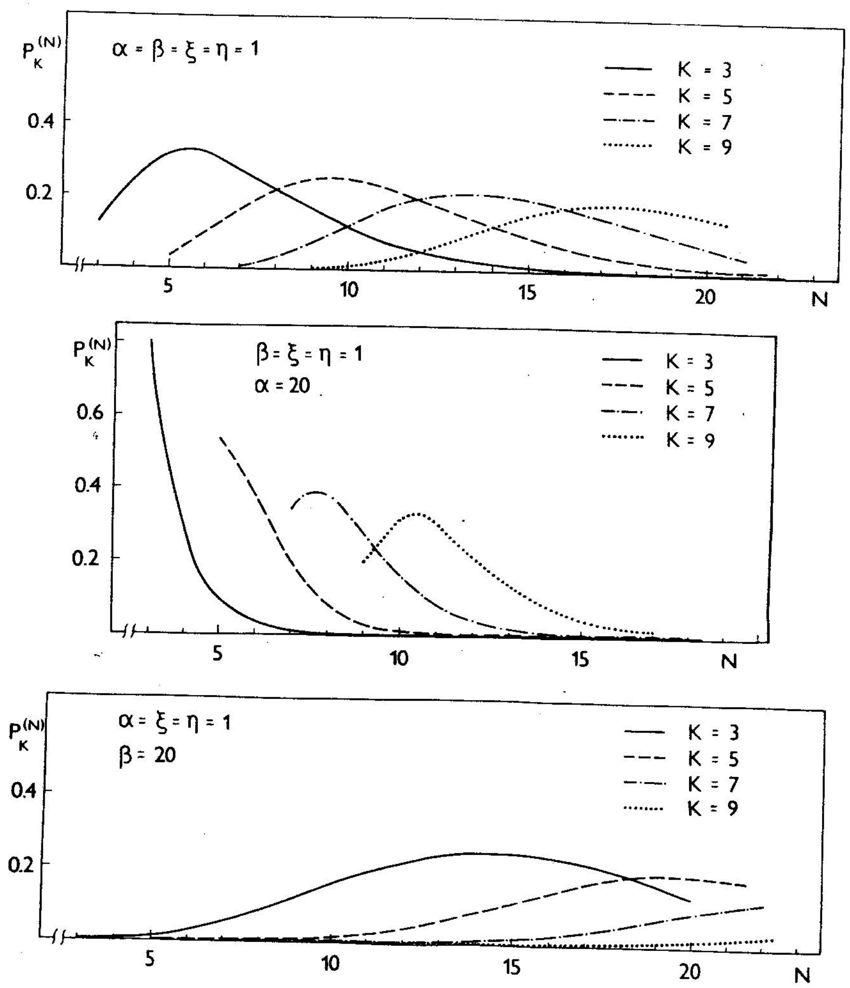

Fig. 3a, b, c. Several dependences of the probability $P_K^{(N)}$ $(\alpha, \beta; \xi, \eta)$, rel. (25), on the number of draws, N.

V. ASYMPTOTIC CASES

1. Let us assume that the starting numbers of all balls are very big, i.e. let there be a sea of balls (quarks) of each colour,

$$\beta_r \to \infty \tag{28a}$$

and let at the same time

$$|c_{fr}| \ll \beta_r \tag{28b}$$

for all c_{fr}'s as well as β_r's. Taking into account only the leading term of the Stirling expansion for the Γ-function we obtain from rel. (21)

$$P_K^{(N)} \to Z(\beta_f)^K \Big(\sum_{(i)} \beta_i\Big)^{N-K} \Big/ (\Sigma' \beta_r)^N \tag{29}$$

where

$$Z = \binom{N}{K}. \tag{30}$$

The probability (29) represents essentially the binomial distribution.

2. In the next step let us consider the limit when the number of draws is very big,

$$N \to \infty. \tag{31}$$

As it is convenient, the number of draws, N, might be identified with any physical quantity (like energy, momentum transfer squared, etc.).

With respect to the relation

$$\lim_{N\to\infty} \binom{N}{K} = \frac{N^K}{K!} \tag{32a}$$

and

$$\lim_{N\to\infty} \left(1 - \frac{p}{N}\right)^N = e^{-p} \tag{32b}$$

we obtain from rel. (29)

$$P_K^{(N)} \to \frac{N^K}{K!} \left(\frac{\beta_f}{\Sigma' \beta_r - \beta_f}\right)^K \exp\left(-N \frac{\beta_f}{\Sigma' \beta_r}\right). \tag{33}$$

The r.h.s. of rel. (33) represents the Poisson distribution; it can be rewritten in different ways in the form of rel. (1),

$$a(1 - x_F)^c \exp(-bP_T), \tag{1}$$

e.g. in such a way that

$$a = \frac{1}{K!} \left(\frac{N \Sigma' \beta_r}{\Sigma' \beta_r - \beta_f}\right)^K, \qquad c = K, \tag{34}$$

$$x_F = \frac{\Sigma' \beta_r - \beta_f}{\Sigma' \beta_r}, \qquad bP_T = N \frac{\beta_f}{\Sigma' \beta_r}. \tag{35}$$

According to the second equality in rel. (34), the parameter c in rel. (1) expresses directly the probability of the number of elementary acts (creations and annihilations) in the intermediate state; moreover, with respect to ref. [18], that

parameter might be simply connected with the slope α_r of the Regge trajectory. Another way of consideration leads simply from rel. (33) only to the leading factor of the exponential behaviour, namely to the expression

$$P_K^{(N)} \to \exp\left(-N \frac{\beta_f}{\Sigma' \beta_r}\right). \tag{36}$$

3. Let the number of balls of each colour approach infinity with the same speed. Then

$$\beta_f / \Sigma' \beta_r \to 1/s \tag{37}$$

and rel. (33) gives

$$P_K^{(N)} \to \frac{N^K}{K!} \left(\frac{1}{s-1}\right)^K \exp\left(-\frac{1}{s} N\right) \tag{33a}$$

and rel. (35) gives

$$x_F = 1 - \frac{1}{s} \tag{38}$$

as well as

$$bP_T = \frac{N}{s} = N(1 - x_F); \tag{39}$$

we see that the Feynman scaling variable x_F and the slope b are directly related to the number of sorts of constituents s. Moreover, let us consider a physical phenomenon which decreases exponentially with (say) energy,

$$\sim \exp(-\lambda E/E_0), \tag{2}$$

where E_0 fixes the energy scale. Now, identifying the number of draws, N, with the energy measured in a convenient scale

$$N \sim E/E_o, \tag{40}$$

the comparison of rel. (36) with (2) gives

$$\lambda \sim 1/s. \tag{41}$$

Rel. (41) suggests an alternative interpretation of the parameters (slopes) which enter into the exponential decrease of the form (2): in the (say) energy range where the exponential law (2) is valid, the parameter λ determines the number of sorts of constituents of the matter which are to be introduced in order to obtain just that exponential decrease.

As far as the statistical approach developed in the present paper is adequate to the physical phenomena under investigation it is worthwhile to emphasize that the details of the dynamics (i.e. the coefficients c_{fr}) do not enter into the asymptotics (29), (33) nor into the related expressions and therefore the asymptotics might fix only the number of sorts of constituents which are to be introduced at those (say) energies.

4. Let us consider the case when the probability is included in $P_K^{(N)}$ that the ball of the f-th colour

a) is drawn in all first α_+ as well as in all last ω_+ draws, as it was done in rel. (23). In the limit $\beta_r \to \infty$ the leading term of rel. (23) again has the form (29) but now (using rel. (27)) the factor Z is given by

$$Z = \binom{N-\alpha_+-\omega_+}{K-\alpha_+-\omega_+}, \tag{42a}$$

which in the limit $N \to \infty$ gives

$$\frac{(N-\alpha_+-\omega_+)^{K-\alpha_+-\omega_+}}{(K-\alpha_+-\omega_+)!} \to \frac{N^{K-\alpha_+-\omega_+}}{(K-\alpha_+-\omega_+)!}. \tag{43a}$$

With $\alpha_+ = \omega_+ = 0$ rel. (32a) is established;

b) is not drawn in all first α_- nor in all last ω_- draws, as it was done in rel. (24). In the limit $\beta_r \to \infty$ the leading term of (24) has again the form (29) but now

$$Z = \binom{N-\alpha_- -\omega_-}{K} \tag{42b}$$

which in the limit $N \to \infty$ gives

$$\frac{(N-\alpha_- -\omega_-)^K}{K!} \to \frac{N^K}{K!}. \tag{43b}$$

c) It can be seen that the introduction of the parameters $\alpha_+, \omega_+, \omega_-$ as well as of the "effective" threshold α_- does not change the functional form of the asymptotics (29) or (33): it influences only their normalization. Due to the fact that the normalization is not very well known from the data, the introduction of those parameters does not help to read off more details of dynamics from the asymptotic formulae. This fact cannot be improved by introducing the draw dependent (or in other way conditioned) coefficients c_{fr} as far as the condition (28b) is valid in the limit (28a).

5. There is another sector of asymptotic formulae which can be reached if in the limit (28a) the inequality (28b) is not valid, i.e. if so many balls (quarks) of some colours are created in the intermediate state that their numbers c_{fr} are proportional to the starting numbers β_r. Then with increasing β_r also the corresponding (not necessarily all) c_{fr}'s increase (in absolute value). However, the form of the corresponding asymptotic formulae depends on additional assumptions; they are not discussed in the present contribution.

We recall that the functional form of the asymptotics might be changed also in such a way that the increasing parameter N is identified not directly with (say) the energy as in rel. (40) but with (e.g.) the logarithm of the energy. Then rel. (36) leads to the power law decrease which with respect to rel. (37) has the form

$$P_K^{(N)} \sim E^{-\mathrm{const}/s}, \tag{44}$$

where s is the number of the sorts of the constituents (quarks etc.) which can be distinguished at those energies. (Some of the aforementioned results have been presented at the 1978 Liblice conference, ref. [19].)

VI. MODIFIED URN MODEL

In the present Section we consider briefly a modification of the urn model which involves the possibility that from the urn no ball at all is drawn; especially the procedure of Sect. II is here slightly modified.

1. Let us assume that

i) There are s sorts of coloured balls (s is an arbitrary positive integer). There are β_r balls of the r-th colour,

$$r = 1, 2, \ldots, s; \tag{45}$$

all β_r are arbitrary. Usually the expression

$$\beta_r/(\beta_1+\beta_2+\ldots+\beta_s)$$

gives the probability of drawing a ball of the r-th colour.

Due to the fact that we want to take into account also the possibility that in some draws there is no ball drawn at all from the urn under consideration, we define the probability that the ball of the r-th colour is drawn as follows

$$b_r\beta_r/(\beta_1+\beta_2+\ldots+\beta_s), \tag{46}$$

where the weights b_r are real and arbitrarily chosen from the interval

$$0 \leq b_r \leq 1; \tag{47}$$

they might be considered as reflecting the fundamental properties of the forces which influence how "easily" a constituent can be "drawn" (in another case, when a cube is thrown they might be related to the circumstance that not each edge is exactly like the others).

If the forces acting on the constituents of the r-th sort are very strong, the probability to observe those constituents (to draw them at a finite draw) is very small; the limit when the aforementioned probability decreases to zero (i.e. the attractive forces increase to infinity) is reflected in rel. (46) (and in its generalizations) by the condition

$$b_r \to 0. \tag{47a}$$

Moreover, let us define the probabilty that in a draw there is not drawn any ball at all from the urn, as follows

$$b_0\beta_0/(\beta_1+\beta_2+\ldots+\beta_s), \tag{48}$$

where the value of the expression $b_0\beta_0$ is determined from the normalization condition, i.e.

$$\frac{(b_1\beta_1+b_2\beta_2+\ldots+b_s\beta_s)+b_0\beta_0}{\beta_1+\beta_2+\ldots+\beta_s}=1, \tag{49}$$

or

$$b_0\beta_0=\Sigma'\beta_r(1-b_r) \tag{50}$$

(the summation Σ' is given by rel. (3)). The introduction of the parameter β_0 is convenient also with respect to the discussion of the asymptotic cases; at present it might be an arbitrary real number.

The procedure just described allows us to consider the case of drawing no ball as the case of drawing the ball of the "zeroth" colour; therefore, formally we shall consider the case of $s+1$ colours with

$$\varphi=0, 1, 2, \ldots, s;$$

the weights b_φ are connected by rel. (49) (one of them depends on the others). To be concrete we assume that all b_r, r given by rel. (45), are known from outer considerations and b_0 is determined by rel. (49).

ii) A ball, say of the φ-th colour is drawn at random and it is put back. Then just $c_{\varphi\varrho}$ ($\gtreqless 0$) new balls are added to the balls of the ϱ-th colour,

$$\varrho=0, 1, 2, \ldots, s.$$

The meaning of the coefficients c_{fr} is known from Sect. II and now the coefficients c_{0r} specify the number of balls which are added to the balls of the r-th colour if no ball is drawn.

The aforementioned procedure allows to create (or annihilate) the constituents in any elementary act in the intermediate state even in the case when no constituent is "drawn": the formulation of the problem in Sect. II does not allow that possibility due to the fact that there a ball of a colour always must be drawn. Therefore, now the "thresholds" might be introduced on a more physical basis.

If in the first draw the ball of the φ-th colour was drawn then in the urn we have $(\beta_r+c_{\varphi r})$ balls of the r-th colour. The expression

$$b_\varrho(\beta_\varrho+c_{\varphi\varrho})/\Sigma'(\beta_r+c_{\varphi r}) \tag{51}$$

gives the probability that the ball of the ϱ-th colour is drawn in the next draw. At the present stage we assume that the coefficients $c_{\varphi r}$ are given, while the

coefficients $c_{\varphi 0}$ are determined by means of the normalization of the probability, namely,

$$\frac{\Sigma' b_r(\beta_r+c_{\varphi r})+b_0(\beta_0+c_{\varphi 0})}{\Sigma'(\beta_r+c_{\varphi r})}=1 \tag{52}$$

or using rel. (50)

$$c_{\varphi 0}=\frac{\Sigma' c_{\varphi r}(1-b_r)}{\Sigma'\beta_r(1-b_r)}\beta_0 \tag{53}$$

(if all $b_r\equiv 1$, then all $c_{\varphi 0}=0$). The fulfilment of the condition (53) assures that the probability under consideration in any number of draws is normalized to unity.

2. Similarly as in Sect. II, in the present case we look for the (modified) probability $P_K^{(N)}$ that in N draws the ball of the φ-th colour is drawn altogether K-times. Using essentially the procedure as it is described in Sect. III and then Rel. (19), we obtain that probability in the following form,

$$\begin{aligned} P_K^{(N)}=\mathcal{P}_K^{(N)}\{z=\varphi,\ j_L=0;\ & b_r \text{ are real and fulfilling Rel. (47), while } b_0 \text{ is given by Rel. (50);} \\ & (\tau_\sigma)_{\min}=\tau_{\sigma-1},\ (\tau_\sigma)_{\max}=K \text{ for } \sigma=1, 2, \ldots, N-K; \\ & (\tau_1)_{\min}=\tau_0=0\}, \end{aligned} \tag{54}$$

say for $K=0, 1, 2, \ldots, N-1$ while for $K=N$ we have

$$P_K^{(N)}=\left(\frac{b_\varphi c_{\varphi\varphi}}{\Sigma' c_{\varphi r}}\right)^N \frac{\Gamma\left(\frac{\beta_\varphi}{c_{\varphi\varphi}}+N\right)}{\Gamma\left(\frac{\beta_\varphi}{c_{\varphi\varphi}}\right)} \frac{\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{\varphi r}}\right)}{\Gamma\left(\frac{\Sigma'\beta_r}{\Sigma' c_{\varphi r}}+N\right)}. \tag{55}$$

Now again the probabilities analogous to those given by Rels. (23) and (24) are easily expressed.

3. The investigation of the limit of the probability (54), (55) when all β's approach infinity leads to the following form of the (what might be called) binomial distribution,

$$\binom{N}{K}\frac{(b_\varphi\beta_\varphi)^K\left(\sum_{(\varrho)} b_\varrho\beta_\varrho\right)^{N-K}}{(\Sigma'\beta_r)^N}, \tag{56}$$

where

$$\sum_{(\varrho)}\equiv\sum_{\substack{\varrho=0\\ \varrho\neq\varphi}}^{s}$$

and the limit $N\to\infty$ leads from expression (56) to

$$\frac{N^K}{K!}\left(\frac{b_\varphi\beta_\varphi}{\sum_{(\varrho)} b_\varrho\beta_\varrho}\right)^K \exp\left(-N\frac{b_\varphi\beta_\varphi}{\Sigma'\beta_r}\right). \tag{57}$$

With the assumption that for $N\to\infty$ the quantity

$$N\frac{b_\varphi\beta_\varphi}{\Sigma'\beta_r}\equiv\varkappa \quad \text{is finite} \tag{58}$$

($\varkappa$ depends on the φ-th sort) the expression (57) can be again rewitten in the standard Poisson form,

$$(K!)^{-1}\varkappa^K \exp(-\varkappa). \tag{59}$$

4. To formulate the Moivre-Laplace limiting theorem let us denote

$$b_\varphi\beta_\varphi/\Sigma'\beta_r=P, \quad \sum_{(\varrho)} b_\varrho\beta_\varrho/\Sigma'\beta_r=Q, \tag{60}$$

$$\sum_{\varrho=0}^{s} b_\varrho\beta_\varrho/\Sigma'\beta_r=R \tag{61}$$

with

$$P+Q=R; \tag{62}$$

moreover let

$$y=(KR-NP)/\sqrt{NPQ}. \tag{63}$$

Now, rel. (56) in the limit $N\to\infty$ together with the conditions

$$\lim_{N\to\infty}(NPQ/R^2)=\infty, \tag{64}$$

$$\lim_{N\to\infty} y \quad \text{is finite} \tag{65}$$

leads to the Gaussian distribution in the form

$$\frac{R^{N+1}}{\sqrt{2\pi NPQ}}\cdot e^{-\frac{1}{2}\cdot y^2}\cdot\left\{1+\frac{1}{\sqrt{N}}\cdot\frac{yR^2(y^2-3)}{6Q\sqrt{PQ}}\cdot\left(1-3\frac{P}{R}+2\frac{P^2}{R^2}\right)+O\left(\frac{1}{N}\right)\right\}. \tag{66}$$

However, taking into account the normalization condition rel. (50) in rel. (61), we obtain $R\equiv1$; this value simplifies then also the expression (66).

5. If all β's approach infinity with the same speed we obtain from (57)

$$\frac{1}{K!}\left(N\frac{b_\varphi}{\sum_{(\varrho)} b_\varrho}\right)^K \exp\left(-N\frac{b_\varphi}{s}\right); \tag{67}$$

in this case the comparison of (67) with the exponential decrease (2) gives

$$\lambda\sim b_\varphi/s \tag{68}$$

instead of rel. (41) (i.e. the slopes are correlated by the number of sorts of constituents as well as by the normalized strength of the forces) while the Feynman scaling variable might be given by

$$x_F=1-b_\varphi/\Sigma'b_r; \tag{69}$$

moreover we have from rel. (50)

$$b_0=s-\Sigma'b_r.$$

6. It is worthwhile to emphasize that the weights specified by the parameters b_φ enter explicitly rels. (68), (69) as well as (63). Especially the limit $x_F\to1$ as well as $\varkappa$ = finite number (with $N\to\infty$) can be achieved now not only by divergent $\Sigma'b_r$ (including essentially the condition that the number of the sorts of constituents increases over any limit) but, for a given φ, $\varphi\neq0$, also by $b_\varphi\to0$ (i.e. rel. (47a)), expressing essentially the fact that even if there exist constituents of the φ-th sort, i.e. $\beta_\varphi\neq0$ (in the last approximations we had $\beta_\varphi\to\infty$), the probability to observe those constituents in finite draws approaches zero: this is just the statistical expression of the confinement of the constituents of that φ-th sort.

VII. CONCLUSIONS

The solution of the generalized urn problem, rel. (21), together with rels. (23) and (24) as well as of the modified urn problem, rel. (54) and others, suggest several physical implications; some special cases are mentioned in Sect. IV. Those solutions can serve also as the basis for the construction of a unified point of view as regards a wide set of physical phenomena where the change of the number of constituents in the intermediate state plays an important rôle.

The fit of the asymptotic leading term (33) with the data gives the easiest possibility to determine the number of the sorts of constituents which influence the process under consideration. On the other hand, if that number is known from other sources, the asymptotic leading term serves to fix the scale of the physical variable which is identified with the number of draws, N, and with respect to rel. (68) it serves to determine the weights b_φ. The approach developed in Sect. VI suggests also a statistical interpretation of the confinement.

In the framework of the statistical approach developed in the present paper the details of the dynamics (i.e. the coefficients c_{fr} or $c_{\varphi r}$ as well as the parameters α, ω or others) can be determined by the exact form (say, using a convenient fitting procedure) of the distribution (21) (or of the other related distributions) as well as by the corrections to the asymptotic leading term, i.e. in the region where the deviations from the "pure" asymptotical behaviour are observed.

The combination of several probabilities $P_K^{(N)}$ for different colours which have been drawn allows to consider the cases when more sorts of objects are observed.

In several cases the statistical distributions discussed in the present paper might replace the distributions based on the Monte Carlo approaches with the advantage that the parameters involved in the present paper offer an immediate physical interpretation.

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NOTE ADDED IN PROOF: Another interpretation of the probability distribution $P_K^{(N)}$, rel. (25) with $N\equiv E_{CM}$ follows from the comparison of Figs. 3a, 3b, 3c of the present paper with Fig. 4 (and with other Figure for the thrust) taken from A. Ali et al., Jet-like distributions from the weak decay of heavy quarks, DESY preprint 78/47, Sept. 1978.

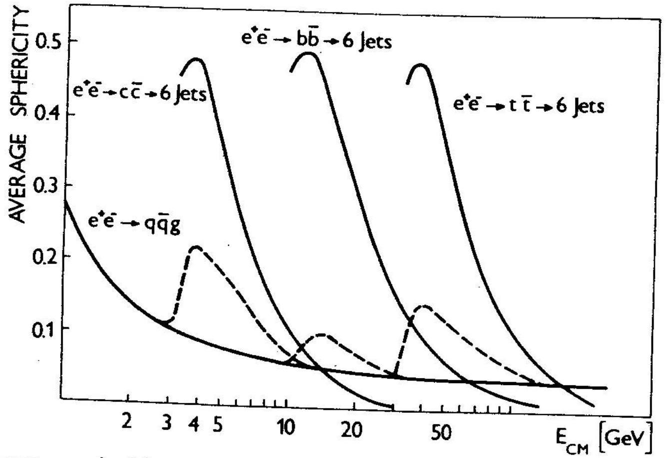

Fig. 4. The average sphericity vs. centre-of-mass energy for several intermediate states of the electron-positron annihilation leading to six jets.

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